A continuous-time version of the Derrida–Retaux model

Bastien Mallein
Joint work with
Yueyun Hu (Université Sorbonne Paris Nord) and Michel Pain (NYU)

LAGA — Université Sorbonne Paris Nord

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From home

Derrida–Retaux model

The Derrida–Retaux model

Let X_0 be a non-negative random variable. We set for all $n \in \mathbb{N}$,

$$X_{n+1} = (X_n^{(1)} + X_n^{(2)} - 1)_+,$$

where $X_n^{(1)}, X_n^{(2)}$ are two independent versions of X_n.

Question

What can be said of the asymptotic behavior of X_n as $n \to \infty$.

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1. Motivation
   - Pinning models
   - Parking in trees
2. A continuous-time version of this model
   - Three intertwined models
   - Results and open questions
3. An exactly solvable version of this model

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Poland–Scheraga model (1966)

Description

- DNA molecule of length N.
- Bounding energy ω_j in each site $j \leq N$.
- S_j distance between the two strands.

$(S_i, 0 \leq i \leq N)$ path with $S_0 = S_N = 0$

$(\omega_i, 0 \leq i \leq N)$ i.i.d.

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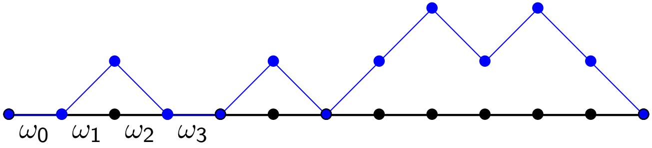

Pinning model on $\mathbb{Z}$

Definition

Polymer measure of length N is defined by

$$P_{N,\omega}(\mathrm{d}S) = \frac{1}{Z_N} \exp\left(\sum_{i=1}^{n} \omega_i \mathbf{1}_{\{S_i = S_{i-1} = 0\}} \right).$$

Remark

If $\omega_i > 0$, the strands are attracted to one another $\omega_i < 0$ they are repulsed.

Random polymer models, G. Giacomin, 2017.

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Définition

Set $F_\omega = \lim_{n\to\infty} N^{-1} \log Z_N$ the free energy of the model.

$$F_\omega = \log 2 \iff \text{unpinned phase}$$
$$F_\omega < \log 2 \iff \text{pinned phase}$$

Open problem

Find for which (ω_j) the model is in the pinned/unpinned phase.
Find the behavior of the free energy around the pinned/unpinned phase transition.

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Hierarchical pinning model (1992)

- Set $B \geq 2$ an integer.
- There is a unique edge at stage 0.
- Each edge is divided into B couple of edges when going from stage k to stage $k+1$.
- Bounding energies are on the smallest edges.

Simplified model introduced by Derrida, Hakim et Vannimenus (1992).

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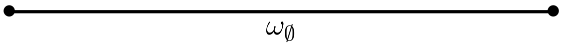

Figure: 0th stage

Hierarchical pinning model

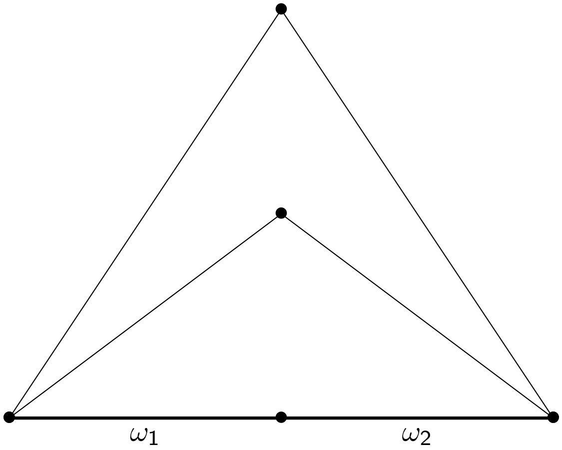

Figure: 1st stage

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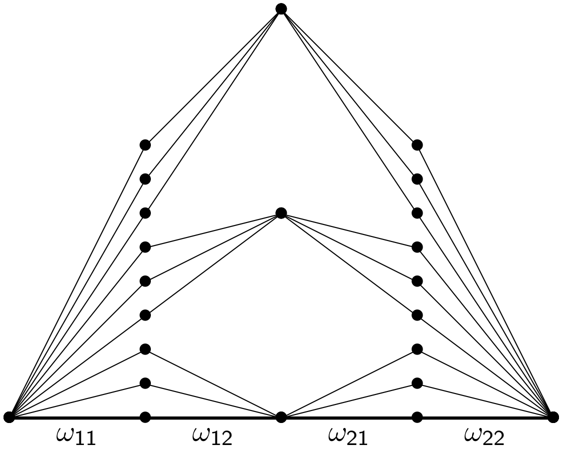

Figure: 2nd stage

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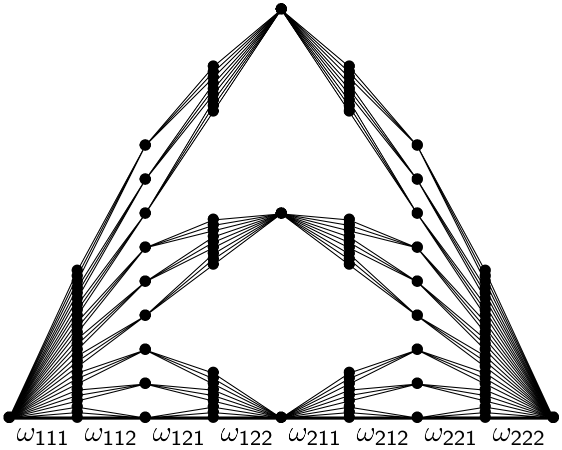

