On kernels of some random operators.

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1. Problem:
Let $\xi(t)$ be a random process with continuous trajectories, $\Phi : C[0, T] \to \mathbb{R}$ and $\eta = \Phi(\xi(\cdot))$ be a random variable. When do we have $\mathcal{P}_\eta \ll \mathfrak{m}$? What are the properties of the density?

2. Another problem:
Let $\mathcal{R} : C[0, T] \to C[0, T]$ be a linear mapping:
$\mathcal{R}(f + g) = \mathcal{R}f + \mathcal{R}g, \quad \mathcal{R}(\alpha f) = \alpha \mathcal{R}f$ and suppose that $\mathcal{R} = \mathcal{R}(\omega)$ is a random linear mapping.

The question is if $\mathcal{R}$ is a.s. an integral operator and what is its domain?
$\mathcal{R}f(x) = \int r(x, y) f(y)\, dy$
What are the properties of the kernel?

Example: Let $w(s)$, $s \geqslant 0$ be a standard Wiener process.

$$\mathcal{R}f(x) = \int_0^t f(x - w(\tau))d\tau,$$

Let us find the expression for the kernel:
$f(x) = \int_{\mathbb{R}} f(y)\delta(x - y)dy$

$$\mathcal{R}f(x) = \int_{\mathbb{R}} f(y) \int_0^t \delta(x - y - w(\tau))\, d\tau\, dy = f * r(t, x),$$

where

$$r(t, x) = \int_0^t \delta(x - w(\tau))\, d\tau.$$

Consider the Fourier transform

$$\widehat{r}(t, p) = \int_0^t e^{ipw(\tau)}\, d\tau.$$

To prove $r(t,\cdot)\in L_2(\mathbb{R})$ a.s., it is sufficient to prove $\widehat{r}(t,\cdot)\in L_2(\Omega\times\mathbb{R},\mathbf{P}\times\mathfrak{m})$ or

$$\int_{\mathbb{R}} dp\,\mathbf{E}\Big|\int_0^t e^{ipw(\tau)}d\tau\Big|^2<\infty$$

For $|p|>1$ we have

$$\mathbf{E}\Big|\int_0^t e^{ipw(\tau)}d\tau\Big|^2=2\mathrm{Re}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\,\mathbf{E}e^{ip(w(\tau_1)-w(\tau_2))}=$$

$$=2\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2 e^{-\frac{p^2}{2}(\tau_1-\tau_2)}\leqslant\frac{C}{p^2}.$$

Additional possibilities: $r(t,\cdot)\in\mathcal{H}_\alpha$ for any $\alpha\in[0,1/2)$, where $\mathcal{H}_\alpha=\{g:\ \mathbf{E}\int(1+|p|^{2\alpha})|\widehat{g}(p)|^2\,dp<\infty\}$.
So we have proved that $r(t,\cdot)\in W_2^\alpha(\mathbb{R})$ a.s.

Let $\xi_x(t), \quad t \geqslant 0, \ x \in \mathbb{R}$ be a solution of a stochastic differential equation

$$d\xi_x(t) = b(\xi_x(t))b'(\xi_x(t))\,dt + b(\xi_x(t))\,dw(t),$$

$$\xi_x(0) = x.$$

Assume that
1. $b \in C_b^2(\mathbb{R})$.
2. $\theta_0 = \inf\limits_{x \in \mathbb{R}} b(x) > 0$.
3. $\exists \quad b_0 > 0$ such that $\lim_{x \to \pm\infty} b(x) = b_0$.
4. $\lim_{x \to \pm\infty} b'(x) = \lim_{x \to \pm\infty} b''(x) = 0$.
5. $b(x) - b_0 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

Suppose that

$$b_0 = 1.$$

The Markov family $\xi_x(t),\ x \in \mathbb{R}$ generates a semigroup $P^t : C(\mathbb{R}) \to C(\mathbb{R})$, where for any $f \in C(\mathbb{R})$

$$u(t,x) = P^t f(x) = \mathbf{E} f(\xi_x(t)).$$

The function $u(t,x)$ satisfies the backward Kolmogorov equation

$$\frac{\partial u}{\partial t} = -\mathcal{A}u,$$

with an initial condition

$$u(0,x) = \lim_{t \downarrow 0} u(t,x) = f(x),$$

where

$$\mathcal{A} = -\frac{1}{2}\frac{d}{dx}\big(b^2(x)\frac{d}{dx}\big) > 0.$$

Define $P^t f(x)$ for $f \in L_2$.

$$P^t f(x) = \mathbf{E} f(\xi_x(t)).$$

Consider the self-adjoint operator $\mathcal{A}$, on the domain $\mathcal{D}(\mathcal{A}) = W_2^2(\mathbb{R})$.
We have

$$P^t = e^{-t\mathcal{A}},$$

and

$$e^{-t\mathcal{A}} f(x) = \mathbf{E} f(\xi_x(t)).$$

We are interested in a probabilistic representation of the resolvent operator.

