Branching processes in random environment with immigration stopped at zero

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Galton-Watson process with immigration

- A Galton-Watson process with immigration :

$$Y_0 = \eta^{(0)}, \qquad Y_{n+1} = \sum_{j=1}^{Y_n} \xi_j^{(n)} + \eta^{(n)},$$

- $$\xi_j^{(n)} \overset{d}{=} \xi,\ \eta^{(n)} \overset{d}{=} \eta$$

  – are i.i.d.

- Offspring generating function

$$f(s) := \mathbf{E} s^{\xi} = \sum_{k=0}^{\infty} \mathbf{P}(\xi = k) s^k, \quad g(s) := \mathbf{E} s^{\eta} = \sum_{k=0}^{\infty} \mathbf{P}(\eta = k) s^k.$$

Galton-Watson processes in random environment with immigration

- Offspring generating functions $f_n(s) := \mathbf{E}s^{\xi^{(n)}}$, $g_n(s) := \mathbf{E}s^{\eta^{(n)}}$ in generations $n = 0, 1, \ldots$ are RANDOM and I.I.D.
-

$$Y_0 = \eta^{(0)}, \quad Y_{n+1} = \sum_{j=1}^{Y_n} \xi_j^{(n)} + \eta^{(n)},$$

where

$$\xi_j^{(n)} \stackrel{d}{=} \xi^{(n)}, \qquad \eta^{(n)} \stackrel{d}{=} \eta^{(0)}$$

are i.i.d. **given** $f_0, f_1, \ldots; g_1, g_2, \ldots$

Galton-Watson processes in random environment with immigration stopped at zero

Assume, without loss of generality that $Y_0 > 0$. Let $W_0 = Y_0$ and for $n \geq 1$

$$W_n := \begin{cases} 0, & \text{if } T_n := \xi_1^{(n)} + \ldots + \xi_{W_{n-1}}^{(n)} = 0, \\ T_n + \eta^{(n)}, & \text{if } T_n > 0. \end{cases}.$$

We call $\mathbf{W}$ as a branching process with immigration stopped at zero.

The quantity

$$\zeta := \min \{n \geq 1 : W_n = 0\}$$

is called a **life period** of the branching process with immigration stopped at zero.

Quenched approach:

The study the behavior of characteristics of a BPIRE for typical realizations of the environment $f_1, f_2, ...; g_1, g_2, ...,$.
Let, as before,

$$\zeta := \min \{n \geq 1 : W_n = 0\}$$

Then

$$\mathbf{P}_{f,g}(\zeta > T) = \mathbf{P}(\zeta > T | f_1, f_2, ...; g_1, g_2, ...)$$

is a random variable on the space of realizations of the environment $f_1, f_2, ...; g_1, g_2, ...$

Annealed approach:

The study the behavior of characteristics of a BPRE performing averaging over possible scenarios $f_1, f_2, ...; g_1, g_2, ...,$ on the space of realizations of the environment:

$$\mathbf{P}(\zeta > T) = \mathbf{E}[\mathbf{P}_{f,g}(\zeta > T)]$$

is a number !

The aim of the talk is to present results on the tail distribution of the random variable

$$\zeta := \min\{n \geq 1 : W_n = 0\}$$

under the annealed approach for branching processes with immigration stopped at zero.

Zubkov (1972) investigated the distribution of life periods for ordinary Galton-Watson processes with immigration.

Classification of Galton-Watson processes in **i.i.d.** random environment

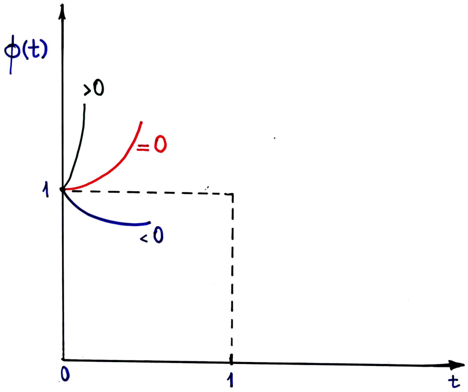
The classification is based on the properties of the moment generating function

$$\Phi(t) = \mathbf{E}e^{tX} = \mathbf{E}e^{t\log f'(1)}$$

of the random variable $X = \log f'(1) \stackrel{d}{=} \log f_n'(1)$.