Figure: 3rd stage

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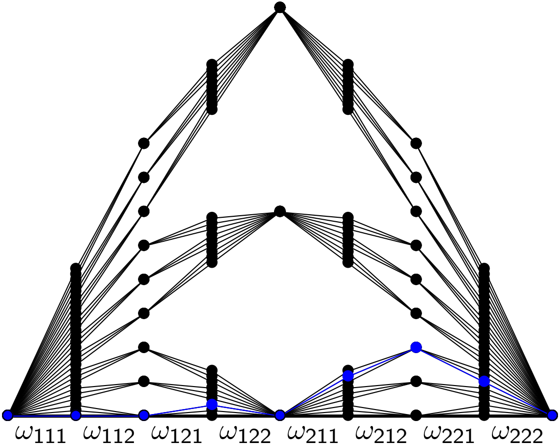

Figure: 3rd stage

Hierarchical pinnning model

Partition function

Set Z_k the sum of the energies of all possible paths at stage k, normalized by B^k. Note that the recursion equation implies

$$Z_{k+1} = \frac{Z_k^{(1)} Z_k^{(2)} + B - 1}{B}.$$

The free energy is $F_\omega = \lim_{k \to \infty} \frac{1}{2^k} \log Z_k$.

Numerous results on that model : Monthus and Garet (2008), Derrida, Giacomin, Lacoin and Toninelli (2009), Lacoin and Toninelli (2009), Giacomin, Lacoin and Toninelli (2010, 2011), Berger and Toninelli (2013)...

Open question

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Derrida–Retaux model (2014)

Set $\bar{X}_k = \log Z_k$. The recursion equation can be rewritten

$$\bar{X}_{k+1} = \log(\exp(\bar{X}_k^{(1)} + \bar{X}_k^{(2)}) + B - 1)/B.$$

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Replace the function $x \mapsto \log((e^x + b - 1)/b)$ by $\max(x, -\log b)$.

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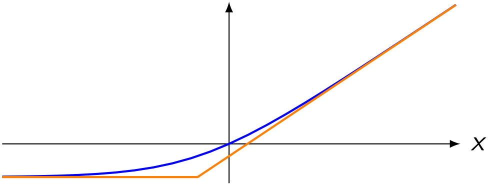

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We obtain the recursion

$$X_{n+1} = (X_n^{(1)} + X_n^{(2)} - 1)_+.$$

Free energy

Assume that $X_0 \sim p\delta_0 + (1-p)\mu$ with μ a probability distribution on $(0, \infty)$. Set

$$F_\infty(p) = \lim_{n\to\infty} 2^{-n}\mathbf{E}(X_n) \in [0, \infty],$$

and define $p_c \in [0, 1]$ such that $F_\infty(p) > 0 \iff p < p_c$. We have $F_\infty = 0$ in the pinned phase, et $F_\infty > 0$ in the unpinned phase.

Conjecture

If $p_c > 0$, we have $F_\infty(p) = \exp(-(c + o(1))(p - p_c)^{-1/2})$ as $p \to p_c$.

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Nevertheless, it is reasonable to define Derrida–Retaux model on a tree.

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Parking in a tree

Collet, Eckmann, Glaser, Martin (1984)

- Let X_0 be an integer-valued random variable.
- Consider a binary tree of height n.
- Put an independent copy of X_0 on each leaf of that tree.
- Then define recursively the value of each internal node to be $(a + b - 1)_+$ if its children have value a and b.

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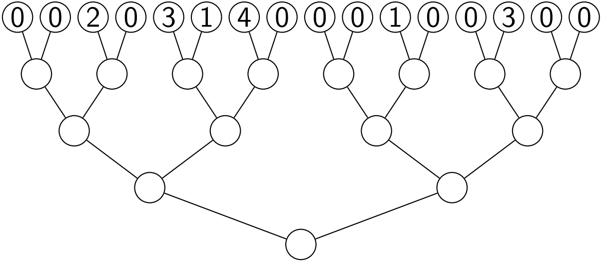

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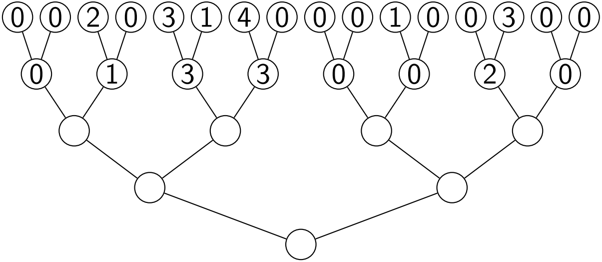

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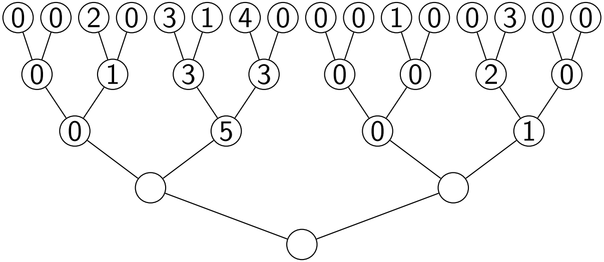

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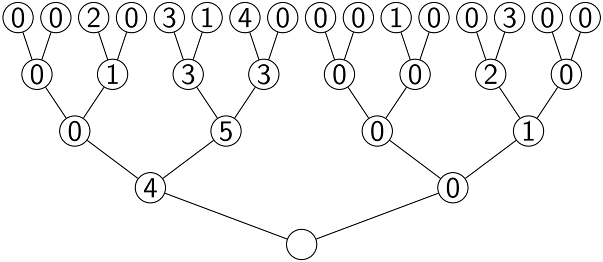