Some ideas:
For any λ such that $\mathrm{Re}\,\lambda < 0$ and any $f \in C(\mathbb{R})$ we have

$$(\mathcal{A} - \lambda I)^{-1} f = \int_0^\infty e^{\lambda\tau} e^{-\tau\mathcal{A}} f \, d\tau$$

$$(\mathcal{A} - \lambda I)^{-1} f(x) = \mathbf{E} \int_0^\infty e^{\lambda\tau} f(\xi_x(\tau)) \, d\tau = \lim_{t\to\infty} \mathbf{E} \int_0^t e^{\lambda\tau} f(\xi_x(\tau)) \, d\tau.$$

Consider a random operator $\mathcal{R}_\lambda^t$, where

$$\mathcal{R}_\lambda^t : f \mapsto \int_0^t e^{\lambda\tau} f(\xi_x(\tau)) \, d\tau.$$

What can we say about $\mathcal{R}_\lambda^t$? If $\lambda = 0$ the kernel is a local time.
$\mathcal{D}(\mathcal{R}_\lambda^t) =$? How to define an operator in the case $\mathrm{Re}\,\lambda > 0$?

Let $\lambda = a + bi$.
First suppose that $a = \mathrm{Re}\,\lambda \leqslant 0$. In this case set

$$\mathcal{R}_\lambda^t f(x) = \int_0^t f(\xi_x(\tau)) e^{\lambda\tau}\, d\tau.$$

Using $f(x) = \int f(y)\delta(x-y)\,dy$ we get

$$\mathcal{R}_\lambda^t f(x) = \int f(y) \int_0^t e^{\lambda\tau}\delta(\xi_x(\tau)-y)\,d\tau\,dy = \int f(y) r_\lambda(t,x,y)\,dy,$$

where

$$r_\lambda(t,x,y) = \int_0^t e^{\lambda\tau}\delta(\xi_x(\tau)-y)\,d\tau.$$

First idea is to use the Fourier transform.

The operator $\mathcal{A} = -\frac{1}{2}\frac{d}{dx}(b^2(x)\frac{d}{dx})$ is self-adjoint,

$$\sigma(\mathcal{A}) = \sigma_{ac}(\mathcal{A}) = [0, \infty).$$

Let $\varphi(x, k)$ be generalized eigenfunctions of the continuous spectrum of the operator $\mathcal{A}$. For any $k \in \mathbb{R}$ it is a solution

$$\mathcal{A}\varphi(x, k) = \frac{k^2}{2}\varphi(x, k)$$

such that

$$\int \varphi(x, k)\overline{\varphi(x, k')}\, dx = \delta(k - k')$$

and

$$\int \varphi(x, k)\overline{\varphi(x', k)}\, dk = \delta(x - x').$$

The choice of $\varphi(x, k)$ is not unique, so we choose $\varphi(x, k)$ such that they have an analytical continuation on k to the upper half-plane and for $\mathrm{Im}\, k > 0$ we have $\lim_{x\to+\infty} \varphi(x, k) = 0$. (If $\mathcal{A} = -\frac{1}{2}\frac{d^2}{dx^2}$ then $\varphi(x, k) = \frac{1}{\sqrt{2\pi}}e^{ikx}$).

The functions $\varphi(x,k)$ define the kernel of an unitary operator $\Psi : L_2(\mathbb{R}) \to L_2(\mathbb{R})$. The operator Ψ weaves the operator $\mathcal{A}$ and the multiplication operator by the function $\frac{k^2}{2}$.
Namely define the unitary operator by

$$\Psi : f(x) \mapsto (L_2) \lim_{M\to\infty} \int_{-M}^{M} f(x)\overline{\varphi(x,k)}\,dx = (\Psi f)(k),$$

$$\Psi^{-1} : g(k) \mapsto (L_2) \lim_{M\to\infty} \int_{-M}^{M} g(k)\varphi(x,k)\,dk = (\Psi^{-1} g)(x).$$

$$\mathcal{A}f = g \quad \Leftrightarrow \quad \frac{k^2}{2}(\Psi f)(k) = (\Psi g)(k).$$

$$F(\mathcal{A})f = g \quad \Leftrightarrow \quad F(\frac{k^2}{2})(\Psi f)(k) = (\Psi g)(k).$$

$$\mathbf{E}\varphi(\xi_x(t),k)=e^{-\frac{k^2t}{2}}\varphi(x,k).$$

$$\mathcal{D}(F(\mathcal{A}))=\{f\in L_2(\mathbb{R}):F(\frac{k^2}{2})(\Psi f)(k)\in L_2(\mathbb{R})\}.$$

Lemma

There exists a constant $L>0$ such that for any $x\in\mathbb{R}$, and $k\in\mathbb{R}$ we have $|\varphi(x,k)|\leqslant L$.

