Maximal Displacement of a Catalytic Branching Random Walk

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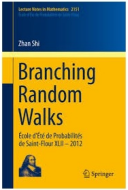
Introduction

The problem of the spread of population of particles (bacteria, individuals and etc.) or, in another context, the propagation of fire or epidemic has been attracting interest of many researchers. It is sufficient to recall the works by A.N.Kolmogorov, I.G.Petrovskiy, N.C.Piskunov and their celebrated KPP-equation (1937), B.A.Sevastyanov (1958), J.Biggins (1978), R.Durrett (1983), F.Comets, S.Popov (2007), M.A.Lifshits (2012), M.Roberts (2013), B.Mallein (2015), Z.Shi (2015), S.Bocharov, S.Harris (2016), E.Neuman, X.Zheng (2017), L.Wang, G.Zong (2019), Y.Nishimori, Y.Shiozawa (2020) and etc.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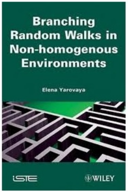
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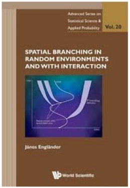
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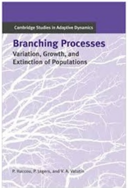
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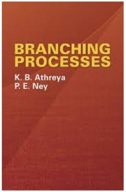
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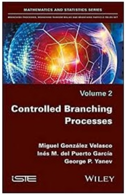
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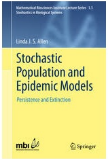
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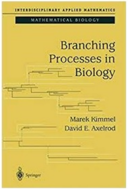
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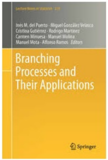
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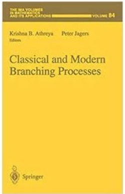
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Peter Jagers
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A frequent assumption in the model, describing the spread of particle population, is a behavioral homogeneity in space.

However, a special approach is needed to investigate the models with spatially non-homogeneous particles evolution.

We are interested in a model of catalytic branching random walk (CBRW) on $\mathbb{Z}^d$, $d \in \mathbb{N}$. There the particles may produce offspring at the presence of catalysts only, located at some fixed points of an integer lattice $\mathbb{Z}^d$, whereas outside these catalysts the particles perform a random walk without branching.

The contribution to the study of CBRW is due to the following researchers

- E.B. Yarovaya (1991,. . . , 2019),
- S.Albeverio, L.Bogachev (1998,. . . , 2000),
- V.A.Vatutin, V.A.Topchij (2003,. . . , 2011),
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- Ph.Carmona, Y.Hu (2014),
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Description of CBRW on $\mathbb{Z}^d$

Catalytic branching random walk (CBRW) is a probabilistic model of the particles population evolution due to offspring production and movement on integer lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$.

- The particles move on $\mathbb{Z}^d$ according to irreducible Markov chain $S = \{S(t), t \geq 0\}$ generated by the infinitesimal matrix $Q = (q(x,y))_{x,y \in \mathbb{Z}^d}$.
- The particles split at the locations of catalysts only.
  $W = \{w_1, \ldots, w_N\} \subset \mathbb{Z}^d$ is a set of catalysts.

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Description of CBRW on $\mathbb{Z}^d$

- When a particle reaches w_k, it spends there random time having exponential distribution with parameter $\beta_k > 0$. Then it either produces a random number ξ_k of offsprings with probability α_k or leaves w_k with probability $1 - \alpha_k$.
- The new particles behave as independent copies of the parent particle.
- At the initial time $t = 0$ there is a single particle on the lattice located at point x.

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Classification of CBRW

Similar to ordinary Galton-Watson branching process, CBRW can be classified as supercritical, critical or subcritical, see E.Vl.Bulinskaya (2014).

As shown in E.Vl.Bulinskaya (2015), only in supercritical CBRW both total and local particle numbers grow exponentially to infinity (with positive probability) as time tends to infinity. The rate of exponential growth is called Malthusian parameter and denoted by $\nu > 0$.

In critical or subcritical CBRW the total number of particles does not change after some time and is a random variable in time-limit, which might be 0.

Notation and assumptions

Let $Z(t)$ be a (random) set of particles existing in CBRW at time $t \geq 0$.
For a particle $z \in Z(t)$ denote by $X^z(t)$ its position at time t.

Let $\mathcal{I}$ stand for the event of infinite number of visits of the catalysts set.

Similarly to Ph.Carmona, Y.Hu (2014) study for $d = 1$ we assume for any d that the random walk is space-homogeneous.
It means that $q(x, y) = q(x - y, 0) = q(0, y - x)$.

Light tails. Assumptions

Assume also that the function

$$H(s) := \sum_{x\in\mathbb{Z}^d} e^{\langle s,x\rangle} q(0,x)$$

is finite for any $s \in \mathbb{R}^d$ where $\langle\cdot,\cdot\rangle$ stands for the inner product of vectors. This assumption is Cramér's condition for the jump value of the random walk S. It is easy to check that the Hessian of H is positive definite and, consequently, H is a convex function. Put also

$$\mathcal{R} = \left\{ r \in \mathbb{R}^d : H(r) = \nu \right\},$$

that is, $\mathcal{R}$ is the level set of the function H for the level ν.