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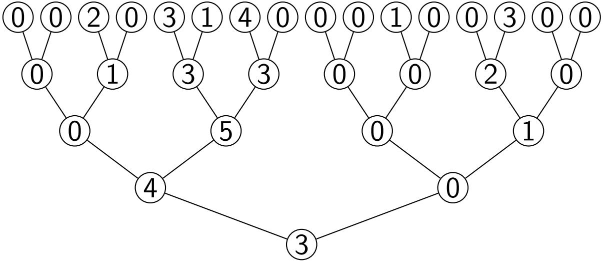

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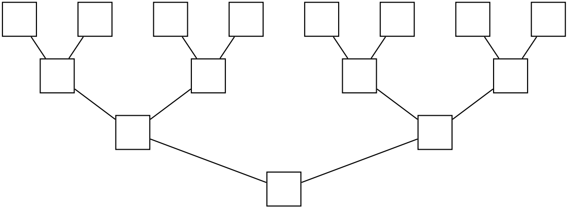

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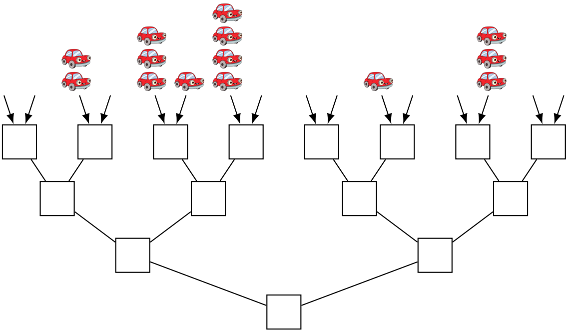

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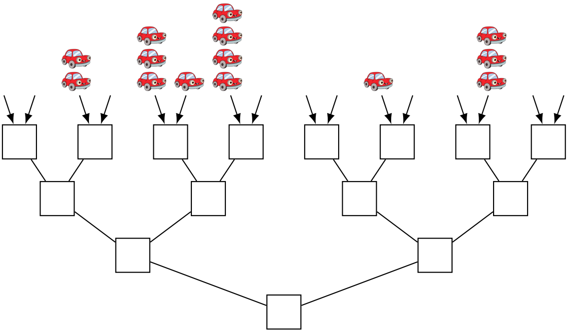

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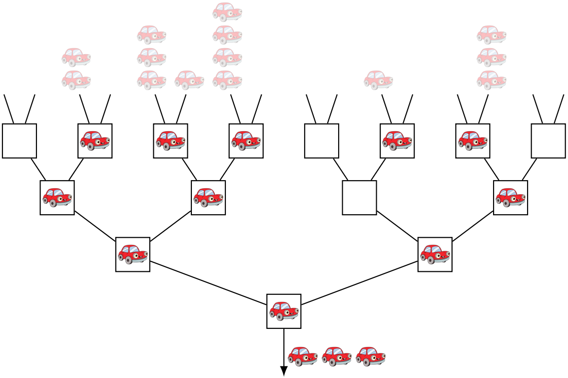

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Theorem (Collet, Eckmann, Glaser, Martin (1984))

Recall that $F_\infty = \lim_{n\to\infty} 2^{-n}\mathbf{E}(X_n)$. *We have*

$$F_\infty = 0 \iff \mathbf{E}(X_0 2^{X_0}) \leq E(2^{X_0}) < \infty.$$

Moreover, if $F_\infty = 0$ *then* $\lim_{n\to\infty} X_n = 0$ *in probability and if* $F_\infty > 0$ *then* $\lim_{n\to\infty} 2^{-n} X_n = F_\infty$ *in probability.*

Open questions

If X_0 is not integer-valued, can we find a similar criterion distinguishing the pinned and unpinned phases ?

Can we show the Derrida–Retaux conjecture on the behavior of the free energy around the pinning transition ?

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Theorem (Chen, Dagard, Derrida, Hu, Lifshits, Shi 2019)

If μ is integer-valued and $\int x^3 2^x \mu(\mathrm{d}x) < \infty$, then

$$F_\infty(p) = \exp\left(-(p-p_c)^{-1/2+o(1)}\right).$$

There exist measure μ such that $\int x^3 2^x \mu(\mathrm{d}x) = \infty$, $p_c > 0$ and

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Continuous-time Derrida–Retaux model

In their article, Derrida and Retaux write that a "good" renormalisation of the DR model should converge in law towards a process $(X_t, t \geq 0)$ whose law $\tilde{\varrho}_t$ satisfies the partial differential equation

$$\forall x > 0, \partial_t \tilde{\varrho}(x) = \partial_x \tilde{\varrho}(x) + \tilde{\varrho} * \tilde{\varrho}(x) - \tilde{\varrho}(x).$$

Problem

This equation is not conservative : even if ϱ_0 is a probability measure, the total mass of ϱ_t will be different from 1 for $t > 0$ large enough.

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Aim

Construct a stochastic process whose density satisfies a PDE close to the one given by Derrida and Retaux.

Remark

Observe that a process version of the Derrida–Retaux model can written as follows: given (U_n) i.i.d. variables with uniform distributions on $[0,1]$, we set

$$\begin{cases} X_{n+1} = X_n + F_n^{-1}(U_{n+1}) - \mathbf{1}_{\{X_n + F_n^{-1}(U_{n+1}) > 0\}} \\ F_{n+1}(x) = \mathbf{P}(X_{n+1} \leq x) \text{ for all } x \geq 0. \end{cases}$$

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Definition

A continuous-time DR model is a solution of the following SDE:

$$\begin{cases} X_t = X_0 - \int_0^t \mathbf{1}_{\{X_s > 0\}} \mathrm{d}s + \int_0^t \int_0^1 F_s^{-1}(u) N(\mathrm{d}s, \mathrm{d}u) \\ F_t(x) = \mathbf{P}(X_t \leq x) \text{ for all } x \geq 0 \end{cases}$$

where N is a PPP on $\mathbb{R}_+ \times [0,1]$ with intensity $\mathrm{d}s\mathrm{d}u$.