Lemma

$\varphi(x,k)\neq 0$ *for* $\mathrm{Im}\, k>0$.

Lemma

Let a function g satisfies the condition $\int_{\mathbb{R}}(1+|k|^{2\alpha})|g(k)|^2\,dk<\infty$ for some $\alpha\in(0,\frac{1}{2})$. Then $\Psi^{-1}g\in W_2^{\alpha}(\mathbb{R})$.

Suppose that $a = \mathrm{Re}\,\lambda \leqslant 0$. In this case

$$r_\lambda(t,x,y) = \int_0^t e^{\lambda\tau}\delta(\xi_x(\tau) - y)\,d\tau.$$

Let us calculate $\Psi r_\lambda(t,x,y)$ with respect to y. We have

$$\delta(\xi_x(\tau) - y) = \int_{\mathbb{R}} \overline{\varphi(y,k)}\varphi(\xi_x(\tau),k)\,dk$$

So that

$$r_\lambda(t,x,y) = \int_{\mathbb{R}} \overline{\varphi(y,k)} \int_0^t \varphi(\xi_x(\tau),k)e^{\lambda\tau}\,d\tau\,dk.$$

We get

$$(\overline{\Psi r_\lambda})(t,x,k) = \int_0^t \varphi(\xi_x(\tau),k)e^{\lambda\tau}\,d\tau.$$

(It is a kernel of the operator $\mathcal{R}_\lambda^t\Psi^{-1} = \mathcal{R}_\lambda^t\Psi^*$.)

Now suppose that $a = \mathrm{Re}\,\lambda > 0$. For $|k| > \sqrt{2a}$ we define Ψr_λ as before. For $|k| < \sqrt{2a}$ set

$$(\overline{\Psi \overline{r_\lambda}})(t,x,k) = -\frac{\varphi(x,k)}{\varphi(0,i|k|)} \int_0^t \varphi(\xi_0(\tau), i|k|) e^{-\lambda\tau}\, d\tau.$$

Define the space $\mathcal{H}_\alpha$ of $L_2(\mathbb{R})$-valued random variables g

$$\mathcal{H}_\alpha = \{g : \ \mathbf{E} \int_{\mathbb{R}} (1+|k|^{2\alpha}) |(\Psi g)(k)|^2\, dk < \infty\},$$

If $g \in \mathcal{H}_\alpha$, then $g \in W_2^\alpha(\mathbb{R})$ a.s.

Theorem

1. For any $\alpha \in [0, \frac{1}{2})$ *there exists the uniform in* $x \in \mathbb{R}$ *limit:*

$$r_\lambda(t, x, \cdot) = (\mathcal{H}_\alpha) \lim_{M \to \infty} r_\lambda(t, x, \cdot, M),$$

where

$$(\Psi r_\lambda)(t, x, k, M) = \mathbf{1}_{[-M,M]}(k) \int_0^t (\Psi r_\lambda)(\tau, x, k) \, d\tau.$$

2. If $a = \mathrm{Re}\, \lambda < 0$ *then for any* $\alpha \in [0, \frac{1}{2})$ *there exists the uniformly in* $x \in \mathbb{R}$ *limit:*

$$r_\lambda(\infty, x, \cdot) = (\mathcal{H}_\alpha) \lim_{t \to \infty} r_\lambda(t, x, \cdot).$$

Theorem

For any fixed $0 \leqslant t < \infty$ *with probability 1 the operator* $\mathcal{R}_\lambda^t$ *is a bounded operator in* $L_2(\mathbb{R})$.

Theorem

There exists a constant $C > 0$ *such that for any* $f \in L_2(\mathbb{R})$ *we have*

$$\mathbf{E}\|\mathcal{R}_\lambda^t f\|_2^2 \leqslant C\|f\|_2^2.$$

Theorem

1. If $\operatorname{Re}\lambda < 0$ *then for any* $f \in L_2(\mathbb{R})$ *we have*

$$\mathbf{E}\int_{\mathbf{R}} r_\lambda(\infty,\cdot,y)f(y)\,dy = (\mathcal{A}-\lambda I)^{-1}f.$$

2. If $\operatorname{Re}\lambda \geqslant 0$ *and* $\lambda \notin \sigma(\mathcal{A})$ *then for any* $f \in L_2(\mathbb{R})$ *we have*

$$(L_2)\lim_{t\to\infty}\mathbf{E}\int_{\mathbb{R}} r_\lambda(t,\cdot,y)f(y)\,dy = (\mathcal{A}-\lambda I)^{-1}f. \qquad (1)$$

