Local probabilities for asymptotically stable random walks in a half space
(jointly with V. Wachtel)

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Outline

Random walk

Consider a random walk $\{S_n, n \geq 1\}$ on $\mathbb{R}^d$, $d \geq 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n), n \geq 1\}$ is a family of independent copies of a random vector

$$X = (X_1, \ldots, X_d) = (X_1, X_{2,d})$$

Let

$$\mathbb{H}^+ := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \colon x_1 > 0\}.$$

For $x \in \overline{\mathbb{H}}$ let

$$\tau_x := \min\{\, n \geq 1 : x + S(n) \notin \mathbb{H}^+ \,\}$$

be the first time the random walk exits the (positive) half space

Green function

Let $\Delta = [0,1)^d$.
We study the asymptotic behaviour of local probabilities

$$p_n(x,y) = \mathbf{P}(x + S_n = y, \tau_x > n), \qquad \text{(lattice case)}, x, y \in \mathbb{Z}^d \cap \mathbb{H}^+$$
$$p_n(x,y) = \mathbf{P}(x + S_n \in y + \Delta, \tau_x > n), \quad \text{(non-lattice case)}, x, y \in \mathbb{H}^+$$

as n increases, and the behaviour of the Green function

$$G(x,y) = \delta(x,y) + \sum_{n=1}^{\infty} p_n(x,y)$$

as x and/or y increase.

Main assumptions

We will assume that

$$\frac{S_n}{c_n} \xrightarrow{d} \zeta^\alpha, \tag{1}$$

where ζ^α is a multivariate stable law of index $\alpha \in (0, 2]$ and c_n is a scaling sequence, which is regularly varying of index $1/\alpha$.

Furthermore, we assume that

$$\mathbf{P}(\zeta^\alpha \in \mathbb{H}) = \mathbf{P}(\zeta_1^\alpha > 0) = \rho \in (0, 1)$$

ensuring that the Spitzer-Doney condition holds

$$\mathbf{P}(S_1(n) > 0) \to \rho, \quad n \to \infty.$$

We will write $X \in \mathcal{D}(d, \alpha, \sigma)$ when the above assumptions hold.

Remark. When $\alpha \in (0, 2)$, convergence $\frac{S_n - d_n}{c_n}$ follows if $\mathbf{P}(|X| > t)$ is regularly varying of index $-\alpha$ and the there exists a measure σ on the unit sphere $\mathbb{S}^{d-1}$ such that for any measurable A on $\mathbb{S}^{d-1}$

$$\frac{\mathbf{P}\left(X > tx, \frac{X}{|X|} \in A\right)}{\mathbf{P}(|X| > x)} \to t^{-\alpha}\sigma(A), \quad x \to \infty.$$

Related works: discrete time

- Vatutin and Wachtel (2009), Doney (2012), Caravenna and Chaumont (2013). One-dimensional case for asymptotically stable random walks.
- Varopoulos(97,99,01). Gaussian estimates for $p_n(x, y)$ for random walks in cones (bounded increments, $\alpha = 2$).
- Varopoulos(01). Gaussian estimates for $p_n(x, y)$ for random walks in Lipschitz domains (existence of sufficiently many moments, $\alpha = 2$).
- Uchiyama (14). Integer valued random walks in half space ($\alpha = 2$). Asymptotics for $G(x, y)$.
- Denisov and Wachtel(15). Random walks in cones ($\alpha = 2$). Asymptotics for $p_n(x, y)$, when either x and y are fixed or x is fixed and y of order c_n and far away from boundary.
- Duraj, Raschel, Tarrago, Wachtel (20). Lattice random walks in convex cones ($\alpha = 2$). Asymptotics for $G(x, y)$ in a half-space, and also asymptotics in convex cones.

