Harris Ergodicity of a Split Transmission Control Protocol

joint work with Sergey Foss

Mikhail Chebunin

Institute of Stochastics, Karlsruhe Institute of Technology

Borovkov meeting

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Introduction

We are considering one of such systems proposed in [1] [1], i.e. we consider a 3-component Markov process in continuous time, the first and second components of which follow their TCP protocols, and the third component describes the accumulated work, with the first component functioning autonomously, and the dynamics of the second component depends on the first and third components.

[1] Baccelli, F., Carofiglio, G., Foss, S.: *Proxy caching in split TCP: dynamics, stability and tail asymptotics*. From Semantics to Computer Science: Essays in Honour of Gilles Kahn. Cambridge University Press, pp. 437-464 (2009).

Introduction

This Markov process describes the operation of an open queuing system (data transmission), in which the intensity of arrival of customers $X(t)$ is given by a left-continuous Markov process that satisfies the equation

$$dX(t) = adt - kX(t)M(dt), \tag{1}$$

where $a > 0,\ k \in (0,1)$ and $M(t)$ is a Poisson process with intensity $\lambda_1 > 0$.

The rate of transmission (leaving) of messages $Y(t)$ depends on the value of work in progress accumulated in the system by the time t

$$Q(t) = \max\left(\sup_{0\le u\le t}\int_u^t (X(v)-Y(v))dv,\right. \qquad (2)$$
$$\left. Q(0)+\int_0^t (X(u)-Y(u))du\right).$$

Namely, if $Q(t) > 0$, then the intensity increment $Y(t)$ satisfies the equation

$$dY(t) = bdt - lY(t)N(dt), \qquad (3)$$

and if $Q(t) = 0$, then the equation

$$dY(t) = b\frac{X(t)}{Y(t)}dt - lY(t)N(dt). \qquad (4)$$

Here $b > 0$, $l \in (0,1)$ and $N(t)$ are another Poisson process with parameter $\lambda_2 > 0$, independent of the first one.

Main result

Theorem 1

If

$$al\lambda_2 < bk\lambda_1, \tag{5}$$

then there exists a distribution π in the three-dimensional positive orthant to which the distribution of the process $(X(t), Y(t), Q(t))$ converges in the total variation metric, i.e.

$$\sup_A |\mathbb{P}((X(t), Y(t), Q(t)) \in A) - \pi(A)| \to 0 \text{ as } t \to \infty$$

for any finite initial values $(X(0), Y(0), Q(0))$. Here the supremum is taken over all Lebesgue measurable sets A in three-dimensional Euclidean space.

Notations

The main step in proving the main theorem is obtaining a similar statement for an embedded Markov chain.

- Let $W(t)$ be a superposition of the processes $M(t)$ and $N(t)$, i.e., a Poisson process, each point of which, independently of the others, belongs either to the process $M(t)$ (with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$) or process $N(t)$ (with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$).

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- Let $0 < T_1 < T_2 < \ldots$ – consecutive points of the $W(t)$ process.
- Denote by $(X_n, Y_n, Q_n) = (X(T_n+0), Y(T_n+0), Q(T_n+0))$, $n \geq 1$ the values of the process $(X(t), Y(t), Q(t))$ at the corresponding embedded times.

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- Let $(X_0, Y_0, Q_0) = (X(0), Y(0), Q(0))$.

Embedded Markov chain

Theorem 2

Under the condition (5), the Markov chain (X_n, Y_n, Q_n) is Harris ergodic and, in particular, there exists a probability measure π^* such that

$$\sup_A |\mathbb{P}((X_n, Y_n, Q_n) \in A) - \pi^*(A)| \to 0 \quad \text{as} \quad t \to \infty \tag{6}$$

for any initial value (X_0, Y_0, Q_0).