Light tails. Assumptions

Finally, let

$\mathcal{O}_\varepsilon := \{x \in \mathbb{R}^d \colon \exists\, r \in \mathcal{R} \text{ s.t. } \langle x, r\rangle > \nu + \varepsilon\}, \varepsilon \geq 0,$

$\mathcal{Q}_\varepsilon := \{x \in \mathbb{R}^d : \langle x, r\rangle < \nu - \varepsilon \ \forall r \in \mathcal{R}\},\ \varepsilon \in [0, \nu),$

$\mathcal{O} := \mathcal{O}_0,\ \mathcal{Q} := \mathcal{Q}_0,\ \mathcal{P} := \partial\mathcal{Q} = \partial\mathcal{O},$

where $\partial\mathcal{S}$ stands for the boundary of set $\mathcal{S} \subset \mathbb{R}^d$.

It follows from the definition of $\mathcal{P}$ that

$\mathcal{P} = \{x \in \mathbb{R}^d : \langle x, r\rangle \leq \nu$ for all $r \in \mathcal{R}$ and $\langle x, r\rangle = \nu$ for at least one $r \in \mathcal{R}\}$.

Note that each set $\mathcal{Q}_\varepsilon$, $\mathcal{Q}$ or $\mathcal{P} \cup \mathcal{Q}$ is convex as an intersection of half-spaces.

Light tails. Main results

Theorem (1)

Let the formulated assumptions be satisfied for a supercritical CBRW on $\mathbb{Z}^d$. Then, for any $x \in \mathbb{Z}^d$, one has

$$\mathsf{P}_x\big(\omega : \forall \varepsilon > 0\ \exists t_0 = t_0(\omega, \varepsilon) \text{ such that } \forall t \geq t_0 \text{ and } \forall v \in Z(t),\ X^v(t)/t \notin \mathcal{O}_\varepsilon\big) = 1,$$

$$\mathsf{P}_x\big(\omega : \forall \varepsilon \in (0, \nu) \exists t_1 = t_1(\omega, \varepsilon) \text{ such that } \forall t \geq t_1\ \exists v \in Z(t),\ X^v(t)/t \notin \mathcal{Q}_\varepsilon \,|\, \mathcal{I}\big) = 1.$$

Light tails. Main results

Theorem 1 means that if we divide the position coordinates of each particle existing in CBRW at time t by t and then let t tend to infinity, then in the limit there are a.s. no particles outside set $\mathcal{P} \cup \mathcal{Q}$ and under condition of infinite number of visits of catalysts there are a.s. particles on $\mathcal{P}$.

In this sense it is natural to call the set $\mathcal{P}$ the propagation front of the particles population.

Light tails. Main results

The next theorem states that each point of $\mathcal{P}$ can be considered as a limiting point for the normalized particles positions in CBRW.

Theorem (2)

Let conditions of Theorem 1 be satisfied. Then, for each $y \in \mathcal{P}$, one has

$$\mathsf{P}_x \left(\omega : \forall t \geq 0 \ \exists v_y = v_y(t,\omega) \in Z(t) \text{ such that } \lim_{t\to\infty} \frac{X^{v_y}(t)}{t} = y \middle| \mathcal{I} \right) = 1.$$

Light tails. Main results

Theorem 3 yields one more way to determine the propagation front $\mathcal{P}$.

Theorem (3)

The set $\mathcal{P}$ can also be specified as
$\mathcal{P} = \{z(r) : r \in \mathcal{R}\}$, where
$z(r) = \frac{\nu}{\langle \nabla H(r), r \rangle} \nabla H(r)$.

Light tails. Main results

Note that our results show that the particles population spreads asymptotically linearly on $\mathbb{Z}^d$ with respect to growing time and the form of the propagation front does not depend on the number of catalysts and their locations but depends only on the value of the Malthusian parameter ν and the function $H(\cdot)$ characterizing the random walk.

In other words, in our limit theorems the normalizing factor of the particles positions is equal to t^{-1} and does not depend on the dimension of the lattice.

Methods of the Study

The many-to-one formula is derived in the most general form in S.Harris, M.Roberts (2017) and being applied for CBRW is of the following form

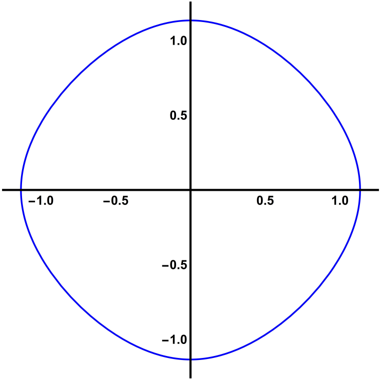
$$\mathsf{E}_x \sum_{v \in N(t)} g(X_v(t)) = \mathsf{E}_x g(S(t)) \times \prod_{k=1}^{N} \exp\{\alpha_k \beta_k (m_k - 1) L(t; w_k)\},$$

where $L(t; y) := \int\limits_0^t \mathbb{I}(S(u) = y)\, du$, $y \in \mathbb{Z}^d$, $t \geq 0$, is the local time of the random walk S at level y, and $g : \mathbb{R}^d \to \mathbb{R}$ is a measurable function.

Methods of the Study

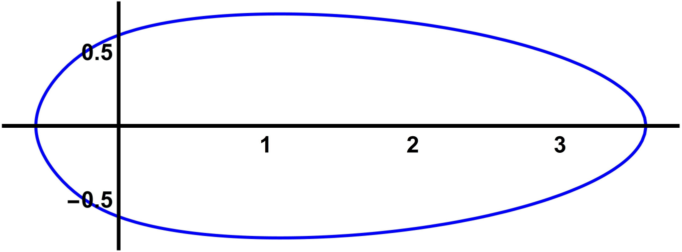
- renewal theorems for systems of renewal equations
- martingale change of measure
- convex analysis
- large deviation theory
- the coupling method
- our results E.Vl.Bulinskaya (2014) and E.Vl.Bulinskaya (2015).

