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ON TRANSIENCE CONDITIONS FOR MARKOV CHAINS AND RANDOM WALKS

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Abstract: We prove a new transience criterion for Markov chains on an arbitrary state space and give a corollary for real-valued chains. We show by example that in the case of a homogeneous random walk with infinite mean the proposed sufficient conditions are close to those necessary. We give a new proof of the well-known criterion for finiteness of the supremum of a random walk.

Keywords: Markov chain, martingale, transience, uniform integrability, test function, random walk

1. Introduction

This paper is a continuation of [1]. Let $X = \{X_n\}_{n\ge 0}$, $X_0 = \text{const}$, be a Markov chain (MC) time-inhomogeneous in general and taking values in a measurable space $(\mathscr{X}, B)$, and let $L : \mathscr{X} \to [0,\infty)$ be a measurable unbounded function. We study conditions under which

$$L(X_n) \to \infty \quad \text{a.s. as } n \to \infty \tag{1.1}$$

for every initial state X_0. If (1.1) holds then we say that the chain is *L-transient* or, simply, *transient.*

In the case of countable MCs, conditions for transience are given in [2] under the additional assumption that the jumps are bounded. In [1], conditions for transience of an MC on an arbitrary state space were proposed in the case of possibly unbounded jumps.

We introduce some conventions and definitions. We assume that the MC $\{X_n\}$ can be represented as a *stochastic recursive sequence* (SRS)

$$X_{n+1} = f_n(X_n, \alpha_n), \quad n \ge 0,$$

where $\{\alpha_n\}$ is a sequence of independent random variables uniformly distributed in $[0,1]$ and $f_n : \mathscr{X} \times [0,1] \to \mathscr{X}$ are measurable functions (the assumption that an MC can be represented as SRS is not too restrictive, see e.g. [3]). Then, for $m = 0, 1, \dots$, we can define an MC $\{X^{(x,m)}_{m+n}\}_{n\ge 0}$ by the initial value $X^{(x,m)}_m = x$ and the recursion

$$X^{(x,m)}_{m+n+1} = f_{m+n}\big(X^{(x,m)}_{m+n}, \alpha_{m+n}\big) \quad \text{for} \quad n = 0, 1, \dots.$$

Further, let $\Delta_{x,m} = L\big(X^{(x,m)}_{m+1}\big) - L(x)$. Given a number $N > 0$, we define the random variables

$$\tau_{x,m}(N) = \min\big\{n \ge 1 : L\big(X^{(x,m)}_{m+n}\big) \ge N\big\}.$$

Write $a^+ = \max(a,0)$ and $a^- = -\min(a,0)$. Denote by $\mathscr{H}$ the class of measurable functions $h : [1,\infty) \to [1,\infty)$ such that the integral $\int_1^\infty (h(t))^{-1}\,dt$ converges and the function $g(t) = \frac{h(t)}{t}$ is nondecreasing and concave.

We recall the main result of [1].

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Theorem 1.1. *Suppose that there exist numbers $N > 0$, $\varepsilon > 0$, and $M > 0$ and a function $h \in \mathscr{H}$ such that*

(1) $\tau_{x,m}(N) < \infty$ *a.s. for all $x \in \mathscr{X}$ and $m \geq 0$;*

(2) *for all $m = 0, 1, 2, \dots$ and all $x \in \mathscr{X}$ such that $L(x) \geq N$, the following holds:*

$$\mathbf{E}\{\Delta_{x,m} \cdot I(\Delta_{x,m} \leq M)\} \geq \varepsilon;$$

(3) *the family of the random variables $\{h(\Delta^-_{x,m}); m \geq 0, L(x) \geq N\}$ is uniformly integrable.*
Then for every $x \in \mathscr{X}$ and every $m \geq 0$

$$\mathbf{P}\big(\lim_{n\to\infty} L\big(X^{(x,m)}_{m+n}\big) = \infty\big) = 1. \tag{1.2}$$

Note that condition (2) of Theorem 1.1 holds only if $\mathbf{E}\Delta^-_{x,m} < \infty$ for all x and m. In Section 2 of this paper, we prove an analogous assertion which is also applicable in the case $\mathbf{E}\Delta^-_{x,m} = \infty$ (Theorem 2.1). Then in Theorem 2.2 we give a more general transience criterion which contains the assertions of Theorems 1.1 and 2.1 as particular cases. Further we show in Remark 4 that condition (1) of Theorem 2.1 is essential.

