Some inequalities in boundary crossing problems for random walks

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Outline

1. Some remarks on the inequalities for the distribution of trajectory supremum

2. Two-sided bounds for the ruin probability

3. Bounds for the expectation of the first exit time from the strip

1. Some remarks on the inequalities for the distribution of trajectory supremum

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables,

$$S_n = X_1 + \ldots + X_n, \qquad S_0 = 0, \qquad \mathbf{E}X < 0.$$

Denote

$$S = \sup_{n \geq 0} S_n, \qquad Q(t) = \mathbf{P}(S \geq t).$$

The function Q is used in many stochastic models (queueing theory, insurance theory, etc.) but its explicit expression is available only for some particular cases. This explains our interest to the lower and upper bounds for this function.

Let $\varphi(\lambda) := \mathbf{E}e^{\lambda X}$ and assume that

$$\varphi(\gamma) < \infty \quad \text{for some } \gamma > 0 \tag{1}$$

(one-sided Cramér condition).
Suppose, in addition, that $\varphi(\gamma) \geq 1$. It is easily seen that the function φ is convex and $\varphi(\mu) = 1$ for some $\mu > 0$.

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It is well-known that

$$Q(t) \leq e^{-\mu t}$$

(Lundberg inequality).

The following inequalities appeared in [Theorem 15.3.5, A.A.Borovkov, Prob. Theory, 2013]): under condition (1)

$$re^{-\mu t} \le Q(t) \le Re^{-\mu t}, \qquad \text{where} \qquad r \le R \le 1, \qquad (2)$$

$$r^{-1} = \sup_{0<t<s} \mathbf{E}(e^{\mu(X-t)}|X > t), \quad R^{-1} = \inf_{0<t<s} \mathbf{E}(e^{\mu(X-t)}|X > t).$$

We denote here $s = \infty$ if $\mathbf{P}(X < t) < 1$ for all $t > 0$, otherwise

$$s = \inf\{t > 0 : \mathbf{P}(X \le t) = 1\}.$$

Recall that this result was obtained under one-sided Cramér condition. It means that the tail distribution $\mathbf{P}(X > t)$ decreases at infinity as exponent or faster.

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If $\mathbf{P}(X > t) = qe^{-\alpha t}, \quad t > 0,$ then $r = R = \dfrac{\alpha}{\alpha - \mu}$.

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Consider now the case when $\mathbf{P}(X > t)$ decreases at infinity faster than exponent. It turned out that $R = 1$ in this case. It means that upper bound in the inequality (2) coincides with the Lundberg inequality.

Theorem

Suppose that
(i) $\mathbf{P}(X \leq C) = 1$ *for some* $C < \infty$*;*
or
(ii) $\mathbf{P}(X < t) < 1$ *for all* t *and*

$$\int_{(t,\infty)} \mathbf{P}(X > y)dy = o(\mathbf{P}(X > t)), \qquad t \to \infty.$$

In both these cases we have $R = 1$*.*

2. Bounds for the ruin probability

Due to this theorem we will use the inequality (2) in the form

$$re^{-\mu t} \le Q(t) \le e^{-\mu t}.$$

As a consequence of this inequality we can present two-sided bounds for the probability to leave the interval (ruin probability in a game setting).
For arbitrary $a > 0, b > 0$ introduce

$$N = \min\{n \ge 1 : S_n \notin (-a, b)\},$$

and let $\beta(a, b) = \mathbf{P}(S_N \ge b)$.

The following inequalities were established in [Lotov, Stat.Prob.Let.,2019]. If $\mathbf{E}X < 0$ then

$$\beta(a,b) \geq \frac{Q(b) - Q(a+b)}{1 - Q(a+b)}.$$

If, in addition, $\mathbf{E}(X^-)^2 < \infty$, where $X^- = \max\{-X, 0\}$, and $Q(t)$ is a convex function for $t > 0$, then

$$\beta(a,b) \leq \frac{Q(b) - Q(a+b+d)}{1 - Q(a+b+d)}, \qquad d := \frac{\mathbf{E}(X^-)^2}{|\mathbf{E}X|}.$$

The certain problem with upper bound in this theorem is the convexity assumption. It is open problem to find sufficient convexity condition for the function Q in general case. At the same time we see that $Q(t)$ is located between two convex functions $re^{-\mu t}$ and $e^{-\mu t}$.