On the necessity of the condition (5) in Theorem 1

Let $\widehat{Y}(t)$ be a process that independently satisfies the equation (3) for all values of $Q(t)$ for $t \geq 0$, and

$$Q^X(t) = \int_0^t X(u)du, \quad Q^{\widehat{Y}}(t) = \int_0^t \widehat{Y}(u)du.$$

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Let $(X^0, \widehat{Y}^0)$ be a two-dimensional limit distribution to which the distributions $(X_n, \widehat{Y}_n)$ converge in the total variation metric as $n \to \infty$. Then for $t \to \infty$

$$\frac{Q^X(t)}{t} \sim \frac{\sum \int_{T_{i-1}}^{T_i} (X_i + au)du}{n} \frac{n}{t} \to \frac{\mathbb{E} \int_0^{T_1} (X^0 + au)du}{\mathbb{E} T_1}$$

$$= \mathbb{E} X^0 + a \frac{\mathbb{E} T_1^2}{2\mathbb{E} T_1} = \mathbb{E} X^0 + \frac{a}{(\lambda_1 + \lambda_2)}.$$

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Since X^0, $\widehat{Y}^0$ are stationary distributions, then

$$\mathbb{E}X^0 = \frac{(1-k)\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2}\left(\mathbb{E}X^0 + \frac{a}{\lambda_1 + \lambda_2}\right) \Rightarrow (\lambda_1 + \lambda_2)\mathbb{E}X^0 + a = a\frac{\lambda_1 + \lambda_2}{k\lambda_1}$$

and inequality holds if and only if the condition (5) holds.

Harris chain

Let $\{X_n\}$ be a time-homogeneous Markov chain taking values in the Polish space $\mathcal{X}$. For some fixed set $V \in \mathcal{B}_{\mathcal{X}}$, we define a random variable

$$\tau_V(x) = \min\{k \geq 1 : X_k(x) \in V\},$$

which is the time of the first hit from the state x to the set V (here $\tau_V(x) = \infty$ if $X_k(x) \notin V$ for all $k \geq 1$).

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Definition 1

The set V is called *recurrent* if $P_x(\tau_V < \infty) = 1$ for all $x \in V$.
It is called *positive recurrent* if $\sup_{x \in V} \mathbb{E}_x \tau_V < \infty$.

Harris chain

Definition 2

A Markov chain $\{X_n\}$ in $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is said to be a *Harris chain* (or *Harris irreducible*) if there exists a set $V \in \mathcal{B}_{\mathcal{X}}$, a probability measure μ on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and numbers $m \geq 1,\ p \in (0,1)$ such that

$$\text{(I)} \quad \mathbb{P}(\tau_V(x) < \infty) = 1\ \forall x \in \mathcal{X}; \quad \sup_{x \in V} \mathbb{E}\tau_V(x) < \infty;$$

$$\text{(II)} \quad \inf_{x \in V} \mathbb{P}(X_m \in B | X_0 = x) \geq p\mu(B)\ \forall B \in \mathcal{B}_{\mathcal{X}}.$$

Harris ergodicity

Let $\tau_V(\mu) = \{k \geq 1 : X_k(\mu) \in V\}$. Obviously, due to (I) $\tau_V(\mu)$ is proper random variable. Denote by $\mathcal{K}$ the set of possible values of $\tau_V(\mu)$, i.e., k_i values such that $\mathbb{P}(\tau_V(\mu) = k_i) > 0$. Let's introduce additionally *non-periodicity* condition for this Markov chain:
there exist $i \geq 1$ and $k_1, k_2, \ldots, k_i \in \mathcal{K}$ such that

$$\text{(III)} \quad \text{g.c.d.}\{m + k_1, m + k_2, \ldots, m + k_i\} = 1,$$

Definition 3

Let conditions (I), (II), (III) be satisfied. Then the Markov chain $\{X_n\}$ is called *Harris ergodic*.

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- (L4) g is eventually bounded by h: $\overline{\lim}_{L(x) \to \infty} g(x)/h(x) < \infty$.

