Asymptotic expansions for first-passage times of random walks.

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(joint work with Denis Denisov and Alexander Tarasov)

Let $S_n = X_1 + X_2 + \ldots + X_n$ be a random walk with i.i.d. incremnets.

We shall always assume that

$$\mathbf{E}X_1 = 0 \quad \text{and} \quad \mathbf{E}X_1^2 \in (0, \infty).$$

Define

$$\tau_x := \inf\{n \geq 1 : x + S_n \leq 0\}, \quad x \geq 0.$$

It is well known that, as $n \to \infty$,

$$\mathbf{P}(\tau_x > n) = V(x)n^{-1/2} + o(n^{-1/2}),$$

where $V(x)$ is proportional to the renewal function of strict decreasing ladder heights.

We want to understand the behaviour of $o(n^{-1/2})$-term.

Let $B(t)$ be a Brownian motion and set

$$\tau_x^{(bm)} := \inf\{t \geq 0 : x + B(t) \leq 0\}.$$

Then, as $t \to \infty$,

$$\begin{aligned}
\mathbf{P}\left(\tau_x^{(bm)} > t\right) = \mathbf{P}\left(\max_{s \leq t} B(s) < x\right) &= \frac{2}{\sqrt{2\pi}} \int_0^{x/\sqrt{t}} e^{-u^2/2} du \\
&= \sqrt{\frac{2}{\pi}} \int_0^{x/\sqrt{t}} \left(\sum_{j=0}^{\infty} \frac{1}{j! 2^j} u^{2j}\right) du \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{\infty} \frac{x^{2j+1}}{j! 2^j (2j+1)} t^{-j-1/2} =: \sum_{j=1}^{\infty} v_j(x) t^{-j+1/2}.
\end{aligned}$$

Remark: $\Delta^j v_j(x) = 0$ for every $j \geq 1$. In other words, every v_j is a polyhartmonic function.

Conjecture:

$$\mathbf{P}(\tau_x > n) = \sum_{j=1}^{\infty} V_j(x) n^{-j+1/2}.$$

This is a formal expansion. The number of accessible terms should depend on the number of finite moments of X_1.

One of the consequences of the Wiener-Hopf factorisation is the following exact expression:

$$1-\mathbf{E}s^{\tau_0} = \exp\left\{-\sum_{n=1}^{\infty}\frac{s^n}{n}\mathbf{P}(S_n \le 0)\right\}.$$

If the distribution of X_1 is continuous and symmetric then $\mathbf{P}(S_n \le 0) = 1/2$. This implies that

$$\sum_{n=0}^{\infty}\mathbf{P}(\tau_0 > n)s^n = \frac{1-\mathbf{E}s^{\tau_0}}{1-s} = (1-s)^{-1/2}$$

and

$$\mathbf{P}(\tau_0 > n) = (-1)^n\binom{-\frac{1}{2}}{n}, \quad n \ge 0.$$

In general case we have

$$\sum_{n=0}^{\infty} \mathbf{P}(\tau_0 > n) s^n = \frac{1 - \mathbf{E} s^{\tau_0}}{1 - s} = (1 - s)^{-1/2} \exp\left\{ \sum_{n=1}^{\infty} \frac{s^n}{n} \Delta_n \right\},$$

where

$$\Delta_n := \frac{1}{2} - \mathbf{P}(S_n \le 0).$$

If $\mathbf{E} X_1 = 0$ and $\mathbf{E} X_1^2 < \infty$ then $\sum \frac{\Delta_n}{n} < \infty$ and, consequently,

$$\mathbf{P}(\tau_0 > n) \sim e^Q a_n, \quad Q := \sum \frac{\Delta_n}{n}.$$

We have also

$$\sum_{n=0}^{\infty} \mathbf{P}(\tau_0 > n) s^n$$

$$= e^Q (1-s)^{-1/2} \exp\left\{ -Q + \sum_{n=1}^{\infty} \frac{s^n}{n} \Delta_n \right\}$$

$$= e^Q (1-s)^{-1/2} + e^Q (1-s)^{-1/2} \left(\exp\left\{ -Q + \sum_{n=1}^{\infty} \frac{s^n}{n} \Delta_n \right\} - 1 \right).$$

In order to determine the behaviour of the remainder, we need to know asymptotic properties of Δ_n. This information can be taken from asymptotic expansions in CLT.