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Thus, we come to the following estimates for $\beta(a, b)$.

Theorem

Assume that $\mathbf{E}X < 0$, X *satisfies condition* (1), *and* $\varphi(\gamma) \geq 1$. *Then*

$$\frac{re^{-\mu b} - e^{-\mu(a+b)}}{1 - re^{-\mu(a+b)}} \leq \beta(a,b).$$

If, in addition, $\mathbf{E}(X^-)^2 < \infty$ *then*

$$\beta(a,b) \leq \frac{e^{-\mu b} - re^{-\mu(a+b+d)}}{1 - re^{-\mu(a+b+d)}}.$$

Now suppose that one-sided Cramér condition does not hold, that is

$$\mathbf{E}e^{\lambda X} = \infty \quad \text{for all} \quad \lambda > 0,$$

and consider random walk $S'_n = X'_1 + \cdots + X'_n$ where

$$X'_i = X_i I_{\{X_i < a+b\}} + (a+b) I_{\{X_i \ge a+b\}}.$$

It is easily seen that the probability $\beta(a,b)$ remains the same after replacement random walk $\{S_n\}$ to $\{S'_n\}$. Clearly, X'_i satisfies one-sided Cramér condition, and other conditions of the previous theorem hold. Thus, the bounds of the previous theorem are valid without Cramér condition as well. Note that, in this case, the numbers μ, r, and d must be defined by the distribution of the truncated random variable

$$X' = X I_{\{X<s\}} + s I_{\{X \ge s\}}, \qquad s = a+b.$$

Let numbers $m_{\pm}^{(k)}$ satisfy following inequalities:

$$m_{-}^{(k)} \le \mathbf{E}|S_N{+}a|^k;\ S_N \le -a)|, \quad m_{+}^{(k)} \le \mathbf{E}((S_N{-}b)^k;\ S_N \ge b).$$

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For example, one can take

$$m_{+}^{(k)} = \mathbf{E}((X_1 - b)^k;\ X_1 \ge b)$$

or

$$m_{+}^{(k)} = \mathbf{E}((X_1 - b)^k;\ X_1 \ge b)$$
$$+\mathbf{E}((X_1 + X_2 - b)^k;\ X_1 \in (-a, b),\ X_1 + X_2 \ge b),$$

and so on.

Theorem

Assume that $\mathbf{E}X = 0$ *and* $\mathbf{E}|X|^3 < \infty$. *Then*

$$\frac{a - KC + m_-^{(1)}}{a+b} \leq \beta(a,b) \leq \frac{a + KC - m_+^{(1)}}{a+b},$$

where $C = \dfrac{\mathbf{E}|X|^3}{\mathbf{E}X^2}$ *and* $0 < K \leq 3$ *is an absolute constant.*

3. Bounds for $\mathbf{E}N$

The problem of finding lower and upper bounds for $\mathbf{E}N$ is interesting and important especially for the sequential probability ratio test (SPRT, Wald, 1947). We remind its setting.

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Suppose that we have i.i.d. observations $Y_1, Y_2, \ldots$ with unknown distribution function F. We have to test two simple hypotheses $\{H_1 : \ F = F_1\}$ versus $\{H_2 : \ F = F_2\}$. The idea is to construct an auxiliary random walk $\{S_n, n \geq 1\}$ with

$$S_n = X_1 + \cdots + X_n, \qquad X_i = \log \frac{f_2(Y_i)}{f_1(Y_i)},$$

where f_1 and f_2 are densities of F_1 and F_2 related to some measure ν.

It is easily seen that $\mathbf{E}_1 X_1 < 0$ and $\mathbf{E}_2 X_1 > 0$ where $\mathbf{E}_j$ and $\mathbf{P}_j$ correspond to the validity of the assumption $F = F_j$.