Continuous-time Derrida–Retaux model

Let us compute the equation satisfied by the law ϱ_t of X_t. Let f be a $\mathcal{C}^1$ function, we have

$$\mathbf{E}(f(X_t)) = \mathbf{E}(f(X_0)) - \int_0^t \mathbf{E}(f'(X_s)\mathbf{1}_{\{X_s>0\}})\mathrm{d}s + \int_0^t \mathbf{E}(f(X_s + X_s') - f(X_s))\mathrm{d}s$$

$$\int f(x)\mathrm{d}\varrho_t(x) = \int f \mathrm{d}\varrho_0(x) - \int_0^t \int f'(x)\mathbf{1}_{\{x>0\}}\mathrm{d}\varrho_s(x)\mathrm{d}s + \int_0^t \int f(y)\mathrm{d}\varrho_s * \varrho_s(y)\mathrm{d}s - \int_0^t \int f(x)\mathrm{d}\varrho_s(x)\mathrm{d}s.$$

As a result, we obtain the PDE

$$\partial_t \varrho = \partial_x \left(\mathbf{1}_{\{x>0\}}\varrho\right) + \varrho * \varrho - \varrho.$$

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$$\begin{aligned}\mathbf{E}(f(X_t)) &= \mathbf{E}(f(X_0)) - \int_0^t \mathbf{E}(f'(X_s)\mathbf{1}_{\{X_s>0\}})\mathrm{d}s \\ &\quad + \int_0^t \mathbf{E}(f(X_s + X_s') - f(X_s))\mathrm{d}s \\ \int f(x)\mathrm{d}\varrho_t(x) &= \int f\mathrm{d}\varrho_0(x) - \int_0^t \int f'(x)\mathbf{1}_{\{x>0\}}\mathrm{d}\varrho_s(x)\mathrm{d}s \\ &\quad + \int_0^t \int f(y)\mathrm{d}\varrho_s * \varrho_s(y)\mathrm{d}s - \int_0^t \int f(x)\mathrm{d}\varrho_s(x)\mathrm{d}s.\end{aligned}$$

As a result, we obtain the PDE

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Figure: Solving the stochastic differential equation

$$X_t = X_0 - \int_0^t \mathbf{1}_{\{X_s > 0\}} \mathrm{d}s + \int_0^t \int_0^1 F_s^{-1}(u) N(\mathrm{d}s, \mathrm{d}u).$$

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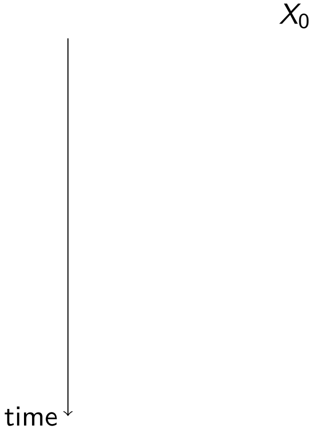

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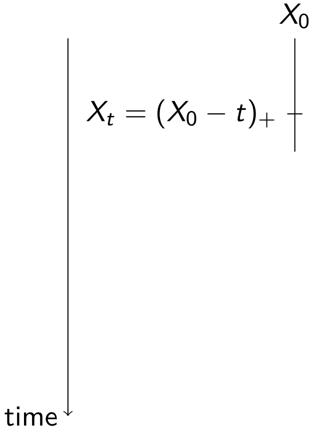

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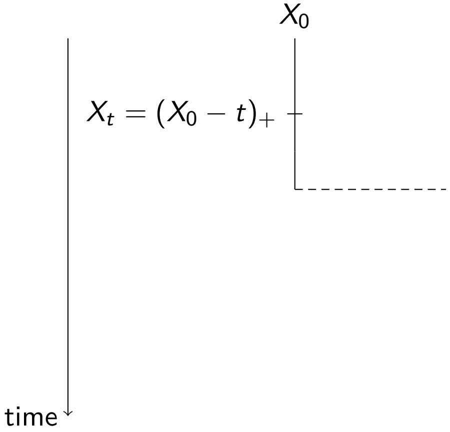

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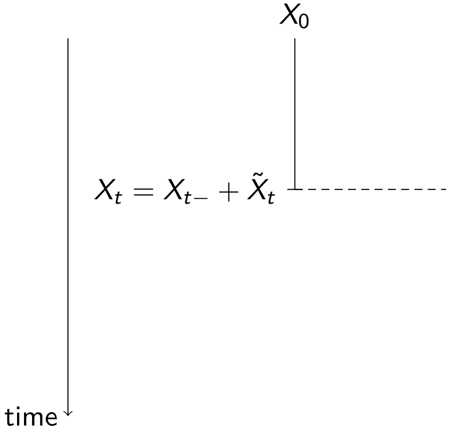

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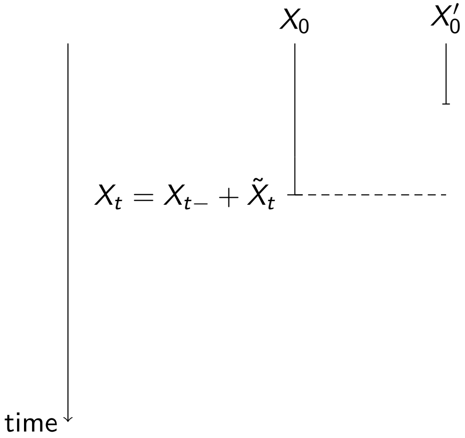