Classification: A BPRE is called

- supercritical if $\Phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) > 0$,
- critical if $\Phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) = 0$,
- subcritical if $\Phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) < 0$.



$$\phi(t) = Ee^{tX}$$

THE CRITICAL CASE $\mathbf{E} \log f'(1) = 0$

Hypothesis A1. The probability generating function $f(s) \overset{d}{=} f_n(s)$ is geometric with probability 1, that is

$$f(s) = \frac{q}{1 - ps}$$

with random $p, q \in (0, 1)$ satisfying $p + q = 1$ and

$$\log f'(1) = \log \frac{p}{q}.$$

Hypothesis A2. There exist real numbers $\kappa \in [0, 1)$ and $\gamma, \sigma \in (0, 1]$ such that, with probability 1

1) the inequality $f(0) \geq \kappa$ is valid;

2) for $g(s) \overset{d}{=} g_n(s)$ the estimate

$$g(s) \leq s^{\gamma}$$

holds for all $s \in (\kappa^{\sigma}, 1]$ with probability 1.

Hypothesis A3. The distribution of X is nonlattice, $\mathbf{E}X^2 < \infty$ and there exists an $\varepsilon > 0$ such that

$$\mathbf{E}\left(\log^{+} g'(1)\right)^{2+\varepsilon} < \infty \quad \text{and} \quad \mathbf{E}\left(X^{+}\log^{+} g'(1)\right)^{1+\varepsilon} < \infty.$$

Theorem

Let Hypotheses A1 - A3 be satisfied. Then there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\zeta > n\right) \sim \frac{C}{\sqrt{n}}$$

as $n \to \infty$.

Theorem

Let Hypotheses A1 - A3 be satisfied. Then there exists a constant $C > 0$ *such that*

$$\mathbf{P}(\zeta > n) \sim \frac{C}{\sqrt{n}}$$

as $n \to \infty$.

Zubkov (1972) for the ordinary critical Galton-Watson process:
Let

$$\theta = \frac{2g'(1)}{f''(1)}.$$

If $\theta \leq 1$ then

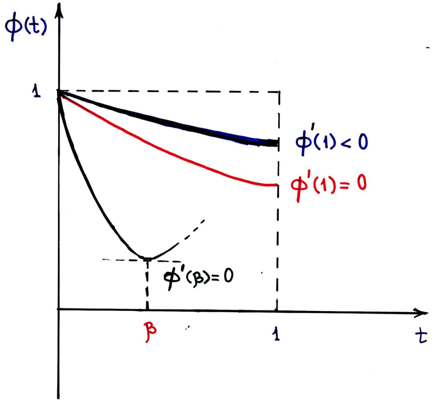
$$\mathbf{P}(\zeta > n) \sim \frac{L(n)}{n^{1-\theta}}$$

as $n \to \infty$. If $\theta > 1$ then

$$\mathbf{P}(\zeta > n) \sim C > 0.$$

THE SUBCRITICAL CASE $\mathbf{E}\log f'(1) < 0$

$\Phi(t) = \mathbf{E}s^{tX}$ - the moment generating function



Subcritical processes ($\mathbf{E}X = \mathbf{E}\log f'(1) < 0$): **three main** sub-cases:

- strongly subcritical, if

$$\Phi'(1) = \mathbf{E}\left[Xe^{X}\right] < 0,$$

- intermediately subcritical, if

$$\Phi'(1) = \mathbf{E}\left[Xe^{X}\right] = 0,$$

- weakly subcritical, if there exists $0 < \beta < 1$ such that

$$\Phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] = 0.$$

CHANGE OF MEASURE

Introduce a new measure $\mathbb{P}$ by setting, for any $n \in \mathbb{N}$ and any measurable bounded function ψ

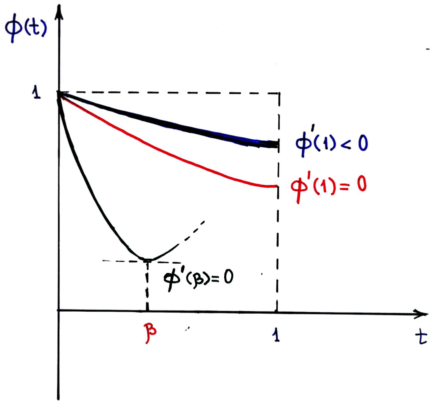
$$\mathbb{E}[\psi(f_1, ..., f_n, g_1, ..., g_n; W_0, ..., W_n)] = \frac{\mathbf{E}[\psi(f_1, ..., f_n, g_1, ..., g_n; W_0, ..., W_n)e^{\delta S_n}]}{\gamma^n},$$

with

$$\gamma := \mathbf{E}[e^{\delta X}].$$

Here $\delta = 1$ for strongly and intermediate subcritical BPIRE and $\delta = \beta$ for weakly subcritical BPIRE.
Observe that $\mathbf{E}[Xe^{\delta X}] = 0$ translates into

$$\mathbb{E}[X] = 0.$$



ϕ(t)
1
ϕ'(1) < 0
ϕ'(1) = 0
ϕ'(β) = 0
β
1
t

Hypothesis B1. The distribution of X is nonlattice and, under $\mathbb{P}$ belongs to the domain of attraction of a two-sided stable law with index $\alpha \in (1, 2]$.

