The probability of exceeding a high boundary by a heavy-tailed branching random walk in varying environment

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Branching Process in Varying Environment

Let $\{\zeta_{n,j}\}_{n,j\geq 0}$ be the sequence of r.v.'s such that

- $\zeta_{n,j},\ n,j\geq 0$ are mutually independent;
- for any $n\geq 0$ r.v.'s $\zeta_{n,1},\zeta_{n,2},\dots$ have common distribution $\mathcal{P}_n$;

Then the process Z_n defined as follows:

$$Z_0=0, \qquad Z_{n+1}=\sum_{j=1}^{Z_n}\zeta_{n,j},\ n\geq 0$$

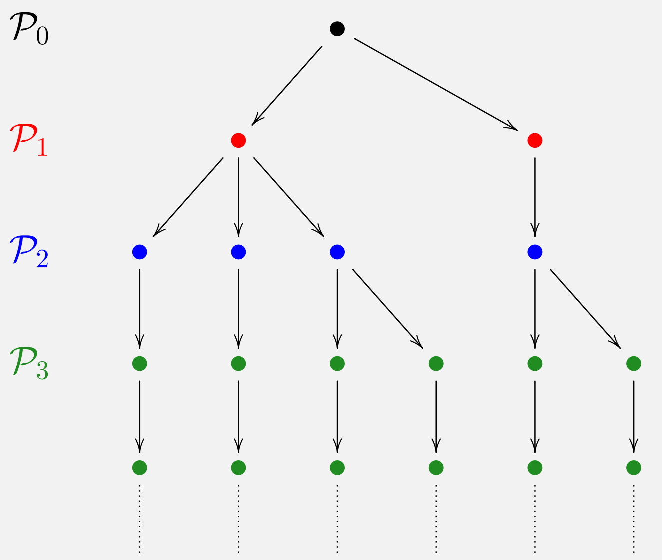
is the *Branching Process in Varying Environment* (BPVE).
Throughout we will assume that

$$\zeta_{n,1}\geq 1 \text{ a.s. for any } n\geq 1 \tag{1}$$

and

$$\mathbb{E}\zeta_{n,1}<\infty \quad \text{for all } n\geq 0 \tag{2}$$

The *genealogical tree* $\mathcal{T}=(\mathcal{V},\mathcal{E})$ for BPVE:



Properties of BPVE with fading

We assume that the following *condition for fading* holds:

$$L:=\prod_{n\geq 0}\mathbb{E}\zeta_{n,1}<\infty. \tag{3}$$

Why "fading"?

Proposition 1

Let (1) and (3) hold. Then there exists $Z_\infty\in L_1(\Omega)$ such that

$$Z_n\to Z_\infty \quad \textit{a.s. and in } L_1$$

and, in particular, $\mathbb{E}Z_\infty=L$.
Moreover, the fading time

$$\nu:=\inf\{n\geq 1: Z_n=Z_{n+1}=\ldots=Z_\infty\}<\infty \quad a.s.$$

In the left picture $\nu=3$ and $Z_\infty=6$.

Let $q_n=\mathbb{P}(\zeta_{n,1}\neq 1)$. What are the conditions for the *finiteness of the moments* of ν and Z_∞?

Proposition 2

Let (1) and (3) hold.

(i) For any nondecresing $g:\mathbb{R}^+\to\mathbb{R}^+$ we have

$$\mathbb{E}g(\nu)<\infty \iff \sum_{n\geq 0}g(n+1)q_n<\infty.$$

(ii) If, in addition to (3),

$$\prod_{n\geq 0}\mathbb{E}\zeta_{n,1}^s<\infty$$

for some $s>1$, then $\mathbb{E}Z_\infty^s<\infty.$

Branching Random Walk

For arbitrary path π in $\mathcal{T}$ started in the root of $\mathcal{T}$ let

$$S(\pi)=\sum_{e\in\pi}\xi_{n(e),j(e)}$$

be a *Branching Random Walk* (BRW), where $n(e)$ is the generation in which edge e ends, and $j(e)$ is the number of particle in generation $n(e)$ in which e ends and $\{\xi_{n,j}\}$ is the sequence of i.i.d. zero-mean r.v.'s with distribution F. We assume also that

$$\sigma(\zeta_{n,j},n,j\geq 0) \text{ and } \sigma(\xi_{n,j},n,j\geq 1) \text{ are independent} \tag{4}$$

For any non-negative function g on $\mathbb{Z}_+$ let

$$S^g(\pi)=S(\pi)-g(|\pi|)$$

be a *g-shifted* BRW.
Our main goal is to find the exact tail asymtotics

$$\mathbb{P}(R_\mu^g>x)\sim ? \quad \text{as } x\to\infty,$$

where $R_\mu^g=\sup_{\pi:|\pi|\leq\mu}S^g(\pi)$ is the rightmost point of g-shifted BRW up to random generation μ

We say that a conting r.v. $\mu\leq\infty$ *does not depend on the future of displacements* if, for any $n\geq 0$ and any events

$$A\in\sigma(\xi_e,n(e)\leq n;\mathbb{I}(\mu\leq n);\mathcal{T}) \text{ and } B\in\sigma(\xi_e,n(e)>n;\mathcal{T}),$$

where $n(e)$ is the number of generation in which edge e ends, the following equality holds

$$\mathbb{P}(AB|\mathcal{T})=\mathbb{P}(A|\mathcal{T})\mathbb{P}(B|\mathcal{T}) \quad \text{a.s.}$$

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Heavy-tailed distributions

F is *heavy-tailed* ($F \in \mathcal{H}$), if

$$\mathbb{E}e^{\lambda\xi} = \infty$$

for all $\lambda > 0$.