Theorem 1 [2]

Theorem

Suppose that the drift of L in $g(x)$ steps satisfies the "drift condition"

$$\mathbb{E}_x \left[L\left(X_{g(x)}\right) - L\left(X_0\right) \right] \le -h(x),$$

where L, g, h satisfy $(\mathrm{L1}) - (\mathrm{L4})$. Let

$$\tau \equiv \tau^{(N)} = \inf\left\{ n > 0 : L\left(X_n\right) \le N \right\}$$

Then there exists $N_0 > 0$, such that for all $N > N_0$ and any $x \in \mathcal{X}$, we have $\mathbb{E}_x \tau < \infty$. Also, $\sup_{L(x) \le N} \mathbb{E}_x \tau < \infty$.

[2] Foss, S., Konstantopoulos, T.: *An overview of some stochastic stability methods.* Journal of Operations Research, Society of Japan. vol. 47, No. 4, pp. 275-303 (2004).

Theorem 29 [3]

Theorem

Let $\{X_n\}$ be a time-homogeneous Markov chain with values in the measurable space $\mathcal{X}$. Let $B \subseteq \mathcal{X}$ be a subset, which is positively recurrent for this chain. Let D be another non-empty set in the space $\mathcal{X}$ and $\tau_D = \min\{n : X_n \in D\}$. Let's pretend that $\sup_{x \in D} \mathbb{E}_x \tau_B < \infty$ and that there is a positive integer N such that $\inf_{x \in B} \mathbb{P}_x(\tau_D \leq N) > 0$. Then the set D is also positively recurrent.

[3] Foss, S., Chernova N.: *Stability of random processes*. Novosibirsk State University (2020).

Ergodicity conditions for random process

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— be some random sequence.

- It is natural to expect the ergodicity of the process X to follow from the ergodicity of the sequence $X_n = X(T_n)$ under fairly general assumptions.

Theorem 3, chapter 7 [4]

Theorem

Assume that the process X admits an embedded Markov chain X_n and the conditions below are fulfilled:
1) X_n satisfies Conditions (I)–(III);
2) $\sup_{x\in\mathcal{X}} \mathbb{E}(T_1|X(0)=x) < \infty$;
3) the distribution of random variable $\hat{\tau}$ has an absolutely continuous component, where

$$\mathbb{P}(\hat{\tau} > t) = \int \mu(dy)\mathbb{P}(T_{\tau_{V(y)}} > t|X(0) \in dy).$$

Then the distribution of the process X converges to the limit distribution in the total variation metric.

[4] Borovkov, A., Foss, S.: *Stochastically recursive sequences and their generalizations*. Siberian Advances in Mathematics, vol. 2, pp. 16-81 (1992).

Main remark

Remark 1

Instead of the Poisson process $W(t)$, we can consider a renewal process in which the distribution of the lengths of time intervals between the moments of jumps has a finite second moment and a continuous component, the distribution density of which is uniformly separated from zero in the neighborhood of the origin, and (regardless of everything else) each moment of the jump belongs either to the process $M(t)$ with probability $p \in (0,1)$, or to the process $N(t)$ with probability $1-p$. The (5) condition then takes the following form:

$$a\left(\frac{1-pk}{pk}+\frac{\mathbb{E}T_1^2}{2(\mathbb{E}T_1)^2}\right) < b\left(\frac{1-(1-p)l}{(1-p)l}+\frac{\mathbb{E}T_1^2}{2(\mathbb{E}T_1)^2}\right).$$

References

- Baccelli, F., Carofiglio, G., Foss, S.: *Proxy caching in split TCP: dynamics, stability and tail asymptotics*. From Semantics to Computer Science: Essays in Honour of Gilles Kahn. Cambridge University Press, pp. 437-464 (2009).
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- Foss, S., Chernova N.: *Stability of random processes*. Novosibirsk State University (2020).
- Borovkov, A., Foss, S.: *Stochastically recursive sequences and their generalizations*. Siberian Advances in Mathematics, vol. 2, pp. 16-81 (1992).
- Foss S., Chebunin M.: *Harris ergodicity of a Split Transmission Control Protocol*, Siberian Electronic Mathematical Reports, Vol. 18, No. 2, pp. 1493â1505 (2021).