Related works: continuous time

- Brownian motion (reflection principle).
- Tamura and Tanaka (2008). Lévy processes in half space. Derivation of $G(x, y)$ based on the Wiener-Hopf factorisation. Not very explcit, as requires Fourier inversion. Explicit (and exact) formula for rotation invariant stable Lévy processes.
- Bogdan, Palmowski, Wang (2018). Rotation invariant stable Lévy processes $(X_t)_{t\geq 0}$ in cones. Asymptotics for

$$\mathbf{P}(x + X_t \in t^{1/\alpha} A, \tau_x > t),$$

when x is fixed.

Motivation and approach

- Main motivation is to use the asymptotics for half spaces to consider asymptotically stable random walks in convex cones.
- Asymptotics for the Green function in half spaces are interesting by themselves.

Approach

- We use the Wiener-Hopf factorisation (similarly to one-dimensional case) to study behaviour of $p_n(x, y)$ for n, x, y in various zones;
- Then we obtain $G(x, y)$ by plugging the asymptotics for $p_n(x, y)$ into the sum

$$\sum_{n \geq 1} p_n(x, y)$$

Outline

One-dimensional case: ladder heights

Let τ and τ^+ be the first weak and strict ladder epochs

$$\tau := \min\{n \ge 1 : S(n) \le 0\} \text{ and } \tau^+ := \min\{n \ge 1 : S(n) > 0\}$$

and $\chi^- = S(\tau), \chi^+ = S(\tau^+)$ be the first ladder heights. Define the sequence of i.i.d.'s $(\chi_n^+)_{n\ge1}$ and $(\chi_n^-)_{n\ge1}$ distributed as χ^+ and χ^- and corresponding renewal functions

$$H(u) = I(u > 0) + \sum_{n=1}^{\infty} \mathbf{P}(\chi_1^+ + \cdots + \chi_n^+ < u)$$

$$V(u) = I(u > 0) + \sum_{n=1}^{\infty} \mathbf{P}(\chi_1^- + \cdots + \chi_n^- < u)$$

and renewal mass functions

$$h(u) = H(u+1) - H(u), \quad v(u) = V(u+1) - V(u).$$

One-dimensional case: Green function

We can find the Green function using the representation, see Spitzer(1964, Chapter 19)

$$G(x, y) = c \sum_{n=0}^{\min(x,y)} v(x-n)h(y-n). \tag{2}$$

Then, we can use the renewal theorem and some information about v, H to obtain the asymptotic behaviour of the Green function.

One-dimensional case: asymptotics for $p_n(0, y)$

Vatutin and Wachtel(2009) analysed the asymptotic behaviour of $p_n(0, y)$ in the stable case, Alili and Doney(1999) in the finite variance case.

Vatutin and Wachtel(2009)

- Small deviations zone ($y = o(c_n)$):

$$p_n(0, y) \sim C \frac{H(y)}{n c_n}, \quad n \to \infty$$

- Normal deviations zone ($A^{-1} c_n \leq y \leq A c_n$)
- Approach: use of a recursion obtained from the Spitzer identity

Doney and Jones(2012)

- Large deviations zone ($y >> c_n$)
- Approach: analysis of the single big jump.

One-dimensional case: asymptotics for $p_n(x, y)$

Asymptotics for $p_n(x, y)$ were obtained by Doney(2012), Caravenna and Chaumont(2013).
The approach is based on a time-dependent extension of formula (2)

$$p_n(x, y) = \sum_{r=0}^{n} \sum_{n=0}^{\min(x,y)} p_r^-(0, x-n) p_{n-r}(0, y-n). \qquad (3)$$

Now using the asymptotics from Vatutin and Wachtel(2009) one can consider several possible case of small and normal deviations. For example, when $x, y = o(c_n)$, see Doney (2012),

$$p_n(x, y) \sim C \frac{V(y)H(x)}{nc_n}.$$

Outline

Half space

Main observation

There exists an extension of the Spitzer identity to half-spaces. For $t \in \mathbb{R}^d$ and $|s| < 1$ the following identity holds (Tamura and Tanaka, 2002)