Given numbers $a > 0$ and $b > 0$, we accept H_1 if $S_N \le -a$ and accept H_2 if $S_N \ge b$. Thus, $\mathbf{P}_1(S_N \ge b)$ and $\mathbf{P}_2(S_N \le -a)$ are the error probabilities which are under study for SPRT.

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The number $\mathbf{E}N$ (usually called average sample number) is very important characteristics of the test due to its known optimality property.

Suppose than H_1 is true. Then

$$\mathbf{E}_1 e^{\lambda X_1} = \int \frac{f_2^\lambda(y)}{f_1^\lambda(y)} f_1(y)\nu(dy) = 1$$

for $\lambda = 1$. Thus, $\mu = 1$ for this case and theorem of previous section provides bounds for the error probability $\mathbf{P}_1(S_N \ge b)$.

Consider random walk of general type and let τ be overshoot, that is

$$\tau = \begin{cases} S_N + a, & \text{if } S_N \le -a, \\ S_N - b, & \text{if } S_N \ge b. \end{cases}$$

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Suppose that $\mu_1 = \mathbf{E} X < 0$ and apply Wald's identity:

$$\mathbf{E}\, S_N = \mu_1 \mathbf{E}\, N = \mathbf{E}\,(-a + \tau;\, S_N \le -a) + \mathbf{E}\,(b + \tau;\, S_N \ge b)$$

$$= a - (a+b)\mathbf{P}(S_N \ge b) + \mathbf{E}\,(|\tau|;\, S_N \le -a) - \mathbf{E}\,(\tau;\, S_N \ge b).$$

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In the case when $\mathbf{E}X = 0$ we suppose that $\mu_2 := \mathbf{E}X^2 < \infty$ and use Wald's identity $\mathbf{E}\, S_N^2 = \mu_2\, \mathbf{E}\, N$. We have here

$$\mathbf{E}\, S_N^2 = a^2\mathbf{P}(S_N \le -a) + b^2\mathbf{P}(S_N \ge b) - 2a\mathbf{E}\,(\tau;\, S_N \le -a)$$

$$+2b\mathbf{E}\,(\tau;\, S_N \ge b) + \mathbf{E}\,(\tau^2;\, S_N \le -a) + \mathbf{E}\,(\tau^2;\, S_N \ge b).$$

Put $\alpha(a,b) = 1 - \beta(a,b)$.
Upper and lower bounds for $\beta(a,b)$ and $\alpha(a,b)$ are presented in Sec.2 in both cases $\mathbf{E}X < 0$ and $\mathbf{E}X = 0$.
Thus, we have to find two-sided inequalities for expectations of the form

$$\mathbf{E}\,(|\tau|^k;\, S_N \le -a), \qquad \text{and} \quad \mathbf{E}\,(\tau^k;\, S_N \ge b).$$

Denote

$$\eta_+ = \min\{n \geq 1 : \; S_n \geq b\}, \qquad \chi_+ = S_{\eta_+} - b,$$

$$\eta_- = \min\{n \geq 1 : \; S_n \leq -a\}, \qquad \chi_- = S_{\eta_-} + a,$$

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then

$$\mathbf{E}\,(\tau^k;\, S_N \geq b) = \mathbf{E}\,(\chi_+^k;\, S_N \geq b) = \mathbf{E}\chi_+^k I_{\{S_N \geq b\}}, \qquad k \geq 1,$$

and, by the Cauchy - Bunyakovsky inequality,

$$\mathbf{E}\,(\tau^k;\, S_N \geq b) \leq \sqrt{\mathbf{E}(I_{\{S_N \geq b\}})^{2k}\mathbf{E}\chi_+^{2k}} = \sqrt{\beta(a,b)\mathbf{E}\chi_+^{2k}},$$

and, similarly,

$$\mathbf{E}\,(|\tau|^k;\, S_N \leq -a) \leq \sqrt{\alpha(a,b)\mathbf{E}|\chi_-|^{2k}}.$$