F is *long-tailed* ($F \in \mathcal{L}$), if $\overline{F}(x) > 0$ for all x and

$$\overline{F}(x+h) \sim \overline{F}(x) \text{ as } x \to \infty$$

for any fixed $h > 0$.

F is *subexponential* ($F \in \mathcal{S}$), if $F \in \mathcal{L}$ and

$$\overline{F * F} \sim 2\overline{F}(x) \text{ as } x \to \infty$$

for any fixed $h > 0$.

F is *strong subexponential* ($F \in \mathcal{S}^*$), if $\overline{F}(x) > 0$ for all x and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)dy \sim 2m_+\overline{F}(x) \text{ as } x \to \infty$$

where $m_+ = \mathbb{E}\xi^+$ must be finite.

Non Uniform Results

For R_μ when $\mu < \infty$ a.s.:

Theorem 1

Let (1), (2) and (4) hold, the σ-algebras $\sigma(\mu; \zeta_j^{(n)}, n \geq 0, j \geq 1)$ and $\sigma(\xi_{n,j}, n, j \geq 1)$ are independent and either

(i) $\mathbb{E}\mu Z_\mu < \infty$ and $F \in \mathcal{S}^*$ or

(ii) $\mathbb{E}Z_\mu(1+\delta)^\mu < \infty$ for some $\delta > 0$ and $F \in \mathcal{S}$.

Then

$$\mathbb{P}(R_\mu^{\hat{c}} > x) \sim \mathbb{E}\eta_\mu \cdot \overline{F}(x) \quad \text{as } x \to \infty$$

for any $c > 0$ where $\mathbb{E}\eta_\nu$ can be expressed as

$$\mathbb{E}\eta_\mu = \sum_{n\geq 1} \mathbb{E}\left[Z_n \mathbb{I}(\mu \geq n)\right].$$

For R_∞:

Theorem 2

Let (1), (3) and (4) hold and $F \in \mathcal{S}^*$. Rhen, given any $c > 0$,

$$\mathbb{P}(R_\infty^{\hat{c}} > x) \sim \frac{L}{c} \cdot \overline{F}_I(x) \quad \text{as } x \to \infty,$$

where $\overline{F}_I(x) = \min\{1, \int_x^\infty \overline{F}(t)dt\}$.

Here $\hat{c}(n) := cn$

Uniform Result

For any nonnegative function g on $\mathbb{Z}_+$ and any counting r.v. μ, let

$$H_\mu^g(x; \widehat{\mathcal{P}}) = \sum_{n\geq 1} \mathbb{E}\left[Z_n \mathbb{I}(\mu \geq n)\right] \overline{F}(x + g(n))$$

where $\widehat{\mathcal{P}} = (\mathcal{P}_0, \mathcal{P}_1, \ldots)$ is the sequence of offspring distributions.

Theorem 3

Let (1), (2) and (4) hold.

(i) Suppose that $F \in \mathcal{L}$. Then, given any independent of the future σ with $\mathbb{E}\eta_\sigma < \infty$, the inequality

$$\mathbb{P}(R_\mu^g > x) \geq (1 + o(1))H_\mu^g(x; \widehat{\mathcal{P}}) \quad \text{as } x \to \infty \tag{5}$$

holds uniformly over all nondecreasing g and all independent of the future $\mu \leq \sigma$ a.s.

(ii) Suppose additionaly that $F \in \mathcal{S}$. Then, given any $N \geq 1$, the equality

$$\mathbb{P}(R_\mu^g > x) = (1 + o(1))H_\mu^g(x; \widehat{\mathcal{P}}) \quad \text{as } x \to \infty \tag{6}$$

holds uniformly over all nondecreasing g and all independent of the future $\mu \leq N$ a.s.

Future Work

For any $c > 0$, let

$$\mathcal{G}_c = \{\text{ nonnegative} g : g(1) \geq c, g(n+1) - g(n) > c, \; n \geq 1\}$$

We believe that the following fact is true

Conjecture

Let (1), (3), (4) and "something else" hold.

(i) Suppose that $F \in \mathcal{L}$. Then, given any $c > 0$, the result (5) holds uniformly over all $g \in \mathcal{G}_c$ and all μ independet of the future.

(ii) Suppose additionally that $F \in \mathcal{S}^*$. Then, given any $c > 0$, the result (6) holds uniformly over all $g \in \mathcal{G}_c$ and all μ independent of the future.

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