Hypothesis B2.

$$\mathbb{E}\left(\log^{+} \frac{f''(1)}{(f'(1))^{2}}\right)^{2} < \infty.$$

Restrictions on the immigration component:

Hypothesis B3.

$$\mathbb{E}\left[\frac{g'(1)}{1-g(0)}\right] < \infty.$$

Theorem

Let Hypotheses B1-B3 be satisfied. Then there exists a generating function

$$\mathcal{H}(r) = \sum_{k=1}^{\infty} h_k s^k, \qquad h_k > 0$$

such that, as $n \to \infty$
1) if the equation $r\mathcal{H}(r) = 1$ *has a root* $1 < r < \gamma^{-1}$, *then*

$$\mathbf{P}(\zeta > n) \sim C(r) r^{-n-1}, \quad C(r) \in (0, \infty);$$

2) if the BPIRE is weakly subcritical and $\gamma^{-1}\mathcal{H}(\gamma^{-1}) < 1,$ *then,*

$$\mathbf{P}(\zeta > n) \sim C \frac{\gamma^n}{n^{1+1/\alpha} l(n)}, \ C \in (0, \infty),$$

$l(n)$ *is a slowly varying function;*
3) if the BPIRE is weakly subcritical and $\gamma^{-1}\mathcal{H}(\gamma^{-1}) = 1,$ *then*

$$\mathbf{P}(\zeta > n) = o(\gamma^n).$$

Branching processes in random environment: survival of a single family only

Branching processes in random environment: survival of a single family only

We consider the situation

$$Y_n = \sum_{i=1}^{Y_{n-1}} \xi_{n1} + 1.$$

Thus, only one immigrant joins each generation.
It will be convenient to assume that if $Y_{n-1} = y_{n-1} > 0$ is the population size of the $(n-1)$th generation of $\mathbf{Y}$ then first

$$\xi_{n1} + \ldots + \xi_{ny_{n-1}}$$

individuals of the nth generation are born and afterwards **exactly one immigrant** enters the population.

Definition 1. All individuals of the nth generation which are children of the immigrant joining the population at moment $i < n$ **constitute the** (i, n)**-clan**.

Definition 2. We say that only a (i, n)-clan survives in $\mathbf{Y}$ to moment n if

$$Y_n^- := \xi_{n1} + \ldots + \xi_{ny_{n-1}} > 0$$

and all Y_n^- particles belong to the (i, n)-clan.

Let $\mathcal{A}_i(n)$ be the event that only the (i, n)-clan survives in $\mathbf{Y}$ to moment n.

We study the asymptotic behavior of the probability $\mathbf{P}\left(\mathcal{A}_i(n)\right)$ as $n \to \infty$ and i varies with n in an appropriate way.

The critical case

Hypothesis C1. The **random** probability generating function $f(s)$ is geometric with probability 1, that is

$$f(s) = \frac{q}{1 - ps}$$

with random $p, q \in (0, 1)$ satisfying $p + q = 1$ and

$$X = \log(p/q).$$

Hypothesis C2. $\mathbf{E}[X] = 0$, $\mathbf{E}\left[X^2\right] \in (0, \infty)$ and $\mathbf{E}\left[e^X\right] < \infty$.

Hypothesis C3. The distribution of X is absolutely continuous.