Light tails. Examples



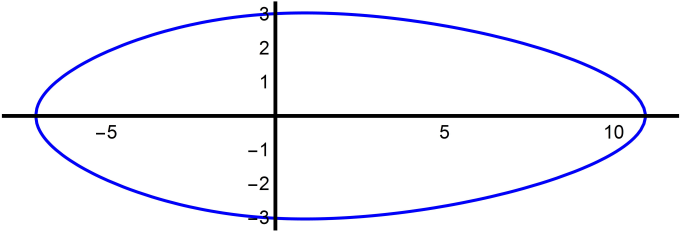
The limiting shape of the front in a simple CBRW on $\mathbb{Z}^2$

Light tails. Examples



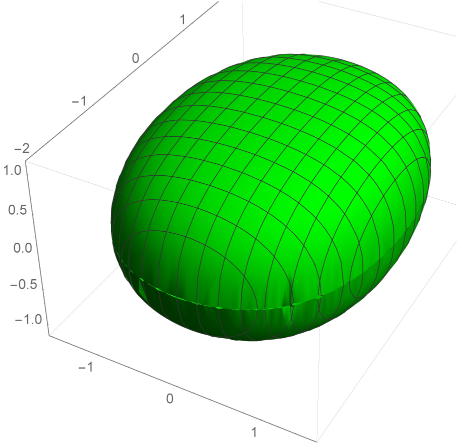
The limiting shape of the front in an asymmetric CBRW on $\mathbb{Z}^2$

Light tails. Examples



The limiting shape of the front in CBRW on $\mathbb{Z}^2$. Coordinates of the jump have the corresponding Poisson and Bernoulli distribution.

Light tails. Examples



The limiting shape of the front in CBRW on $\mathbb{Z}^3$

Semi-exponential distribution of jump. Assumptions

Let components of the jump vector $Y = (Y_1, \dots, Y_d)$ of the random walk S have semi-exponential distribution, i.e. for any $i = 1, \dots, d$ and $y \in \mathbb{Z}_+$ one has

$$\mathsf{P}(Y_i > y) = L_i^{(1,+)}(y) \exp\left\{-y^{\gamma_i^+} L_i^{(2,+)}(y)\right\} := R_i^+(y),$$

$$\mathsf{P}(Y_i < -y) = L_i^{(1,-)}(y) \exp\left\{-y^{\gamma_i^-} L_i^{(2,-)}(y)\right\} := R_i^-(y).$$

Symbol "+" refers to the right tail, whereas symbol "−" refers to the left one. For $i = 1, \dots, d$ and $\kappa \in \{+, -\}$, functions $L_i^{(1,\kappa)}(y)$ and $L_i^{(2,\kappa)}(y)$, $y \in \mathbb{Z}_+$, vary slowly and $\gamma_i^\kappa \in (0, 1)$.

Semi-exponential distribution of jump. Assumptions

It follows from the latter assumptions that $-\ln R_i^\kappa(y)$, $y \in \mathbb{Z}_+$, is a regularly varying function of index γ_i^κ. It is known that there exists an asymptotically uniquely determined inverse function $R_i^{-1,\kappa}(s)$, $s \geq 0$, in the sense that $-\ln R_i^\kappa\left(R_i^{-1,\kappa}(y)\right) \sim y$, $R_i^{-1,\kappa}(-\ln R_i^\kappa(y)) \sim y$ as $y \to \infty$, $y \in \mathbb{Z}_+$, and

$$R_i^{-1,\kappa}(s) = s^{1/\gamma_i^\kappa} L_i^{(3,\kappa)}(s),$$

where function $L_i^{(3,\kappa)}(s)$, $s \geq 0$, varies slowly.

Assume that for each $x = (x_1, \dots, x_d)$, $x_i \neq 0$, $i = 1, \dots, d$, one has

$$\mathsf{P}_0\big(\operatorname{sgn}(x)S(u)/R^{-1,\kappa(x)}(t) \in [|x|, +\infty)\big) = (1 + \delta(u,t))$$

$$\times h(u) \prod_{i=1}^{d} \left(\mathsf{P}\left(\operatorname{sgn}(x_i) Y_i \geq |x_i| R_i^{-1,\kappa(x_i)}(t)\right)\right)^{(1-\varepsilon_i(u,t))},$$

where $h(u) = h(u, x)$, $u \geq 0$, is a positive non-decreasing function such that $h(u) \sim cu^d$, $u \to \infty$, for some constant $c > 0$.

For each $i = 1, \dots, d$, the non-negative function $\varepsilon_i(u, t) = \varepsilon_i(u, t, x) \to 0$, as $t \to \infty$, uniformly in $u/t \in [0, 1]$, and for all t large enough the inequality $\varepsilon_i(u_1, t) \leq \varepsilon_i(u_2, t)$ is valid, whenever $u_1 \leq u_2$, $u_1, u_2 \in [0, t]$.

Function $\delta(u, t) = \delta(u, t, x) \to 0$, as $t \to \infty$, uniformly in $u/t \in [0, 1]$.

The notation $\operatorname{sgn}(x)S(u)/R^{-1,\kappa(x)}(t)$ is a vector in $\mathbb{R}^d$ with ith coordinate $\operatorname{sgn}(x_i)S_i(t)/R_i^{-1,\kappa(x_i)}(t)$, and $[|x|,+\infty) := [|x_1|,+\infty) \times \ldots \times [|x_d|,+\infty)$. Here $\kappa(x_i) =$"$+$", if $x_i \geq 0$, and $\kappa(x_i) =$"$-$", if $x_i < 0$.