In Section 3, we state corollaries to Theorem 2.1 for real-valued Markov chains (Theorem 3.1 and Corollary 3.1). Then we recall the well-known criteria for a homogeneous random walk to tend to infinity (Theorems 3.3 and 3.4) and show by example that the sufficient conditions in Theorem 2.1 are close to those necessary. In the Appendix, we give proofs of Theorem 3.1 and Corollary 3.1 as well as new proofs of Theorems 3.3 and 3.4.

2. Statements and Proofs of Transience Criteria

Theorem 2.1. *Suppose that there exist numbers $N > 0$, $\varepsilon > 0$, and $M > 0$ and a measurable function $h \in \mathscr{H}$ such that*

(1) *for all $x \in \mathscr{X}$ and $m \geq 0$,*

$$\limsup_{n\to\infty} L\big(X^{(x,m)}_{m+n}\big) = \infty \quad \text{a.s.};$$

(2) *for all $m = 0, 1, 2, \dots$ and all $x \in \mathscr{X}$ such that $L(x) \geq N$, the following inequality holds:*

$$\mathbf{E}\min\big(\Delta^+_{x,m}, M\big) \geq (1+\varepsilon)\mathbf{E}\min\big(\Delta^-_{x,m}, M\big);$$

(3) *$g(M) \geq 1 + \varepsilon$, and for all $m = 0, 1, 2, \dots$, all $x \in \mathscr{X}$ such that $L(x) \geq N$, and all $t \geq M$,*

$$\mathbf{P}(\Delta_{x,m} > t) \geq g(t)\mathbf{P}(\Delta_{x,m} < -t).$$

Then for every $x \in \mathscr{X}$ and every $m \geq 0$

$$\mathbf{P}\big(\lim_{n\to\infty} L\big(X^{(x,m)}_{m+n}\big) = \infty\big) = 1. \tag{2.1}$$

Remark 1. We assume that g is concave and nondecreasing (see above) to simplify the statement. This assumption is technical and may be relaxed in various ways. For example, the assumption of concavity of the function can be replaced by the assumption of its "slow variation."

Remark 2. For the class of so-called ψ-irreducible MCs, the following general transience criterion is known (see, e.g., [4, p. 174] for both the result and the definition of ψ-irreducibility). *Suppose that an MC X is ψ-irreducible. It is transient if and only if there exist a bounded nonnegative function V and a set C with $\psi(C) > 0$ such that*

$$\mathbf{E}(V(X_1) - V(x)) \geq 0$$

for all $x \notin C$ and $\psi(D) > 0$, where $D = \{x \ : \ V(x) > \sup_{y \in C} V(y)\}$.

Theorem 1.1 and Theorem 2.1 can be regarded as constructive analogs of this criterion: In these theorems, we construct the function V explicitly (namely, $Y_n = V\big(X_n^x\big)$ where Y_n is defined in (2.4)).

Remark 3. The following example shows that condition (1) in Theorem 2.1 is not superfluous; i.e., conditions (2) and (3) do not imply (1) in general.

Let$\{\xi_n\}_{n\ge 0}$ be independent identically distributed random variables. Define the distribution of ξ_1 as follows: for $t \ge e$

$$\mathbf{P}(\xi_1 > t) = \frac{1}{2\log t}, \quad \mathbf{P}(\xi_1 < -t) = \frac{1}{2\log^3 t}.$$

Define the MC $\{X(n)\}_{n\ge 0}$ on the real axis by $X(0) = x_0$ and if $X(n) = x \equiv l + y$, where l is an integer and $0 \le y < 1$. Then $X(n+1) = l + \frac{1+y}{2}$ with probability $\frac{3}{4}p_y$, $X(n+1) = l + (1 - 2(1-y))^+$ with probability $\frac{1}{4}p_y$, and $X(n+1) = x + \xi_n$ with probability $q_y \equiv 1 - p_y$. Assume that $1 > q_y > 0$ decreases in $y \in [0, 1)$ and

$$\sum_{k=0}^{\infty} q_{1-2^{-k}} < \infty.$$

It is not difficult to verify that this MC satisfies conditions (2) and (3) of Theorem 2.1 for $L(x) = x^+$, $M = 10$, and $N = 0$. We show that $\sup X(n) < \infty$ a.s. To this end, it suffices to show that, with probability 1, only finitely many events $\{|X(n+1) - X(n)| > 1\}$ occur. In turn, this happens if there exists $\alpha > 0$ such that, for every initial value $X(0) = x_0 \in [0, 1)$, we have $\mathbf{P}(X(n) \in [0, 1) \text{ for all } n) \ge \alpha$. In view of monotonicity of q_x, it is sufficient to show that this probability is positive when $x_0 = 0$.