3. If $\lambda \in \sigma(\mathcal{A})$ *then (1) holds for any* $f \in \mathcal{D}(\mathcal{A}-\lambda I)^{-1}$.

Some generalizations

$$\mathcal{A} = -\frac{1}{2}\frac{d}{dx}(b^2(x)\frac{d}{dx}) + V(x),$$

where $V(x)$ is a rapidly decreasing potential.

$$\sigma(\mathcal{A}) = \{-k_1^2, \ldots, -k_N^2\} \cup [0, \infty).$$

$$L_2(\mathbb{R}) = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\} \oplus H_{ac}^{\mathcal{A}}$$

$$\varphi_j \in L_2(\mathbb{R}),\ j = 1, 2, \ldots, N; \quad \varphi(\cdot, k) \notin L_2(\mathbb{R}).$$

Using $\varphi(x, k)$ we define an isometric operator

$$\Psi : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \quad \Psi^* : L_2(\mathbb{R}) \to H_{ac}^{\mathcal{A}}.$$

$$\Psi\Psi^* = Id \quad \Psi^*\Psi = \mathrm{P}_{ac}^{\mathcal{A}}$$

the projector onto an absolutely continuous subspace.

In this case for $\mathrm{Re}\,\lambda \leqslant 0$ set

$$\mathcal{R}_\lambda^t f(x) = \int_0^t e^{\lambda\tau} \big(\mathrm{P}_{ac}^{\mathcal{A}} f\big)(\xi_x(\tau)) e^{-\int_0^\tau V(\xi_x(s))\,ds}\, d\tau,$$

where $\mathrm{P}_{ac}^{\mathcal{A}}$ is the projector onto an absolutely continuous subspace. The kernel $t_{ac}^{\mathcal{A}}$ of the projector $\mathrm{P}_{ac}^{\mathcal{A}}$ has the form

$$t_{ac}^{\mathcal{A}}(x,y) = \int \varphi(x,k)\overline{\varphi(y,k)}\, dk$$

We get

$$\overline{\Psi r_\lambda}(t,x,k) = \int_0^t e^{\lambda\tau} \varphi(\xi_x(\tau),k) e^{-\int_0^\tau V(\xi_x(s))\,ds}\, d\tau.$$

We use the identity

$$\mathbf{E}\varphi(\xi_x(\tau),k) e^{-\int_0^\tau V(\xi_x(s))\,ds} = e^{-\frac{k^2\tau}{2}} \varphi(x,k).$$

Non-self-adjoint case.

$$\mathcal{B} = -\frac{1}{2}b^2(x)\frac{d^2}{dx^2} - a(x)\frac{d}{dx} + W(x)$$

$bb' - a \in L_1(\mathbb{R})$
The operator $\mathcal{B}$ is similar to a self-adjoint one. Let $\mathcal{A}$ be self-adjoint.

$$\mathcal{B} = \mathcal{K}^{-1}\mathcal{A}\mathcal{K}, \quad \|\mathcal{K}\|_{L_2\to L_2} < \infty, \ \|\mathcal{K}^{-1}\|_{L_2\to L_2} < \infty$$

$$F(\mathcal{B}) = \mathcal{K}^{-1}F(\mathcal{A})\mathcal{K}.$$

The operator $\mathcal{B}$ is self-adjoint in the scalar product

$$\langle u, v\rangle_K = \langle Ku, Kv\rangle.$$

Consider the operator

$$\mathcal{B} = -\frac{1}{2} b^2(x) \frac{d^2}{dx^2} - a(x) \frac{d}{dx} + W(x).$$

This operator is similar to the operator

$$\mathcal{A} = -\frac{1}{2} \frac{d}{dx} (b^2(x) \frac{d}{dx}) + W(x) + V(x),$$

where $\mathcal{K}^{-1} f(x) = G(x) f(x)$

$$G(x) = \exp(\int_0^x H(y)\, dy), \quad H(y) = \frac{b(y) b'(y) - a(y)}{b^2(y)}.$$

Let $\xi_x(t),\ \ t \geqslant 0,\ x \in \mathbb{R}$ be a solution of the stochastic differential equation

$$d\xi_x(t) = a(\xi_x(t))\,dt + b(\xi_x(t))\,dw(t),\ \ \xi_x(0) = x.$$

In this case

$$\mathcal{R}^t_\lambda f(x) = \int_0^t e^{\lambda\tau}\big(\mathrm{P}^{\mathcal{B}}_{ac} f\big)(\xi_x(\tau))e^{-\int_0^\tau W(\xi_x(s))\,ds}\,d\tau,$$

where $\mathrm{P}^{\mathcal{B}}_{ac} = \mathcal{K}^{-1}\mathrm{P}^{\mathcal{A}}_{ac}\mathcal{K}$ is the projector on the absolute continuous subspace.