In the paper E.Vl.Bulinskaya, 2020 (Mathematical Population Studies: an International Journal of Mathematical Demography) we show that the important condition above is satisfied, in particular, for the case of independent coordinates of the walk jump when the absolute values of positive and negative components have discrete Weibull distribution.

Define the following sets in $\mathbb{R}^d$

$$\mathcal{O}_\varepsilon := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i|^{\gamma_i^{\kappa(x_i)}} > \nu + \varepsilon \right\}, \quad \varepsilon \geq 0,$$

$$\mathcal{Q}_\varepsilon := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i|^{\gamma_i^{\kappa(x_i)}} < \nu - \varepsilon \right\}, \ \varepsilon \in [0, \nu),$$

$$\mathcal{O} := \mathcal{O}_0, \quad \mathcal{Q} := \mathcal{Q}_0,$$

$$\mathcal{P} := \partial\mathcal{O} = \partial\mathcal{Q} = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i|^{\gamma_i^{\kappa(x_i)}} = \nu \right\}.$$

Semi-exponential distribution of jump. Results

Stipulate that $X^v(u)/R^{-1,\kappa}(t)$ is a vector in $\mathbb{R}^d$ with ith coordinate equal to $X_i^v(u)/R_i^{-1,\kappa(X_i^v(u))}(t)$.

Theorem (4)

Let the conditions above be satisfied for a supercritical CBRW on $\mathbb{Z}^d$. Then for any $z \in \mathbb{Z}^d$ we have
$\mathsf{P}_z(\omega : \forall \varepsilon > 0\ \exists t_1 = t_1(\omega, \varepsilon)$ s.t. $\forall t \geq t_1$ and $\forall v \in Z(t),\ X^v(t)/R^{-1,\kappa}(t) \notin \mathcal{O}_\varepsilon) = 1$,
$\mathsf{P}_z(\omega : \forall \varepsilon \in (0, \nu)\ \exists t_2 = t_2(\omega, \varepsilon)$ s.t. $\forall t \geq t_2$ $\exists v \in Z(t),\ X^v(t)/R^{-1,\kappa}(t) \notin \mathcal{Q}_\varepsilon | \mathcal{I}) = 1$.

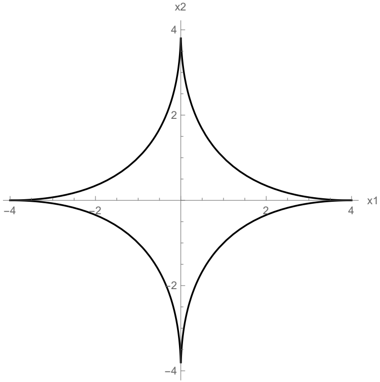
Semi-exponential distribution of jump. Results

The following result states that each point of the surface $\mathcal{P}$ can be considered as a limit point for normalized locations of particles in CBRW, i.e. the surface $\mathcal{P}$ is minimal in a sense.

Theorem (5)

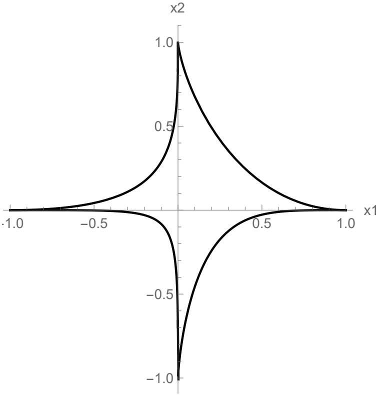
Let conditions of Theorem 4 be satisfied. Then, for each $z \in \mathbb{Z}^d$ and $y \in \mathcal{P}$, one has
$\mathsf{P}_z\left(\omega : \forall t \geq 0\ \exists v_y = v_y(t,\omega) \in Z(t)\right.$ s.t.
$\left.\lim_{t\to\infty} X^{v_y}(t)/R^{-1,\kappa}(t) = y \middle| \mathcal{I}\right) = 1.$

Semi-exponential distribution. Examples



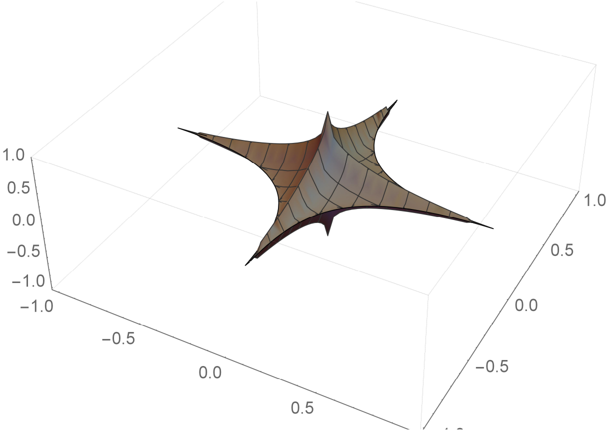
The limit shape of the front in a symmetric CBRW on $\mathbb{Z}^2$

Semi-exponential distribution. Examples



The limit shape of the front in a non-symmetric CBRW on $\mathbb{Z}^2$

Semi-exponential distribution. Examples



The limit shape of the front in a CBRW on $\mathbb{Z}^3$

Let the components of the vector jump $Y = (Y_1, \ldots, Y_d)$ of the random walk S have regularly varying tails, i.e., for each $i = 1, \ldots, d$ and $y \in \mathbb{Z}_+$,

$$\mathsf{P}(Y_i \geq y) = y^{-\gamma_i^+} L_i^{(1,+)}(y) =: R_i^+(y),$$
$$\mathsf{P}(Y_i \leq -y) = y^{-\gamma_i^-} L_i^{(1,-)}(y) =: R_i^-(y).$$

The symbol "$+$" refers to the right tail, whereas the symbol "$-$" marks the left one. For $i = 1, \ldots, d$ and $\kappa \in \{+, -\}$ the function $L_i^{(1,\kappa)}(y)$, $y \in \mathbb{Z}_+$, is slowly varying and $\gamma_i^\kappa \in (0, +\infty)$.