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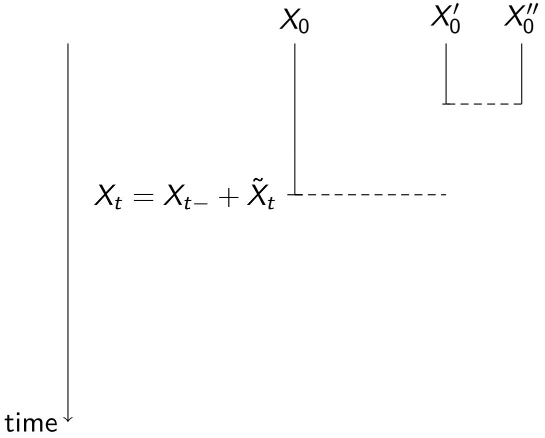

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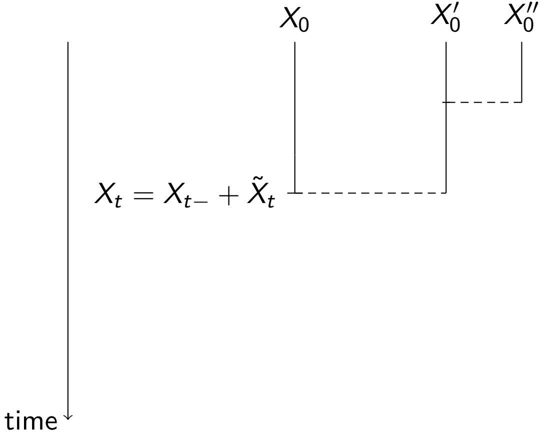

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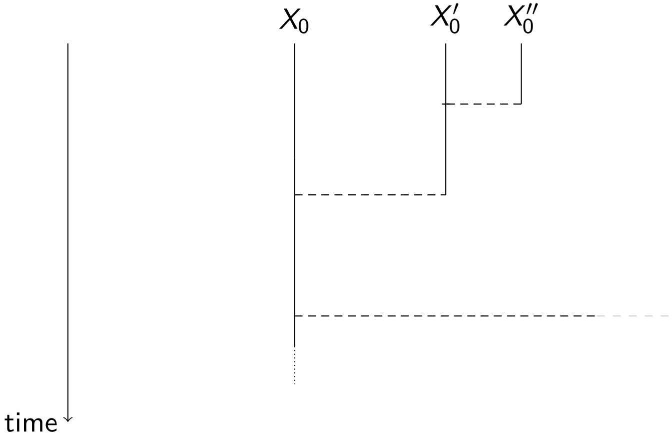

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The tree construction

- Take a **Yule tree** of height t.
- Start with i.i.d. amout of paint on each leaf of the tree, sampled according to the law of X_0.
- Paint down the branches with a quantity 1 of paint per unit of branch length, until no more paint is left.
- When two painters meet, they put their remaining paint in common.
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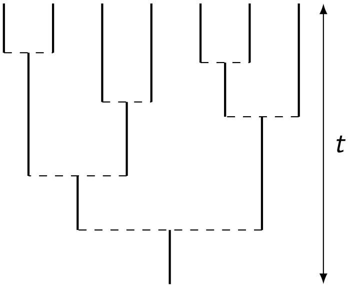

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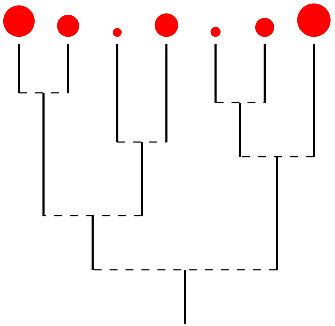

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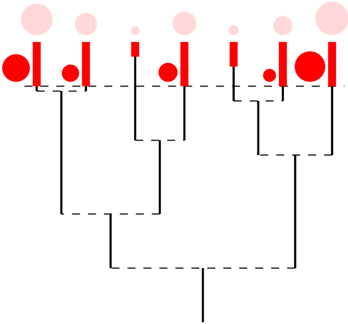

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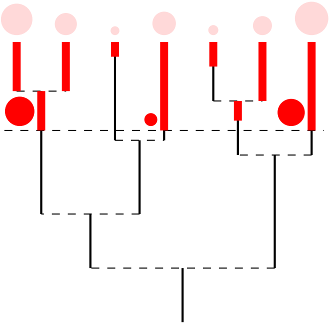

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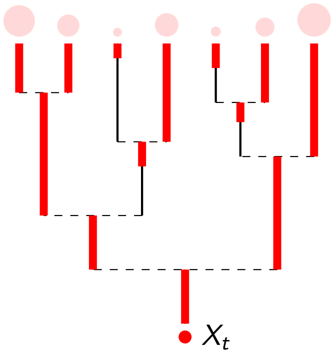

Connexion between the results

Proposition

There exists a unique weak solution to the McKean-Vlasov type SDE, with same law at time t as in the tree-painting scheme of height t.

Proposition

There exists a unique weak solution to the PDE

$$\partial_t \varrho = \partial_x (\mathbf{1}_{\{x>0\}} \varrho) + \varrho * \varrho - \varrho,$$

Moreover, ϱ_t is the law of the solution of the SDE at time t.

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Some general results in continuous settings

Proposition

- *The free energy* $F_\infty = \lim_{t\to\infty} e^{-t}\mathbf{E}(X_t)$ *exists.*
- *If* $F_\infty > 0$*, then* $\lim_{t\to\infty} e^{-t} X_t = F_\infty \mathrm{Exp}(1)$ *in law.*

Proof.

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Solving this ODE we have $\mathbf{E}(X_t) = e^t(\mathbf{E}(X_0) + \int_0^t e^{-s}\mathbf{P}(X_s > 0)\mathrm{d}s)$, hence the limit exists.