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}[e^{it\cdot S(n)}; \tau_0 > n] = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{E}\left[e^{it\cdot S(n)}; S_1(n) > 0\right]\right\}.$$

Using this identity one can extend the recursion from Vatutin and Wachtel (2009). For $y = (y_1, \ldots, y_d)$ put

$$B_n(y) = \mathbf{P}(S_1(n) \in (0, y_1), S_2(n) < y_2, \ldots, S_d(n) < y_d; \tau_0 > n)$$

Then,

$$nB_n(y) = \mathbf{P}(S(n) < y, S_1(n) > 0) + \sum_{k=1}^{n-1} \int_{\mathbb{H}^+} \mathbf{P}(S_k < y - z, S_1(k) > 0) dB_{n-k}(z$$

Half space: approach

We can use the same approach now:

1. Find asymptotics or upper bounds for $p_n(0, y)$.
2. Extend formula (3) to the multidimensional case to find the asymptotic behaviour of $p_n(x, y)$.
3. Find the asymptotics for $G(x, y)$ by analysing the sum of $p_n(x, y)$.

Small deviations (start at the origin)

Here we modified the first part of arguments of Vatutin and Wachtel (2009), but very similar considerations will work just fine.

1. Use concentration inequalities to obtain a rough estimate of $p_n(0, y)$;
2. Make these inequalities sharp by using the above recursions;
3. Apply recursions again to obtain the exact asymptotics.

Theorem

Suppose $X \in \mathcal{D}(d, \alpha, \sigma)$ and the distribution of X is lattice. Then

$$p_n(0, y) \sim C g_{\alpha,\sigma}\left(0, \frac{y_{2,d}}{c_n}\right)\frac{H(y_1)}{nc_n^d}, \quad \text{as } n \to \infty,$$

uniformly in $y_1 \in (0, \delta_n c_n]$, where $\delta_n \to 0$ as $n \to \infty$.

Here, $g_{\alpha,\sigma}$ is the density of the limiting law ζ^α.

Small deviations (arbitrary starting point)

We make use of the following extension of (3)

$$p_n(x,y) = \sum_{k=0}^{n} \sum_{z_1=1}^{x_1\wedge y_1} \sum_{z_2,\dots,z_d} p_k^+(0, z-x) p_{n-k}(0, y-z). \qquad (4)$$

Using the asymptotics with start at the origin we obtain

Theorem

Let $x, y \in \mathbb{H}^+ \cap \mathbb{Z}^d$. Suppose $X \in \mathcal{D}(d, \alpha, \sigma)$ and the distribution of X is lattice. Then

$$p_n(x,y) \sim V(x_1)H(y_1)\frac{g_{\alpha,\sigma}\left(0, \frac{y_{2,d}-x_{2,d}}{c_n}\right)}{nc_n^d}$$

uniformly in $x_1, y_1 \in (0, \delta_n c_n]$ such that $|x-y| \le Ac_n$, where $\delta_n \to 0$ as $n \to \infty$ and A is a fixed constant.

Large deviations (stable case)

This was the main difficulty. Arguments of Doney and Jones (2012) can be extended only when $|x| >> c_n$. However, we need $x_1 >> c_n$ to cover all zones.

We will impose one of the following assumptions

- (global, for $\alpha = 2$) There exists a regularly varying ϕ of index $-\alpha$ such that it is sufficient to assume that

$$\mathbf{P}(|X| > x) \le \phi(|x|). \tag{5}$$

- (local) There exists regularly varying of index $-\alpha$ function ϕ such that

$$\mathbf{P}(X \in x + \Delta) \le \frac{\phi(|x|)}{|x|^d} =: g(|x|). \tag{6}$$

  and

$$a_1\phi(t) \le \mathbf{P}(|X| > t) \le a_2\phi(t), \quad t \in \mathbb{R}, \tag{7}$$

  for some positive constants a_1, a_2.