Theorem

If Hypotheses C1-C2 are valid then
1) for any fixed i

$$\lim_{n\to\infty} n^{3/2}\mathbf{P}\left(\mathcal{A}_i(n)\right) = w_i \in (0,\infty)\,;$$

2) for any fixed N

$$\lim_{n\to\infty} n^{1/2}\mathbf{P}\left(\mathcal{A}_{n-N}(n)\right) = r_N \in (0,\infty)\,;$$

3) if, in addition, Hypotheses C3 is valid and $\min\left(i, n-i\right) \to \infty$ *then*

$$\lim i^{1/2}\left(n-i\right)^{3/2}\mathbf{P}\left(\mathcal{A}_i(n)\right) = K \in (0,\infty)\,.$$

Subcritical case

Hypothesis SubC1. The generating function $f(s)$ is geometric with probability 1, that is

$$f(s) = \frac{q}{1 - ps} = \frac{1}{1 + m(F)(1 - s)}$$

with random $p, q \in (0, 1)$ satisfying $p + q = 1.$

Hypothesis SubC2.The BPRE is subcritical, i.e.

$$-\infty < \mathbf{E}X < 0$$

Introduce a new measure $\mathbb{P}$ by setting, for any $n \in \mathbb{N}$ and any measurable bounded function $\psi : \Delta^n \times \mathbb{N}_0^{n+1} \to \mathbb{R}$

$$\mathbb{E}[\psi(F_1, \cdots, F_n, Y_0, \cdots, Y_n)] := \gamma^{-n} \mathbf{E}[\psi(F_1, \cdots, F_n, Y_0, \cdots, Y_n) e^{\delta S_n}],$$

with

$$\gamma := \mathbf{E}[e^{\delta X}],$$

where $\delta = 1$ for strongly and intermediate subcritical BPIRE and $\delta = \beta$ for weakly subcritical BPIRE.

Observe that $\mathbf{E}[X e^{\delta X}] = 0$ translates into

$$\mathbb{E}[X] = 0.$$

Hypothesis SubC3. If a BPIRE is **either intermediate** or **weakly subcritical** then the distribution of X belongs with respect to $\mathbb{P}$ to the domain of attraction of a two-sided stable law with index $\alpha \in (1, 2]$.

Since $\mathbb{E}[X] = 0$, Hypothesis **SubC3** provides existence of an increasing sequence of positive numbers

$$c_n = n^{1/\alpha} l_1(n)$$

with slowly varying sequence $l_1(1), l_1(2), \ldots$ such that, the distribution law of S_n/c_n converges weakly, as $n \to \infty$ to the mentioned two-sided stable law. Besides, under this condition there exists a number $\rho \in (0, 1)$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(S_n > 0\right) = \rho.$$

Recall that $\mathcal{A}_i(n)$ is the event that only the (i, n)-clan survives in $\mathbf{Y}$ at moment n.

Strongly subcritical case.

Theorem

Let $\mathbf{Y}$ *be a* ***strongly subcritical*** *BPIRE satisfying Hypotheses* ***SubC1****. Then*
1) for any fixed N

$$\lim_{n\to\infty} \mathbf{P}\left(\mathcal{A}_{n-N}(n)\right) =: r_N \in (0,\infty)\,;$$

2) there exists a constant $R \in (0,\infty)$ *such that*

$$\lim_{n-i\to\infty} \gamma^{-(n-i)}\mathbf{P}\left(\mathcal{A}_i(n)\right) = R.$$

Intermediate subcritical case.

Theorem

Let $\mathbf{Y}$ be an ***intermediate*** *subcritical BPIRE meeting Hypotheses* ***SubC1*** *and* ***SubC2****. Then*
1) for any fixed N

$$\lim_{n\to\infty}\mathbf{P}\left(\mathcal{A}_{n-N}(n)\right)=:r_N\in(0,\infty)\,;$$

2) there exist a slowly varying function $l(n)$ and a constant $R\in(0,\infty)$ such that

$$\lim_{n-i\to\infty}\gamma^{-(n-i)}\,(n-i)^{\rho}\,l(n-i)\mathbf{P}\left(\mathcal{A}_i(n)\right)=R.$$

Intermediate subcritical case.

Theorem

Let $\mathbf{Y}$ *be a weakly subcritical BPIRE meeting Hypotheses* ***SubC1*** *and* ***SubC2****.*
Then
1) for any fixed N

$$\lim_{n\to\infty}\mathbf{P}\left(\mathcal{A}_{n-N}(n)\right)=r_N\in(0,\infty);$$

2) for any fixed i *there exists a constant* $R_i\in(0,\infty)$ *such that*

$$\lim_{n-i\to\infty}\gamma^{-(n-i)}(n-i)\,c_{n-i}\mathbf{P}\left(\mathcal{A}_i(n)\right)=R_i.$$

3) there exists a constant $R\in(0,\infty)$ *such that*

$$\lim_{\min(i,n-i)\to\infty}\gamma^{-(n-i)}(n-i)\,c_{n-i}\mathbf{P}\left(\mathcal{A}_i(n)\right)=R.$$

THANKS FOR YOUR ATTENTION!