We have

$$\mathbf{P}(X(n+1) \in [0, 1) \mid X(n) = x \in [0, 1)) = p_x.$$

Introduce the auxiliary Markov chain: $\widetilde{X}(0) = 0$ and if $\widetilde{X}(n) = 1 - 2^{-m}$ then $\widetilde{X}(n+1) = 1 - 2^{-m-1}$ with probability $\frac{3}{4}$ and $\widetilde{X}(n+1) = (1 - 2^{-m+1})^+$ with probability $\frac{1}{4}$. Then we may take

$$\alpha = \mathbf{E}\Big(\prod_{n\ge 0} p_{\widetilde{X}(n)}\Big). \tag{2.2}$$

Let the function L be such that $L(1 - 2^{-m}) = m$ for all nonnegative integers m. Then $Y_n = L(\widetilde{X}(n))$, $n \ge 0$, is a random walk with reflection at zero: $Y(n+1) = Y(n) + 1$ with probability $\frac{3}{4}$ and $Y(n+1) = (Y(n) - 1)^+$ with probability $\frac{1}{4}$. Therefore,

$$\frac{Y(n)}{n} \to \frac{1}{2} \quad \text{a.s.}$$

and $\mathbf{E}\eta(k) \to 2$, where

$$\eta(k) = \sum_{n=0}^{\infty} \mathbf{I}(Y(n) = k) = \sum_{n=0}^{\infty} \mathbf{I}(\widetilde{X}(n) = 1 - 2^{-k}).$$

Then

$$\alpha = \mathbf{E} \prod_{k=0}^{\infty} p_{1-2^{-k}}^{\eta(k)} = \mathbf{E}\Big(\exp\Big(\sum_k \eta(k) \log p_{1-2^{-k}}\Big)\Big)$$

and α is positive if

$$\sum_k \eta(k) q_{1-2^{-k}} < \infty \qquad \text{a.s.}$$

The latter is implied by convergence of the series

$$\sum_k \mathbf{E}\eta(k) q_{1-2^{-k}}.$$

Remark 4. Verification of condition (1) of Theorem 2.1 is difficult. Therefore, it is useful to give sufficient conditions for (1) to hold.

Lemma 2.1. *Suppose that there exist numbers $N > 0$, $\delta > 0$, and $d > 0$ such that*
(1) $\tau_{x,m}(N) < \infty$ *a.s. for all* $x \in \mathscr{X}$ *and* $m \geq 0$*;*
(2) *for all* $m = 0, 1, 2, \dots$ *and all* $x \in \mathscr{X}$ *such that* $L(x) \geq N$

$$\mathbf{P}\left(\Delta_{x,m} \geq d\right) \geq \delta.$$

Then for every nonnegative integer m *and every* $x \in \mathscr{X}$

$$\mathbf{P}\bigl(\limsup_{n\to\infty} L\bigl(X_{m+n}^{(x,m)}\bigr) = \infty\bigr) = 1. \tag{2.3}$$

We omit the proof of the lemma since it repeats almost word for word that of Lemma 2.1 in [1].

Proof of Theorem 2.1. The first part of the proof is almost the same as that of Theorem 1.1 (see [1]). The proof is the same for all $m \geq 0$; so we restrict exposition to the case $m = 0$.

Pick an arbitrary $x \in \mathscr{X}$ and let $C > 0$. Define

$$Y_n = \int\limits_{1+\frac{(L(X_n^{(x)})-N)^+}{C}}^{\infty} \frac{dt}{h(t)}. \tag{2.4}$$

It is sufficient to prove that for a suitable choice of $C > 0$

$$\text{the sequence } \{Y_n\}_{n\geq 0} \text{ forms a positive supermartingale.} \tag{2.5}$$

Indeed, if this is the case then by the familiar theorem the sequence $\{Y_n\}$ converges a.s. and, by condition (1) of the theorem, the latter is equivalent to the convergence $L\bigl(X_n^{(x)}\bigr) \to \infty$ a.s.