The latter assumptions imply that $R_i^\kappa(y)$, $y \in \mathbb{Z}_+$, is a regularly varying function of index $-\gamma_i^\kappa$.

It is known (see, e.g., E.Seneta (1976)) that there exists an asymptotically uniquely determined inverse function $R_i^{-1,\kappa}(s)$, $s \geq 0$, in the sense that $1/R_i^\kappa\left(R_i^{-1,\kappa}(y)\right) \sim y$, $R_i^{-1,\kappa}(1/R_i^\kappa(y)) \sim y$ as $y \to \infty$, $y \in \mathbb{Z}_+$, and

$$R_i^{-1,\kappa}(s) = s^{1/\gamma_i^\kappa} L_i^{(2,\kappa)}(s),$$

where the function $L_i^{(2,\kappa)}(s)$, $s \geq 0$, varies slowly.

Heavy tails. Independent jump coordinates. Assumptions

Let the following condition be true

$$\mathsf{P}_0\Big(\operatorname{sgn}(x)S(u)/R^{-1,\kappa(x)}(e^{\nu t}) \in [|x|\,, +\infty)\Big) = h(u)$$
$$\times\, (1+\delta(u,t,x)) \prod_{i=1}^{d} \mathsf{P}\left(\operatorname{sgn}(x_i)Y_i \geq |x_i| R_i^{-1,\kappa(x_i)}(e^{\nu t})\right).$$

Here $h(u)$, $u \geq 0$, is a positive non-decreasing function such that $h(u) \sim cu^d$, $u \to \infty$, for some constant $c > 0$.

Function $\delta(u,t,x) \to 0$ as $t \to \infty$ uniformly in $u/t \in [0,1]$ and $\min\{x_1,\ldots,x_d\} \geq \varepsilon$, for each $\varepsilon > 0$.

Write $\operatorname{sgn}(x)S(u)/R^{-1,\kappa(x)}(e^{\nu t})$ for a vector in $\mathbb{R}^d$ with ith coordinate $\operatorname{sgn}(x_i)S_i(t)/R_i^{-1,\kappa(x_i)}(e^{\nu t})$ and $[|x|,+\infty) := [|x_1|,+\infty) \times \ldots \times [|x_d|,+\infty)$. Here $\kappa(x_i) = $ "+" whenever $x_i \geq 0$ and $\kappa(x_i) = $ "−"

Define the following sets in $\mathbb{R}^d$

$$\mathcal{O}_\varepsilon := \left\{ x \in \mathbb{R}^d : \prod_{i=1}^{d} |x_i|^{\gamma_i^{\kappa(x_i)}} \geq \varepsilon \right\}, \quad \varepsilon \geq 0,$$

$$V\left(\vec{\lambda}^{+}, \vec{\lambda}^{-}\right) := \left[-\left(\lambda_1^{-}\right)^{-1/\gamma_1^{-}}, \left(\lambda_1^{+}\right)^{-1/\gamma_1^{+}}\right] \times \ldots \times \left[-\left(\lambda_d^{-}\right)^{-1/\gamma_d^{-}}, \left(\lambda_d^{+}\right)^{-1/\gamma_d^{+}}\right],$$

$$\vec{\lambda}^{\kappa} = \left(\lambda_1^{\kappa}, \ldots, \lambda_d^{\kappa}\right), \ \lambda_i^{\kappa} \geq 0,$$

$$\Lambda\left(\vec{\lambda}^{+}, \vec{\lambda}^{-}\right) = \bigcup_{i=1}^{d} \Big\{ x_j = 0, j \neq i, j = 1, \ldots, d, \ x_i \in \left[-\left(\lambda_i^{-}\right)^{-1/\gamma_i^{-}}, \left(\lambda_i^{+}\right)^{-1/\gamma_i^{+}}\right] \Big\}.$$

Stipulate that $X^z(u)/R^{-1,\kappa}(e^{\nu t})$ is a vector in $\mathbb{R}^d$ with ith coordinate $X_i^z(u)/R_i^{-1,\kappa(X_i^y(u))}(e^{\nu t})$.

Theorem (6)

Let the listed above conditions be satisfied for supercritical CBRW on $\mathbb{Z}^d$.
Then for any starting point $x \in \mathbb{Z}^d$ one has
$\mathsf{P}_x(\omega : \forall \varepsilon > 0 \; \exists t_1 = t_1(\omega, \varepsilon)$ s.t. $\forall t \geq t_1$ and $\forall z \in Z(t), \; X^z(t)/R^{-1,\kappa}(e^{\nu t}) \notin \mathcal{O}_\varepsilon) = 1.$

Hence, in the time-limit a.s. all the particles in CBRW with properly normalized positions are inside the surface $\mathbb{R}^d \setminus \mathcal{O}_\varepsilon$.