We further use the following inequalities.
1) If $\mathbf{E}X < 0$, $\mathbf{E}(X^-)^{k+1} < \infty$, then

$$\mathbf{E}|\chi_-|^k \le L_k := \frac{k+2}{k+1} \frac{\mathbf{E}(X^-)^{k+1}}{|\mathbf{E}X|}, \qquad k \ge 1,$$

uniformly in $a < 0$ (G.Lorden, 1970);

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uniformly in $a < 0$ (G.Lorden, 1970);

2) If $\mathbf{E}X = 0$, $\mathbf{E}X^{k+2} < \infty$, then

$$\mathbf{E}\chi_+^k \le M_k := C\frac{k+2}{k+1}\cdot\frac{\mathbf{E}|X|^{k+2}}{\mathbf{E}X^2}, \qquad \mathbf{E}|\chi_-|^k \le M_k, \quad C \le 2,$$

uniformly in $a < 0$, $b > 0$ (A. Mogulskii, 1973).

We come to the following upper and lower bounds.

Theorem

Suppose that $\mathbf{E}X < 0$ *and* $\mathbf{E}(X^-)^3 < \infty$. *Then*

$$|\mu_1|\mathbf{E}\, N \leq a - (a+b)\beta(a,b) + \sqrt{\alpha(a,b)L_2} - m_+^{(1)}.$$

Theorem

Suppose that $\mathbf{E}X < 0$ *and one-sided Cramér condition* (1) *holds. Then*

$$|\mu_1|\mathbf{E}\, N \geq a - (a+b)\beta(a,b) - \mu^{-1}e^{-\mu b} + m_-^{(1)}.$$

Remind that arbitrary lower bounds for the expectation of overshoots can be taken as $m_\pm$:

$$m_-^{(1)} \leq |\mathbf{E}(S_N + a;\, S_N \leq -a)|, \qquad m_+^{(1)} \leq \mathbf{E}(S_N - b;\, S_N \geq b).$$

Next consider random walk with $\mathbf{E}X = 0$. Such a situation may arise also when considering so-called operating characteristics of SPRT which corresponds to the calculation $\mathbf{P}(S_N \leq -a)$ and $\mathbf{E}N$ under validity of some third hypothesis.

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Suppose that $\mu_2 := \mathbf{E}X^2 < \infty$ and use Wald's identity $\mathbf{E}\,S_N^2 = \mu_2\,\mathbf{E}\,N$. Recall that

$$\mathbf{E}\,S_N^2 = a^2\mathbf{P}(S_N \le -a) + b^2\mathbf{P}(S_N \ge b) - 2a\mathbf{E}\,(\tau;\, S_N \le -a)$$

$$+2b\mathbf{E}\,(\tau;\, S_N \ge b) + \mathbf{E}\,(\tau^2;\, S_N \le -a) + \mathbf{E}\,(\tau^2;\, S_N \ge b).$$

Put $\alpha(a,b) = 1 - \beta(a,b)$. After estimation of each term of this expression we come to the following inequalities.

Theorem

Suppose that $\mathbf{E}X = 0$ *and* $\mathbf{E}X^4 < \infty$. *Then*

$$\mu_2 \mathbf{E}\, N \le a^2\alpha(a,b) + b^2\beta(a,b) + 2M_1^{1/2}\Big(a\sqrt{\alpha(a,b)}$$

$$+b\sqrt{\beta(a,b)}\Big) + M_2^{1/2}\Big(\sqrt{\alpha(a,b)} + \sqrt{\beta(a,b)}\Big).$$

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Suppose that $\mathbf{E}X = 0$ *and* $\mathbf{E}X^2 < \infty$. *Then*

$$\mu_2 \mathbf{E}\, N \ge a^2\alpha(a,b) + b^2\beta(a,b) + 2am_-^{(1)} + 2bm_+^{(1)} + m_-^{(2)} + m_+^{(2)}.$$

Thank you for attention!