We thus prove (2.5). Since we consider a Markov chain, it is enough to show that the inequality

$$\mathbf{E}\bigl\{Y_{n+1} - Y_n \mid X_n^{(x)}\bigr\} \leq 0 \quad \text{a.s.} \tag{2.6}$$

is valid for all n.

The proof of (2.6) is carried out for all n in a similar way. Therefore, to simplify notation, we confine exposition to the case $n = 0$.

The inequality $\mathbf{E}\{Y_1 - Y_0\} \leq 0$ is obvious when x is such that $L(x) \leq N$. Therefore, we further consider only the case $z \equiv L(x) - N = \text{const} > 0$. Let

$$A(x) = Y_1 - Y_0 = \int\limits_{1+(z+\Delta_x)^+/C}^{1+z/C} \frac{dt}{h(t)},$$

where $\Delta_x \equiv \Delta_{x,0}$. Then

$$\mathbf{E}(A(x)) = \int\limits_{-\infty}^{+\infty} \int\limits_{1+(z+u)^+/C}^{1+z/C} \frac{dt}{h(t)} \mathbf{P}(\Delta_x \in du).$$

Integrating this expression by parts we obtain

$$E \equiv C\mathbf{E}(A(x)) = -\int\limits_0^{\infty} \frac{\mathbf{P}(\Delta_x > u)}{h(1 + \frac{z+u}{C})}\, du + \int\limits_0^{z} \frac{\mathbf{P}(\Delta_x < -u)}{h(1 + \frac{z-u}{C})}\, du.$$

It is sufficient to show that $E \leq 0$.

We introduce the convention $0/0 = 1/0 = \infty$ and rewrite condition (2) in the more convenient form:

$$\frac{\mathbf{E}\min(\Delta_x^+, M)}{\mathbf{E}\min(\Delta_x^-, M)} = \frac{\int_0^M \mathbf{P}(\Delta_x > u)\,du}{\int_0^M \mathbf{P}(\Delta_x < -u)\,du} \geq 1 + \varepsilon. \tag{2.7}$$

In the proof we need some positive constants r, R and the constant C to satisfy several constraints. Let

$$T(\alpha) = \sup_{t \geq 0} \frac{g(1 + \alpha t)}{g(1 + t)}, \quad \alpha > 1$$

(the concavity of g implies finiteness of the number $T(\alpha)$ for all $\alpha > 1$ and the convergence $T(\alpha) \to 1$ as $\alpha \to 1$).

Choose $R \in (0, 1)$ so small that

$$\frac{h(1 + 2R)}{h(1)} \leq 1 + \varepsilon, \tag{2.8}$$

and choose $r \in (0, 1)$ so small that for $\alpha = \frac{1+r}{1-r}$

$$(\alpha T(\alpha) - 1) \leq \varepsilon. \tag{2.9}$$

Note that since $h \in \mathscr{H}$, we have

$$1 \leq \frac{h(1 + \alpha t)}{h(1 + t)} = \frac{1 + \alpha t}{1 + t} \frac{g(1 + \alpha t)}{g(1 + t)} \leq \alpha T(\alpha)$$

for all $\alpha > 1$ and $t > 0$; hence (2.9) implies

$$\frac{h(1 + (1 + r)u)}{h(1 + (1 - r)u)} \leq 1 + \varepsilon \tag{2.10}$$

for every $u > 0$ (put $t = (1 - r)u$).

Now, pick C so large that the following inequalities hold:

$$g(rRC/2) \geq 2(1 + \varepsilon) \tag{2.11}$$

and

$$C > \frac{\max(M, 1 + R(1 + r), 16K)}{Rr} \tag{2.12}$$

where $K = \int_1^\infty (h(t))^{-1}\,dt$.

We now consider the two cases: (a) $0 < z \leq RC$ and (b) $z > RC$.