Theorem (7)

Let conditions of Theorem 6 be valid. Moreover, let $\mathsf{E}\xi_k \ln(\xi_k + 1) < \infty$ for any $k = 1, \ldots, N$. Then there exists a function $\varphi(\lambda; x)$, $\lambda \geq 0$, $x \in \mathbb{Z}^d$, such that, for any $\lambda_i^\kappa \geq 0$, $\kappa \in \{+, -\}$, $i = 1, \ldots, d$, and $x \in \mathbb{Z}^d$, one has

$$\mathsf{P}_x\left(\forall z \in Z(t), X^z(t)/R^{-1,\kappa}(e^{\nu t}) \in V\left(\vec{\lambda}^+, \vec{\lambda}^-\right)\right) \to \varphi\left(\sum_{i,\kappa} \lambda_i^\kappa; x\right), \quad t \to \infty.$$

It gives the distribution of the limiting surface Λ (the limiting shape of the front) containing all the particles with the normalized positions.

Here $\varphi(\lambda; x) \in (0,1)$, $\varphi(0; x) = 1$ and $\varphi(\lambda; x)$ tends to the probability of the local extinction of the particles population in CBRW as $\lambda \to +\infty$, for each fixed $x \in \mathbb{Z}^d$.

Moreover, as $x \in \mathbb{Z}^d \setminus W$, the function $\varphi(\lambda; x)$, $\lambda \geq 0$, admits the following representation

$$\varphi(\lambda; x) = \sum_{k=1}^{N} \int_0^\infty \varphi(\lambda e^{-\nu u}; w_k)\, d_{W_k} F_{x,w_k}(u)$$

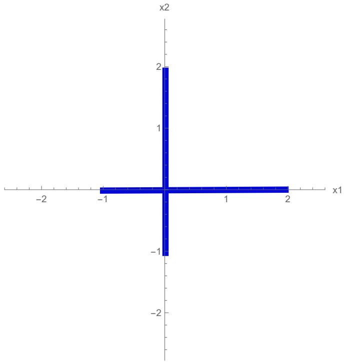
$$+1 - \sum_{k=1}^{N} w_k F_{x,w_k}(\infty),$$

where the functions $\varphi(\,\cdot\,; w_j)$, $j = 1, \ldots, N$, satisfy the system of integral equations

$$\varphi(\lambda; w_j) = \alpha_j \int_0^\infty f_j(\varphi(\lambda e^{-\nu u}; w_j))\, dG_j(u)$$
$$+ (1-\alpha_j) \sum_{k=1}^{N} \int_0^\infty \varphi(\lambda e^{-\nu u}; w_k)\, dG_{j,k}(u)$$
$$+ (1-\alpha_j)\left(1 - \sum_{k=1}^{N} w_k F_{w_j,w_k}(\infty)\right).$$

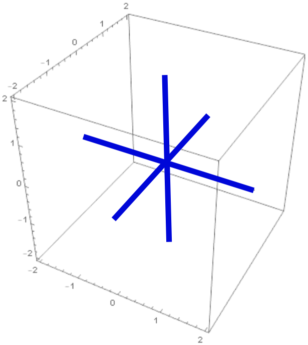
This system has a unique solution in a certain function class.

Heavy tails. Independent jump coordinates. Examples



The limiting shape of the front in CBRW on $\mathbb{Z}^2$

Heavy tails. Independent jump coordinates. Examples



The limiting shape of the front in CBRW on $\mathbb{Z}^3$

Heavy tails. Isotropic case. Assumptions

For a set $A \subset \mathbb{R}^d$ separated from the origin by a ball of a radius $b > 0$, we assume that
$\mathsf{P}\left(Y^1 \in uA\right) \sim V(u)F(A), \quad \text{as} \quad u \to \infty,$
where $V(u) = u^{-\gamma}L_1(u)$, $u \geq 0$, is a regularly varying function of index $-\gamma$, $\gamma \in (0, +\infty)$, $L_1(u)$, $u \geq 0$, is a slowly varying function at infinity and $F(A)$ is a functional defined on a suitable class of sets such that, as $u \to \infty$,
$\mathsf{P}\left(Y^1 \in s + uA\right) \sim \mathsf{P}\left(Y^1 \in uA\right)$ for $\|s\| = o(u)$.
The latter property simply expresses the continuity of the functional F: we have $F(v + A) \sim F(A)$, as $\|v\| \to 0$.

Heavy tails. Isotropic case. Assumptions

According to E.Seneta (1976) there exists an asymptotically uniquely determined inverse function $V^{-1}(u) = u^{1/\gamma}L_2(u)$, $u \geq 0$, where L_2 is a slowly varying function at infinity.

Set $N(t) := V^{-1}\left(e^{\nu t}\right) = e^{\nu t/\gamma}L_2(e^{\nu t})$, $t \geq 0$.

We also assume that the normalizing factor $N(t)$, $t \geq 0$, belongs to the maximum jump approximation zone of the random walk S, i.e.

$$\mathsf{P}(S(u)/N(t) \in A) \sim qu\mathsf{P}(Y^1 \in N(t)A), \; t \to \infty,$$

uniformly in $u/t \in [0,1]$. Broad sufficient conditions for its validity can be found, e.g., in Borovkov and Borovkov (2008).