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An exactly solvable version of this model

We look for solution of the PDE taking the form

$$\varrho_t(\mathrm{d}x) = p_t \delta_0(\mathrm{d}x) + (1 - p_t)\lambda_t e^{-\lambda_t x}\mathrm{d}x.$$

The PDE can then be rewritten as

$$\begin{cases} p' = (1-p)(\lambda - p) \\ \lambda' = -\lambda(1-p) \end{cases}$$

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Mixtures of exponential random variables are stable under the action of the PDE (interpreting Dirac mass at 0 as exponential random variable with infinite parameter).

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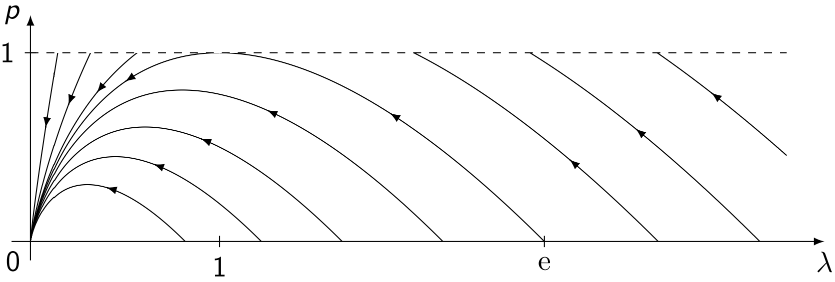

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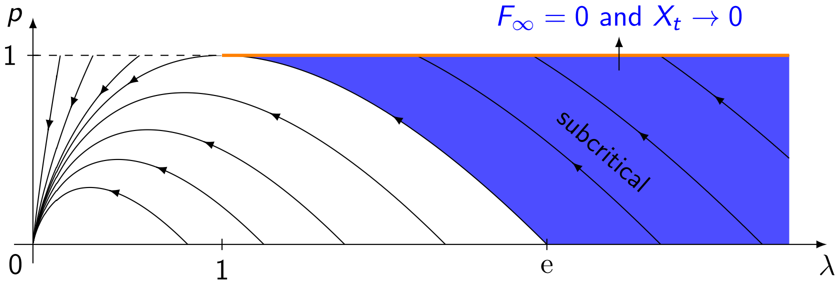

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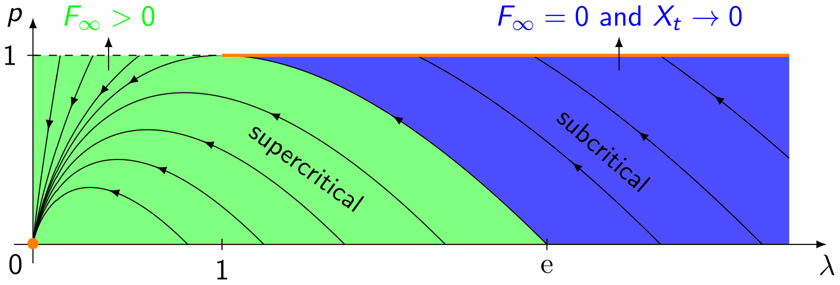

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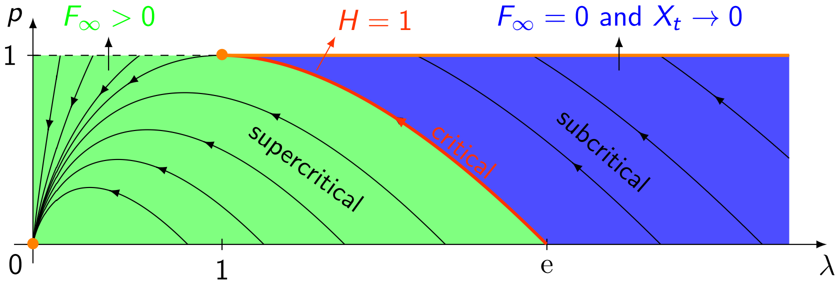

Asymptotic behavior of the solutions

Proposition

Let (p, λ) be a solution of the differential system.

1. *If (p_0, λ_0) in the supercritical phase, then*
$$\lambda_t \sim K e^{-t} \quad \text{and} \quad p_t \sim K t e^{-t}.$$
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The Derrida–Retaux conjecture

Recall that

$$F_\infty = \lim_{t\to\infty} e^{-t}\mathbf{E}(X_t) = \lim_{t\to\infty} e^{-t}\frac{1-p_t}{\lambda_t}.$$

With careful analysis of the differential equation, one proves

Theorem

Fix $\lambda \in (0, e)$ and set $p_c = \lambda \log \lambda - \log \lambda$, we have

$$F_\infty(p,\lambda) \sim C \exp(-\pi\sqrt{2\lambda}(p_c - p)^{-1/2}) \text{ as } p \uparrow p_c \text{ if } \lambda > 1$$
$$F_\infty(p,\lambda) \sim C(1-p)^{2/3} \exp(-\tfrac{\pi}{\sqrt{2}}(1-p)^{-1/2}) \text{ as } p \uparrow 1 \text{ if } \lambda = 1$$
$$F_\infty(p,\lambda) \sim C(1-p)^{1/(1-\lambda)} \text{ as } p \uparrow 1 \text{ if } \lambda < 1$$

The Derrida–Retaux conjecture

Recall that

$$F_\infty = \lim_{t\to\infty} e^{-t}\mathbf{E}(X_t) = \lim_{t\to\infty} e^{-t}\frac{1-p_t}{\lambda_t}.$$