In the first case, estimate E from above as

$$E \leq -\int_0^{RC} \frac{\mathbf{P}(\Delta_x > u)}{h(1 + \frac{z+u}{C})}\,du + \int_0^{RC} \frac{\mathbf{P}(\Delta_x < -u)}{h(1 + \frac{z-u}{C})}\,du \equiv E_1 + E_2 + E_3 + E_4,$$

where

$$E_1 = -\int_{M+0}^{RC} \frac{\mathbf{P}(\Delta_x > u)}{h(1 + \frac{z+u}{C})}\,du, \quad E_2 = -\int_0^{M} \frac{\mathbf{P}(\Delta_x > u)}{h(1 + \frac{z+u}{C})}\,du,$$

$$E_3 = \int_0^{M} \frac{\mathbf{P}(\Delta_x < -u)}{h(1 + \frac{z-u}{C})}\,du, \quad E_4 = \int_{M+0}^{RC} \frac{\mathbf{P}(\Delta_x < -u)}{h(1 + \frac{z-u}{C})}\,du.$$

In view of monotonicity of h, these quantities admit the following upper bounds:

$$E_1 \le -\frac{1}{h(1+2R)} \int\limits_{M+0}^{RC} \mathbf{P}(\Delta_x > u)\, du, \quad E_2 \le -\frac{1}{h(1+2R)} \int\limits_{0}^{M} \mathbf{P}(\Delta_x > u)\, du,$$

$$E_3 \le \frac{1}{h(1)} \int\limits_{0}^{M} \mathbf{P}(\Delta_x < -u)\, du, \quad E_4 \le \frac{1}{h(1)} \int\limits_{M+0}^{RC} \mathbf{P}(\Delta_x < -u)\, du.$$

By inequality (2.8) and condition (3),

$$E_1 + E_4 \le \int\limits_{M+0}^{RC} \left(-\frac{\mathbf{P}(\Delta_x > u)}{(1+\varepsilon)h(1)} + \frac{\mathbf{P}(\Delta_x > u)}{g(u)h(1)} \right) du \le 0.$$

By inequalities (2.8) and (2.7),

$$E_2 + E_3 \le \int\limits_{0}^{M} \left(-\frac{\mathbf{P}(\Delta_x > u)}{(1+\varepsilon)h(1)} + \frac{\mathbf{P}(\Delta_x < -u)}{h(1)} \right) du \le 0.$$

Hence, we obtain $E \le 0$, as required.

In the second case, we have

$$E \le \left(\int\limits_{0}^{rz} + \int\limits_{rz+0}^{z} \right) \left(-\frac{\mathbf{P}(\Delta_x > u)}{h(1+\frac{z+u}{C})} + \frac{\mathbf{P}(\Delta_x < -u)}{h(1+\frac{z-u}{C})} \right) du \equiv J_1 + J_2.$$

By the monotonicity of h, the first summand may be estimated as follows:

$$J_1 \le \int\limits_{0}^{rz} \left(-\frac{\mathbf{P}(\Delta_x > u)}{h(1+\frac{z+rz}{C})} + \frac{\mathbf{P}(\Delta_x < -u)}{h(1+\frac{z-rz}{C})} \right) du.$$

Inequality (2.10) implies that

$$J_1 \le \frac{1}{h(1+\frac{z+rz}{C})} \int\limits_{0}^{rz} \left(-\mathbf{P}(\Delta_x > u) + (1+\varepsilon)\mathbf{P}(\Delta_x < -u) \right) du.$$

Represent the last integral as a sum of three integrals and estimate them. By (2.7),

$$\int\limits_{0}^{M} \left(-\mathbf{P}(\Delta_x > u) + (1+\varepsilon)\mathbf{P}(\Delta_x < -u) \right) du \le 0.$$

From condition (3) of the theorem, we have

$$\int\limits_{M+0}^{rz/2} \left(-\mathbf{P}(\Delta_x > u) + (1+\varepsilon)\mathbf{P}(\Delta_x < -u) \right) du \le 0.$$

In view of the monotonicity of g and (2.11), for $z > RC$

$$g(rz/2) \ge g(rRC/2) \ge 2(1+\varepsilon),$$

and thus

$$\int\limits_{rz/2+0}^{rz} (-\mathbf{P}(\Delta_x > u)/2 + (1+\varepsilon)\mathbf{P}(\Delta_x < -u))\, du \le 0.$$