Heavy tails. Isotropic case. Main result

Theorem (8)

Let assumptions above be satisfied for supercritical CBRW on $\mathbb{Z}^d$. Then for the function $\varphi(\lambda; x)$, $\lambda \geq 0$, $x \in \mathbb{Z}^d$, introduced above, and for any $x \in \mathbb{Z}^d$, one has, as $t \to \infty$,

$$\mathsf{P}_x \left(\forall z \in Z(t) : X^z(t)/N(t) \notin A\right) \to \varphi\left(F(A); x\right).$$

Invoking different sets A under specific assumptions on common distribution of the components of the random walk jump Y leads to more detailed description for the limiting shape of the front of the particles population.

For its proof we use the system of equations derived by us

$$E_{w_i}(t;\mathcal{U}) = \alpha_i \int_0^t (1 - f_i(1 - E_{w_i}(t-s;\mathcal{U})))\, dG_i(s)$$

$$+(1-\alpha_i)\sum_{j=1}^{N}\int_0^t E_{w_j}(t-s;\mathcal{U})dG_{i,j}(s) + I_{w_i}(t;\mathcal{U}),$$

where $E_{w_i}(t;\mathcal{U}) := \mathsf{P}_{w_i}(\exists z \in Z(t) : X^z(t) \in \mathcal{U})$, $t \geq 0$, $i = 1, \ldots, N$, $\mathcal{U} \subset \mathbb{R}^d$, $W \cap \mathcal{U} = \varnothing$, and

$$I_{w_i}(t;\mathcal{U}) = \sum_{y\notin W}(1-\alpha_i)\frac{q(w_i,y)}{q}$$

$$\times\int_0^t \mathsf{P}_y(S(t-s)\in\mathcal{U},\, {}_{w_k}\tau_{y,w_k} > t-s, k=1,\dots,N)\, dG_i(s)$$

Difficulties arising in analysis of the system of equations consist in that we substitute a set which depends on time instead of vector $\mathcal{U}$. For this reason the equations are no longer of convolution-type and the renewal theory is not applicable.

Maximal displacement of critical and subcritical CBRWs

Since the particle population in critical and subcritical CBRW on $\mathbb{Z}^d$ locally degenerates (it also degenerates globally whenever the random walk is recurrent), the problem of the spread of a particle cloud on the lattice is naturally reformulated in the following way: what is the maximal displacement of particles in CBRW from the origin for the whole history of the particle population existence?

Let $M_t := \max\{X^z(t), z \in Z(t)\}$ be the maximum of CBRW at time t and $M := \max\{M_t, t \geq 0\}$ be the maximal displacement for its whole history.

In the following 4 theorems we consider a simple random walk S on $\mathbb{Z}$. It means that $\frac{q(x,x+1)}{-q(x,x)} = p$, $\frac{q(x,x-1)}{-q(x,x)} = q$, $q(x,y) = 0$, $|x-y| \geq 2$, where $p+q=1$ and $p, q \in (0,1)$.

Such a random walk is symmetric, whenever $p = q$, and asymmetric otherwise. In one jump a particle performing a simple random walk on $\mathbb{Z}$ moves to the nearest point to the right with probability p and to the nearest point to the left with probability q. A simple random walk on $\mathbb{Z}$ is recurrent if and only if it is symmetric.

The asymptotic results in Theorems 9–12 hold true under wider assumptions of any finite number of catalysts and an arbitrary starting point. The difference consists in the constants arising in the asymptotics. However, for the sake of brevity we concentrate on the case of a single catalyst located at the origin being the starting point as well.

Theorem (9)

Let $f'(1) = 1$ and $f''(1) = \sigma^2 \in (0, \infty)$, where $f(s) = \mathsf{E}s^{\xi}$, $s \in [0, 1]$, for CBRW on $\mathbb{Z}$, in which the random walk S is simple and symmetric. Then

$$\mathsf{P}_0\left(M > x\right) \sim \frac{\sqrt{1-\alpha}}{\sqrt{\alpha\sigma^2\sqrt{x}}}, \quad x \to \infty.$$

The result of Theorem 9 is a counterpart of the main result of S.P.Lalley, Y.Shao (2015) derived for the model of a critical BRW on $\mathbb{Z}$.

However in the latter model the decay rate of the probability $\mathsf{P}_0\,(M > x)$ has an order $1/x^2$, as $x \to \infty$.

Therefore, the particles in the critical CBRW manage to go father away from the origin before returning to it and, possibly, dying, than in the model of BRW, in which the particles may die at any point.

Theorem 10 gives the solution to the same problem as in Theorem 9. The only difference is that now we consider a subcritical CBRW on $\mathbb{Z}$.

Theorem (10)

Let $m = f'(1) < 1$ for a CBRW on $\mathbb{Z}$, in which the random walk S is simple and symmetric. Then

$$\mathsf{P}_0\left(M > x\right) \sim \frac{1-\alpha}{2\alpha(1-m)x}, \quad x \to \infty.$$

The result of Theorem 11 is a counterpart of the main result of E.Neuman, X.Zheng (2017) devoted to a subcritical BRW on $\mathbb{Z}$. However, in the latter case the probability $\mathsf{P}_0\left(M > x\right)$ decays exponentially-fast that differs importantly from us.

Theorems 9 and 10 are focused on the case of a simple symmetric random walk on $\mathbb{Z}$.

The two following theorems are devoted to investigation of critical and subcritical CBRW in which the random walk is a simple and asymmetric one, i.e. it has a drift to the right, whenever $p > q$, or to the left, whenever $p < q$.

Because of a drift the random walk is no longer a recurrent one.

Correspondingly, the criticality condition of CBRW changes as well.