With careful analysis of the differential equation, one proves

Theorem

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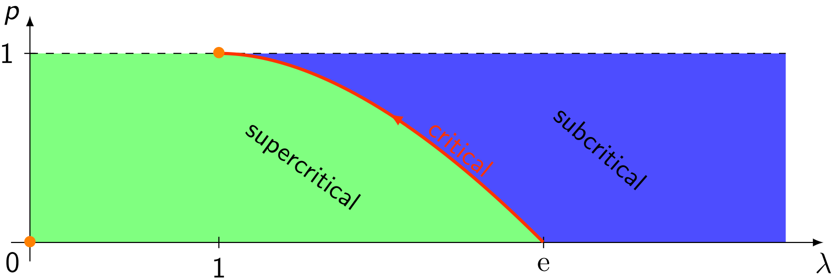
On the phase diagram



$$\begin{cases} p' = (1-p)(\lambda - p) \\ \lambda' = -(1-p)\lambda \end{cases}$$

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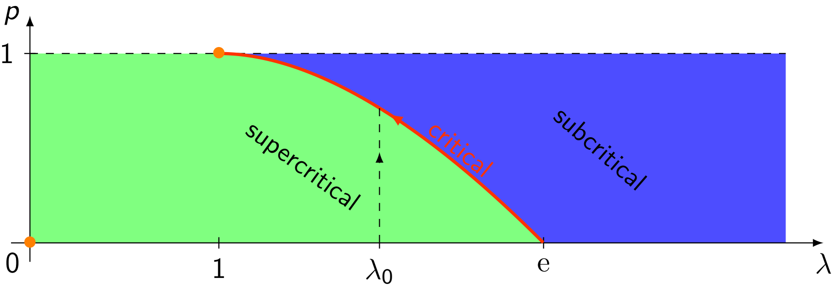

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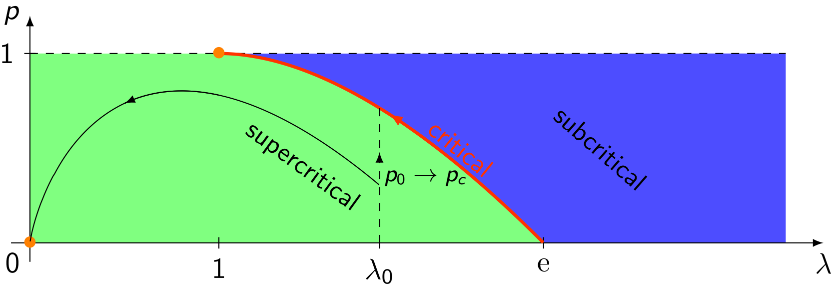

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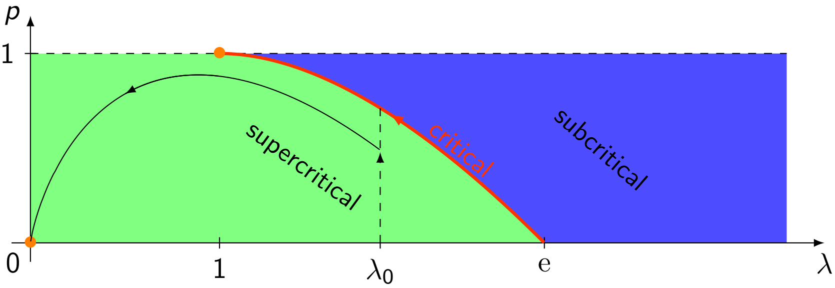

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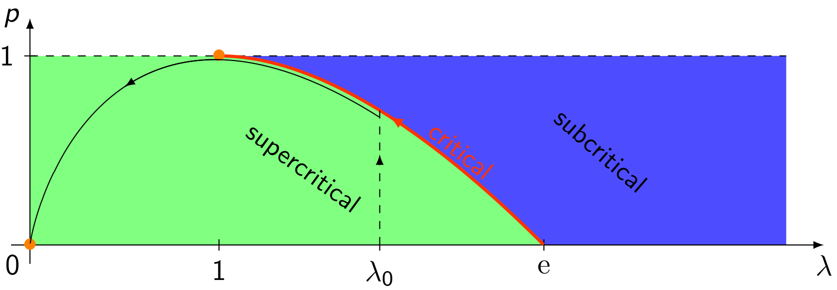

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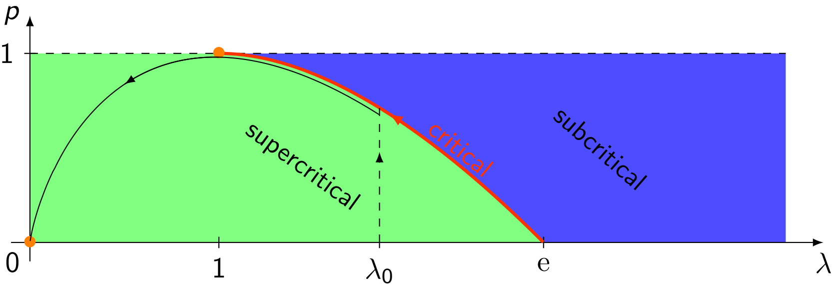

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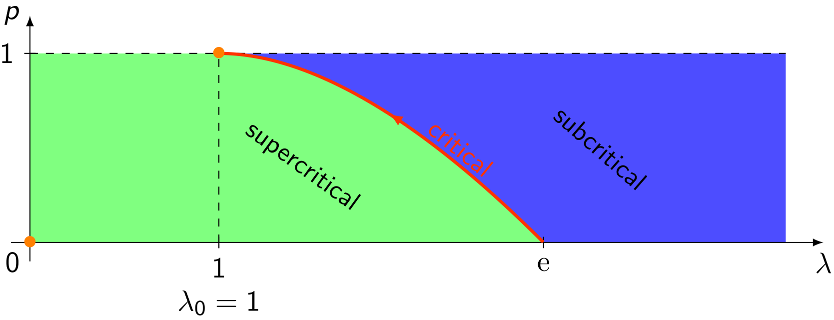