As a result, we obtain the upper bound

$$J_1 \le -\frac{1}{2h(1+\frac{z+rz}{C})} \int\limits_{rz/2}^{rz} \mathbf{P}(\Delta_x < -u)\, du \le -\frac{rz\mathbf{P}(\Delta_x < -rz)}{4h(1+\frac{z+rz}{C})}. \tag{2.13}$$

For J_2, we have the following inequality:

$$J_2 \le K\mathbf{P}(\Delta_x > rz). \tag{2.14}$$

It follows from (2.12) that $g(rz) \ge g(1+z(1+r)/C)$. Indeed, the function g is nondecreasing and, since $z > RC$,

$$rz - (1+z(1+r)/C) = z\frac{rC-(1+r)}{C} - 1 > RrC - R(1+r) - 1 > 0.$$

It follows from (2.13) and (2.14) that

$$E \le -\frac{rz\mathbf{P}(\Delta_x < -rz)}{4(1+\frac{z+rz}{C})g(rz)} + K\mathbf{P}(\Delta_x > rz).$$

Furthermore, since $R < 1$, $r < 1$, and $z > RC$, we have the inequality $rz/4(1+(z+rz)/C) > K$. Indeed,

$$rz - 4K(1+z(1+r)/C) > (r - 8K/C)z - 4K > rRC/2 - 4K > 0$$

by (2.12). Hence,

$$E \le -K\frac{\mathbf{P}(\Delta_x < -rz)}{g(rz)} + K\mathbf{P}(\Delta_x > rz) \le 0$$

by condition (3). Theorem 2.1 is thus proved.

We now combine Theorems 1.1 and 2.1. Denote the distribution of the r.v. $\Delta_{x,m}$ by $\mu_{x,m}(\cdot) = \mathbf{P}(\Delta_{x,m} \in \cdot)$.

Theorem 2.2. *Suppose that condition* (1) *of Theorem* 2.1 *holds. Suppose further that for all $x \in \mathscr{X}$ and all $m = 0, 1, 2, \ldots$ there exists a representation*

$$\mu_{x,m} = c^{(1)}_{x,m}\mu^{(1)}_{x,m} + c^{(2)}_{x,m}\mu^{(2)}_{x,m},$$

where $\mu^{(1)}_{x,m}$ and $\mu^{(2)}_{x,m}$ are probability measures, $c^{(1)}_{x,m}, c^{(2)}_{x,m} \ge 0$, $c^{(1)}_{x,m} + c^{(2)}_{x,m} = 1$, and that there exist numbers $N > 0$, $\varepsilon > 0$, and $M > 0$ and a function $h \in \mathscr{H}$ such that

(a) *the random variables $\Delta^{(1)}_{x,m}$ with distribution $\mu^{(1)}_{x,m}$ satisfy conditions* (2) *and* (3) *of Theorem* 1.1;

(b) *the random variables $\Delta^{(2)}_{x,m}$ with distribution $\mu^{(2)}_{x,m}$ satisfy conditions* (2) *and* (3) *of Theorem* 2.1.

Then for every $x \in \mathscr{X}$ and every $m \ge 0$

$$\mathbf{P}\big(\lim_{n\to\infty} L\big(X^{(x,m)}_{m+n}\big) = \infty\big) = 1. \tag{2.15}$$

Proof. As in the proof of Theorem 2.1, it is sufficient to show that the inequality $\mathbf{E}\,(Y_1 - Y_0) \le 0$ holds, where

$$Y_0 = \int\limits_{1+\frac{(L(x)-N)^+}{C}}^{\infty} \frac{dt}{h(t)}, \quad Y_1 = \int\limits_{1+\frac{(L(X^{(x)}_1)-N)^+}{C}}^{\infty} \frac{dt}{h(t)}.$$

Let

$$A(x,u) = \int\limits_{1+\frac{(L(u)-N)^+}{C}}^{1+\frac{(L(x)-N)^+}{C}} \frac{dt}{h(t)}.$$

Then

$$\mathbf{E}\{Y_1 - Y_0\} = \int\limits_{-\infty}^{+\infty} A(x,u)\mu(du) = c^{(1)} \int\limits_{-\infty}^{+\infty} A(x,u)\mu^{(1)}(du) + c^{(1)} \int\limits_{-\infty}^{+\infty} A(x,u)\mu^{(2)}(du),$$

where the first integral is nonnegative by Theorem 1.1 and the second is nonnegative by Theorem 2.1. Theorem 2.2 is thus proved.