Large deviations (arbitrary starting point)

Theorem

Suppose there exists a regularly varying ϕ *such that* (5) *holds. Then there exists* $c_0 > 0$ *and* $C_0 > 0$ *such that for* $x, y \in \mathbb{H}^+$ *with* $|x - y| > c_n$ *we have*

$$p_n(x, y) \le \frac{C_0}{n c_n^d} \left(\mathrm{e}^{-c_o \frac{|x-y|}{c_n}} 1_{\{\alpha=2\}} + n\phi(|x - y|) \right) H(y_1 + 1) V(x_1 + 1). \quad (8)$$

If, in addition (6) *and* (7) *hold then*

$$p_n(x, y) \le C_0 \left(\frac{1}{n c_n^d} \mathrm{e}^{-c_o \frac{|x-y|}{c_n}} 1_{\{\alpha=2\}} + g(|x - y|) \right) H(y_1 + 1) V(x_1 + 1). \quad (9)$$

Remark This theorem can be improved to obtain the sharp asymptotics if, for example,

$$\mathbf{P}(X = y) \text{ is regularly varying.}$$

Comments on large deviations

This seems to be the most difficult part. The proof relies on the following large deviations bounds from Berger(2019).

Lemma

Let $X \sim \mathcal{D}(\alpha, \sigma)$. Suppose that there exists a regularly varying φ such that (5) holds. Then, there exist constants c_H and C_H such that for $|x| \geq c_n$ we have,

$$\mathbf{P}(S(n) \in x + \Delta) \leq \frac{C_H}{c_n^d} \left(n\phi(|x|) + e^{-c_H(|x|/c_n)^2} 1_{\{\alpha=2\}} \right). \qquad (10)$$

If, in addition, (6) holds, then

$$\mathbf{P}(S(n) \in x + \Delta) \leq C_H n\phi(|x|)|x|^{-d} + \frac{1}{c_n^d} e^{-c_H(|x|/c_n)^2} 1_{\{\alpha=2\}}. \qquad (11)$$

Comments on large deviations (start at the origin)

Using the representation

$$nB_n(y) = \mathbf{P}(S(n) < y, S_1(n) > 0) + \sum_{k=1}^{n-1} \int_{\mathbb{H}^+} \mathbf{P}(S_k < y - z, S_1(k) > 0) dB_{n-k}(z)$$

we proceed by induction, plugging in the above large deviations estimates at each step.

Asymptotics for $G(x, y)$ near the boundary

There are 2 cases

1. Near to the boundary case: $y_1 = o(|y|)$. In this case we use small deviations asymptotics $p_n(x, y)$;
2. Far from the boundary case: y_1 is of order $|y|$. In this case we use normal deviations bounds.

In both these cases we need large deviations bounds as will be explained now.

Asymptotics for $G(x, y)$(start at the origin)

We can now represent for large A,

$$G(x,y) = \underbrace{\sum_{n: c_n \le A^{-1}|y|} p_n(x,y)}_{\text{large deviations}} + \underbrace{\sum_{n: A^{-1}|y| < c_n} p_n(x,y)}_{\text{small/normal deviations}}.$$

Now we can use the above results to analyse the sum. We can show that the large deviations sum is smaller than

$$A^{-1}\frac{H(y_1)V(x_1)}{|x-y|^d}$$

and hence negligible under assumptions in the theorem.