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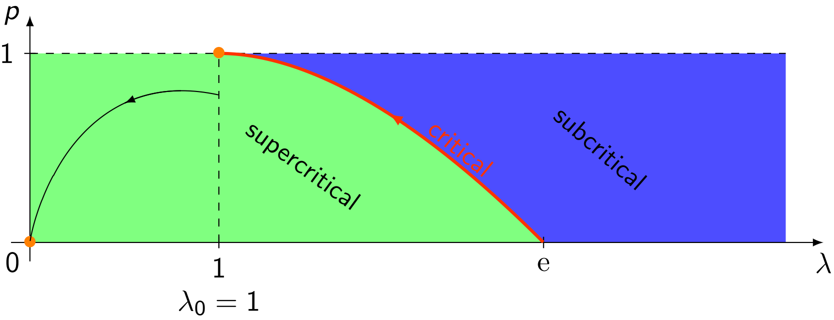

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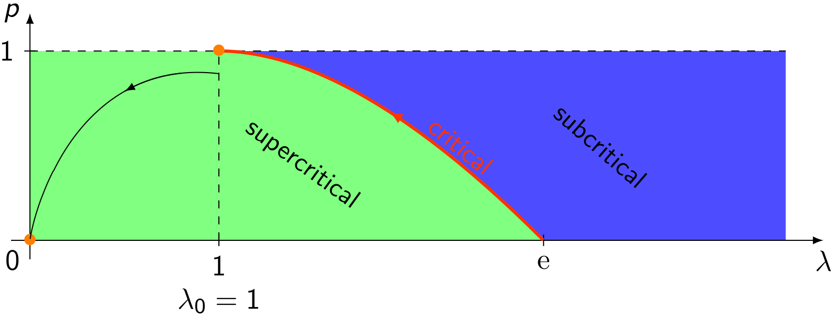

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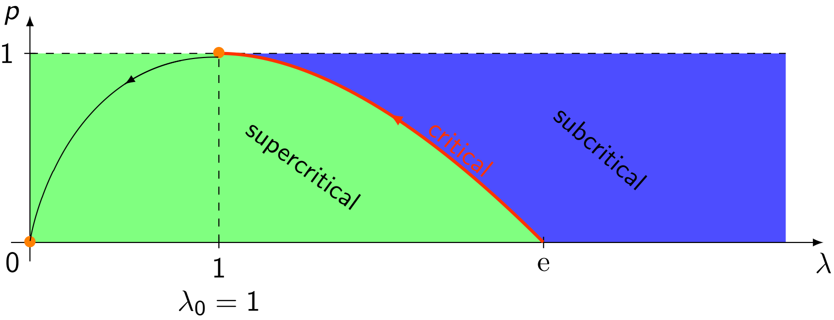

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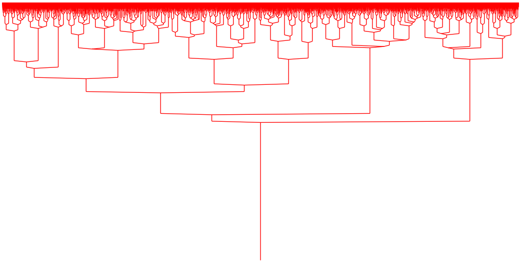
Behavior at criticality

One can also look at the asymptotic behavior of the continuous-time DR model on the line:

- On the event $X_t > 0$, $N_t = O(t^2)$ leaves will contribute to the mass of paint at the origin.
- The total mass M_t of paint that was put on these leaves is approximately $c_\star N_t$.
- The tree of the origin of the paint scales towards a time-inhomogeneous branching Markov process: particles grow mass linearly, and a particle of mass m splits at rate $2m/(1-t)^2$.



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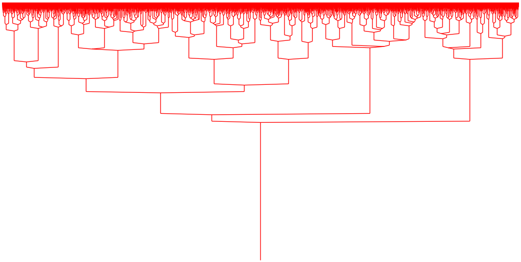

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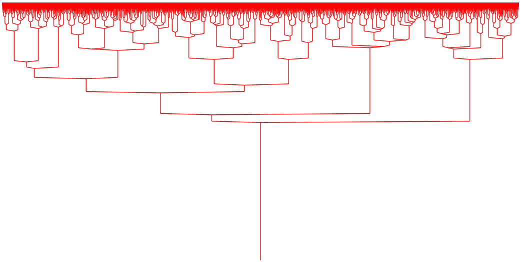

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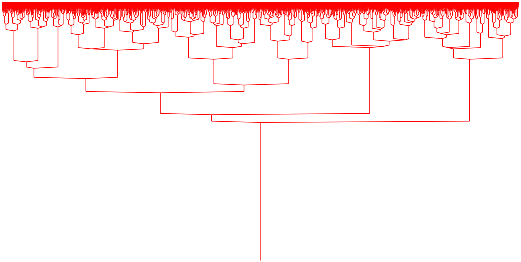

Final open questions

1. Can the results given for exponential variables be extended to general initial measures ?
2. Can the results be extended to the original Derrida–Retaux model ?
3. Can a similar behavior be observed for a probability distribution satisfying
$$\partial_t \varrho = \partial_x (a\varrho) + \varrho * \varrho - \varrho,$$
where a is a smooth function satisfying $a(0) = 0$ and $a(x) \sim 1$ as $x \to \infty$?

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Thank you for your attention!