3. Markov Chains on the Real Axis (Random Walks)

Corollaries to Theorem 2.1. We now assume that the MC X is time-homogeneous and real-valued (i.e., it is a random walk). Following the traditional notation, we write S_n instead of X_n. We define the random walk $S_n^{(x)}$ recursively by

$$S_0^{(x)} = x, \quad S_{n+1}^{(x)} = S_n^{(x)} + \xi_{n,S_n},$$

where $\{\xi_{n,y}\}$ is a family of independent random variables and the distribution of $\xi_{n,y}$ does not depend on n. The random walk $S_n^{(x)}$ is *homogeneous* if $\{\xi_{n,y}\}$ is a family of independent identically distributed random variables. In this case, the index y may be omitted.

For the test function $L(x) = x^+$, Theorem 2.1 implies the following

Theorem 3.1. *Suppose that there exist numbers $N > 0$ and $M > 0$ and a measurable function $h \in \mathscr{H}$ such that*

(1) *for all x, with probability* 1,

$$\limsup_{n\to\infty} S_n^{(x)} = \infty; \tag{3.1}$$

(2) *as $y \to \infty$*

$$\inf_{x\geq N} \mathbf{E}\min\bigl(\xi_x^+, y\bigr) \to \infty; \tag{3.2}$$

(3) *for all $x \geq N$ and $t \geq M$, the following inequalities hold:*

$$\mathbf{P}(\xi_x > t) \geq \frac{h(t)}{t}\mathbf{P}(\xi_x < -t).$$

Then for all x

$$\mathbf{P}\Bigl(\lim_{n\to\infty} S_n^{(x)} = \infty\Bigr) = 1. \tag{3.3}$$

In the case of a homogeneous random walk Theorem 3.1 implies the following

Corollary 3.1. *Let $\mathbf{E}\xi_1^+ = \mathbf{E}\xi_1^- = \infty$, and let there exist a number $N > 1$ and a function $h \in \mathscr{H}$ such that for every $t > N$*

$$\frac{\mathbf{P}(\xi_1 > t)}{\mathbf{P}(\xi_1 < -t)} \geq \frac{h(t)}{t}. \tag{3.4}$$

Then

$$\mathbf{P}(\lim_{n\to\infty} S_n = \infty) = 1. \tag{3.5}$$

We give proofs of these results in the Appendix.

Classification of homogeneous random walks with infinite mean. The first edition of Feller's textbook contained the following assertion: if $\mathbf{E}\xi_1^+ = \mathbf{E}\xi_1^- = \infty$ then $\limsup S_n = \infty$ and $\liminf S_n = -\infty$ a.s. (see [5, II, Chapter XII.2, Theorem 2]). However, Rogozin [6] showed (by an example with stable distributions) that this assertion is incorrect.

Let us recall the well-known classification of homogeneous random walks (see [7, Chapter XII.2, Theorem 1]). Let $\tau_+ = \inf\{n > 0 : S_n \geq 0\}$ and $\tau_- = \inf\{n \geq 0 : S_n < 0\}$.

Theorem 3.2. *There are exactly three possibilities:*
(a) $\lim S_n = \infty$ *a.s.; in this case*

$$\mathbf{E}\tau_+ < \infty \quad \textit{and} \quad \mathbf{P}(\tau_- = \infty) > 0;$$

(b) $\lim S_n = -\infty$ *a.s.; in this case*

$$\mathbf{E}\tau_- < \infty \quad \textit{and} \quad \mathbf{P}(\tau_+ = \infty) > 0;$$

(c) $\limsup S_n = \infty$ *and* $\liminf S_n = -\infty$ *a.s.; in this case*

$$\mathbf{E}\tau_+ = \mathbf{E}\tau_- = \infty \quad \textit{and} \quad \mathbf{P}(\tau_+ < \infty) = \mathbf{P}(\tau_- < \infty) = 1.$$

In the particular case when the mean does not exist, Kesten (see [8, Corollary 3, p. 1195]) obtained the following

Theorem 3.3. *If* $\mathbf{E}|\xi_1| = \infty$ *then one of the following three cases occurs:*
(a) $\mathbf{P}(\lim S_n/n = +\infty) = 1$;
(b) $\mathbf{P}(\lim S_n/n = -\infty) = 1$;
(c) $\mathbf{P}(\limsup S_n/n = +\infty) = \mathbf{P}(\liminf S_n/n = -\infty) = 1$.