Asymptotics for $G(x, y)$, near the boundary

In this case, as $x_1, y_1 = o(|y|) = o(c_n)$, we can analyse the main term as follows

$$\sum_{n:A^{-1}|x-y|<c_n} p_n(x,y) \sim C \sum_{n:A^{-1}|x-y|<c_n} g_{\alpha,\sigma}\left(0, \frac{y_{2,d}-x_{2,d}}{c_n}\right) \frac{H(y_1)V(x_1)}{nc_n^d}$$

$$\sim CH(y_1)V(x_1) \int_{A^{-1}|x-y|}^{\infty} g_{\alpha,\sigma}\left(0, \frac{y_{2,d}-x_{2,d}}{t}\right) \frac{dt}{t^{d+1}}$$

$$= \frac{H(y_1)V(x_1)}{|x-y|^d} \int_{A^{-1}}^{\infty} g_{\alpha,\sigma}\left(0, \frac{y_{2,d}-x_{2,d}}{|x-y|}t\right) \frac{dt}{t^{d+1}}$$

When the large deviations part is negligible, letting A to infinity, we obtain

$$G(x,y) \sim C\frac{H(y_1)V(x_1)}{|x-y|^d} \int_0^{\infty} g_{\alpha,\sigma}\left(0, \frac{y_{2,d}-x_{2,d}}{|x-y|}t\right) \frac{dt}{t^{d+1}}$$

Asymptotics for $G(x, y)$ near the boundary

Theorem

Let $X \in \mathcal{D}(d, \alpha, \sigma)$ be a lattice random variable. Suppose there exists a regularly varying ϕ such that (6) and (7) hold. Then,

$$G(x, y) \sim C \frac{H(y_1)V(x_1)}{|x-y|^d} \int_0^\infty g_{\alpha,\sigma}\left(0, \frac{y_{2,d} - x_{2,d}}{|x-y|} t\right) t^{d-1} dt$$

for $x_1, y_1 = o(|x-y|)$. In particular, in the isotopic case, that is when the limiting density σ is uniform on the unit sphere,

$$G(x, y) \sim C \frac{H(y_1)V(x_1)}{|x-y|^d}, \quad |x-y| \to \infty,$$

for $x_1, y_1 = o(|x-y|)$.

A similar result holds in the non-lattice case.

Finite varaince case near the boundary

Theorem

Let $X \in \mathcal{D}(d, 2, \sigma)$ be a non-lattice random variable. Assume that one of the following assumptions hold

1. $d = 2$ *and* $\mathbf{E}[|X^2| \ln |X|] < \infty$;
2. $d \geq 3$ *and* $\mathbf{E}[|X|^d]$.

$$G(x, y) \sim C \frac{\int_{y_1}^{y_1+1} H(u) du V(x_1)}{|x-y|^d}, \quad |x-y| \to \infty,$$

for $x_1, y_1 = o(|x-y|)$.

Normal deviations (preliminary result)

Here we need a preliminary result, which is an extension of one-dimensional results due to Bolthausen(1976) and Doney(1986)

Theorem

Let $X \in \mathcal{D}(d, \alpha, \sigma)$*. Then there exists a random vector* M *on* $\mathbb{H}^+$ *with density* $p_M(v)$ *such that, for all* $u \in \mathbb{R}^d$*,*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{S_n}{c_n} \in u + \Delta \mid \tau^- > n\right) = \mathbf{P}(M \in u + \Delta) = \int_{u+\Delta} p_M(v)dv.$$

Normal deviations(in progress)

Again, arguments of Vatutin and Wachtel (2009) can be extended without difficulties. For start at 0 we obtain

Theorem

Suppose X is asymptotically stable and the distribution of X is lattice. Then, for every $r > 0$,

$$c_n^d \mathbf{P}(S_n = y \mid \tau_0^- > n) - p_M(y/c_n) \to 0 \text{ as } n \to \infty \tag{12}$$

uniformly in $\mathbb{H}^+$.

- Asymptotics for $p_n(0, y)$ will follow immediately by unconditioning.
- Similarly to the above we obtain asymptotics for $p_n(x, y)$.
- Then summing up and getting rid of the large deviations terms the asymptotics for $G(x, y)$ can be obtained.

Further questions

- Complete normal deviations case (in progress).
- Optimal moment conditions in finite variance case.
- Use the results to study asymptotically stable random walks in convex cones.
- Lévy processes in cones.