Now $r := 1 - {}_{\varnothing}F_{0,0}(\infty) \in (0, 1)$, and the criticality of CBRW implies that $\alpha m + (1-\alpha)(1-r) = 1$ which is equivalent to $m = 1 + r\alpha^{-1}(1-\alpha)$.

In the next theorem we estimate the distribution tail of the random variable M for a critical CBRW on $\mathbb{Z}$, in which the underlying random walk is simple and asymmetric.

Theorem (11)

Let $m = 1 + r\alpha^{-1}(1-\alpha)$ and $f''(1) = \sigma^2 \in (0,\infty)$ for a CBRW on $\mathbb{Z}$, in which the random walk is simple and asymmetric. Then

$$\mathsf{P}_0\left(M > x\right) \sim \frac{\sqrt{2(1-\alpha)(q-p)}}{\sqrt{\alpha\sigma^2}} \left(\frac{p}{q}\right)^{\frac{x+1}{2}}, \text{ if } p < q,$$

$$\mathsf{P}_0\left(M > x\right) \to s_0, \text{ if } p > q,$$

as $x \to \infty$, where $s_0 \in (0,1)$ is a unique solution to equation $\alpha(1 - f(1-s)) + (2q(1-\alpha) - 1)\,s + (1-\alpha)(p-q) = 0$ with respect to unknown variable s, $s \in [0,1]$.

The following result contains solution to the same problem which is the subject of Theorem 11, but now for subcritical CBRW on $\mathbb{Z}$.

Theorem (12)

Let $m < 1 + r\alpha^{-1}(1-\alpha)$, for CBRW on $\mathbb{Z}$, in which the random walk is simple and asymmetric. Then

$\mathsf{P}_0\left(M > x\right) \sim \frac{(1-\alpha)(q-p)}{1-2p(1-\alpha)-\alpha m}\left(\frac{p}{q}\right)^{x+1}$, whenever $p < q$,

$\mathsf{P}_0\left(M > x\right) \to s_0$, whenever $p > q$,

as $x \to \infty$, where $s_0 \in (0,1)$ is a unique root of equation $\alpha(1 - f(1-s)) + (2q(1-\alpha) - 1)\,s + (1-\alpha)(p-q) = 0$ with respect to unknown variable s, $s \in [0,1]$.

The results of Theorems 11 and 12 are expected.

Namely, if the random walk S has a drift to the left ($p < q$), then the particles in CBRW do not manage to go far away to the right, since they drift to the left.

Conversely, if the random walk S has a drift to the right ($p > q$), then there are particles in CBRW which will go away to the right to "infinity", and therefore $M = \infty$ with positive probability s_0.

The results obtained by us are the first investigation in the domain of description of population propagation in critical and subcritical CBRW. It is worthwhile to note that visible differences in the propagation of particle population in supercritical CBRW and supercritical BRW were revealed only in the second term of the asymptotic expansions for their corresponding maximums, see Ph.Carmona, Y.Hu (2014), B.Mallein (2016) and E.Vl.Bulinskaya (2019). Meanwhile, as shown in our investigations, in critical and subcritical CBRWs and the corresponding critical and subcritical BRWs the differences are noticeable already in the first asymptotic approximation of the probability.

List of our publications in this direction (2013-2020)

1. E.Vl. Bulinskaya. Subcritical catalytic branching random walk with finite or infinite variance of the offspring number Proceedings of the Steklov Institute of Mathematics, 282 (2013), no. 1, p. 62–72.

2. E.Vl. Bulinskaya. Finiteness of hitting times under taboo. Statistics and Probability Letters, 85 (2014), no. 1, p. 15-19.

3. E.Vl. Bulinskaya. Complete classification of catalytic branching processes. Theory of Probability and its Applications, 59 (2014), no. 4, p. 545–566.

4. E.Vl. Bulinskaya. Strong and weak convergence of the population size in a supercritical catalytic branching process. Doklady Mathematics, 92 (2015), no. 3, p. 714–718.

5. E.Vl. Bulinskaya. Spread of a catalytic branching random walk on a multidimensional lattice. Stochastic processes and their applications, 128 (2018), no. 7, p. 2325-2340.

6. E.Vl. Bulinskaya. Maximum of a catalytic branching random walk. Russian Mathematical Surveys, 74 (2019), no. 3, p. 546–548.

7. E.Vl. Bulinskaya. Multidimensional catalytic branching random walk with regularly varying tails. Proceedings of the 2019 2nd International Conference on Mathematics and Statistics. Prague, Czech Republic, July 08-10, 2019. ACM International Conference Proceeding Series, New York, United States, 2019, p. 6-13.

8. E.Vl. Bulinskaya. Fluctuations of the Propagation Front of a Catalytic Branching Walk. Theory of Probability and its Applications, 64 (2020), no. 4, p. 513-534.

9. E.Vl. Bulinskaya. Isotropic multidimensional catalytic branching random walk with regularly varying tails. Computer Research and Modeling, 11 (2019), no. 6, p. 1033-1039.

10. E.Vl. Bulinskaya. Maximum of catalytic branching random walk with regularly varying tails. Journal of Theoretical Probability, (2020), DOI 10.1007/s10959-020-01009-w.

11. E.Vl. Bulinskaya. Catalytic branching random walk with semi-exponential increments. Mathematical Population Studies: an International Journal of Mathematical Demography, (2020), DOI 10.1080/08898480.2020.1767424.

12. E.Vl. Bulinskaya. On the maximal displacement of catalytic branching random walk, Siberian Electronic Mathematical Reports, 17(2020), p.1088-1099.