Difficulty: One deals here with convolutions of sequences, which have zero total sum. Thus, one can not use standard subexponential estimates for convolutions.

Theorem 1. Assume that $\mathbf{E}|X_1|^r$ is finite for some integer $r \geq 3$. Assume also that either the distribution of X_1 is lattice or $\limsup_{|t|\to\infty} \left|\mathbf{E}e^{itX_1}\right| < 1$. Then there exist numbers $\nu_1, \nu_2, \ldots, \nu_{\lfloor \frac{r-1}{2} \rfloor}$ such that

$$\mathbf{P}(\tau_0 > n) = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} \nu_j a_n^{(j)} + o\left(\frac{\log n}{n^{(r-1)/2}}\right),$$

where

$$a_n^{(j)} = (-1)^n \binom{j - \frac{3}{2}}{n}.$$

Using asymptotic expansions, we get

$$\frac{\Delta_n}{n} = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} p_j n^{-j-1/2} + h_n, \quad h_n = o(n^{-r/2})$$

and

$$\sum_{n=1}^{\infty} \frac{s^n}{n} \Delta_n = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} p_j \sum_{n=1}^{\infty} n^{-j-1/2} s^n + \sum_{n=1}^{\infty} h_n s^n.$$

If one changes the basis of the expansion, then one gets a simpler expression:

$$\frac{\Delta_n}{n} = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} q_j a_n^{(j+1)} + \tilde{h}_n, \quad \tilde{h}_n = o(n^{-r/2})$$

and

$$\sum_{n=1}^{\infty} \frac{s^n}{n} \Delta_n = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} q_j ((1-s)^{j-1/2} - 1) + \tilde{H}(s).$$

Theorem 2. Assume that S_n is left-continuous and that $\mathbf{E}|X_1|^r$ is finite for some integer $r \geq 3$. Then there exist polynomials $V_1(x), V_2(x), \dots, V_{\lfloor \frac{r-1}{2} \rfloor}(x)$ (every V_k is of degree $2k-1$) such that

$$\mathbf{P}(\tau_x > n) = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} V_j(x) a_n^{(j)} + o\left(\frac{x^{r-1}}{n^{(r-1)/2}} \right)$$

uniformly in $x = o(\sqrt{n})$.

The proof of this result is based on the formula

$$\mathbf{P}(\tau_x = n) = \frac{x}{n} \mathbf{P}(S_n = -x)$$

and on asymptotic expansions in the local CLT.

Theorem 3. Assume that S_n is lattice and that $\mathbf{E}|X_1|^r$ is finite for some integer $r \geq 3$. Then there exist functions $V_1(x), V_2(x), \ldots, V_{\lfloor \frac{r-1}{2} \rfloor}(x)$ such that, for every fixed x,

$$\mathbf{P}(\tau_x > n) = \sum_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} V_j(x) a_n^{(j)} + o\left(\frac{\log n}{n^{(r-1)/2}}\right).$$

Koroljuk(1962) derived some expansions for the probability $\mathbf{P}\left(\max_{k\le n} S_k < x\sqrt{n}\right)$ for every fixed $x > 0$.

Borovkov (1962) obtained a full expansion for $\mathbf{P}\left(\max_{k\le n} S_k < x\right)$ for $x = o(n)$ under the assumption that X_1 has finite exponential moments. The remainder term in that result is of order $O(e^{-\gamma x})$ with some $\gamma > 0$. Thus, Borovkov's expansion is not applicable in the case of bounded x.

Nagaev (1970) has derived an expansion for $\mathbf{P}\left(\max_{k\le n} S_k < x\sqrt{n}\right)$ under the assumption that the moment of order m is finite. The remainder term in that paper is given by

$$O\left(\min\left\{\frac{1}{\sqrt{n}}, (1+(x/\sqrt{n})^{1-m})n^{1-m/2}\log^2 n\right\}\right).$$

This implies that his expansion is also not applicable in the case of fixed x.

All coefficients in that expansions are polynomials in x!

Consider a substochastic transition kernel

$$P(x, dy) = \mathbf{P}(x + S_1 \in dy, x + S_1 > 0).$$

Then

$$P^n f(x) = \mathbf{E}[f(x + S_n); \tau_x > n], \quad x > 0.$$

It is well known that $V(x)$ is harmonic:

$$(P - I)V = 0.$$

Theorem 4. Every function V_j defined in Theorem 3 is polyharmonic of order j:

$$(P - I)^j V_j = 0.$$