Further, Erickson (see [9]) gave simple conditions which are equivalent to each of the above three cases. Put

$$J_+ = \int_0^\infty \frac{x}{\mathbf{E}\min(\xi_1^-, x)} \mathbf{P}(\xi_1 \in dx), \quad J_- = \int_0^\infty \frac{x}{\mathbf{E}\min(\xi_1^+, x)} \mathbf{P}(\xi_1 \in -dx).$$

Theorem 3.4 [9]. *If* $\mathbf{E}|\xi_1| = \infty$ *then*
(a) $\mathbf{P}(\lim S_n/n = +\infty) = 1 \iff J_- < \infty$;
(b) $\mathbf{P}(\lim S_n/n = -\infty) = 1 \iff J_+ < \infty$;
(c) $\mathbf{P}(\limsup S_n/n = +\infty) = \mathbf{P}(\liminf S_n/n = -\infty) = 1 \iff J_+ = J_- = \infty$.

In the Appendix we give new and, in our opinion, shorter and more direct proofs of Theorems 3.3 and 3.4.

For distributions with regularly varying tails, the following assertion was established in [10, Theorem 2.3, Section 1] (as a corollary to another result).

Proposition 3.1. *Suppose that the following conditions hold:*

$$\mathbf{P}(\xi_1 > t) \geq W(t) \equiv t^{-\alpha} L_W(t), \quad \mathbf{P}(\xi_1 < -t) \leq V(t) \equiv t^{-\beta} L_V(t), \tag{3.6}$$

where $L_V(t)$ *and* $L_W(t)$ *are slowly varying functions,* $\alpha < 1$, *and* $V(t) = o(W(t))$. *Then the convergence of the series*

$$\sum_n V(W^{(-1)}(1/n)) < \infty \tag{3.7}$$

implies the finiteness of $\inf_{k \geq 0} S_k$ *and the convergence* $S_n \to \infty$ *a.s.*

It is not difficult to see that Theorem 3.4 implies Proposition 3.1.

We now show by example that the sufficient conditions of Theorem 2.1 and Corollary 3.1 are close to those necessary.

Example 1. Consider a random walk $S_n = \xi_1 + \cdots + \xi_n$ where for $t \ge e$ and $0 \le \alpha < 1$

$$\mathbf{P}(\xi_1 > t) = \frac{C}{t^\alpha \log^\beta t}, \quad \mathbf{P}(\xi_1 < -t) = \frac{C}{t^\alpha \log^\gamma t}.$$

We have (see [7, Chapter VIII.9, Theorem 1]) the following property of regularly varying functions:

$$\frac{t\mathbf{P}(\xi_1 > t)}{\mathbf{E}\min(\xi_1^+, t)} \to 1 - \alpha \quad \text{as } t \to \infty.$$

Since $\alpha < 1$, the condition $J_- < \infty$ can be reformulated as

$$\int_0^\infty \frac{1}{\mathbf{P}(\xi_1 > t)} \mathbf{P}(\xi_1 \in -dt) < \infty.$$

The latter is equivalent to convergence of the integral

$$\int_e^\infty \left(\frac{\alpha}{t \log^{\gamma-\beta} t} + \frac{\gamma}{t \log^{1+\gamma-\beta} t} \right) dt. \tag{3.8}$$

Thus, if $0 < \alpha < 1$ then the random walk is transient if and only if $\beta + 1 < \gamma$. Corollary 3.1 gives the same condition $\beta + 1 < \gamma$ as sufficient. In the case $\alpha = 0$, it follows from (3.8) that the random walk is transient if and only if $\beta < \gamma$. In this case Corollary 3.1 gives top-heavy sufficient conditions. However, if we apply Theorem 2.1 directly by taking $L(t) = \log(1 + t^+)$ as a test function, we obtain the condition $\beta < \gamma$ as sufficient.

4. Appendix

Proof of Theorem 3.1. We check the conditions of Theorem 2.1. Condition (1) follows from (3.1). Consider the test function $L(t) = t^+$. Then $\Delta_{x,m} \equiv \Delta_x = (S_1^x)^+ - x^+$ for $x \in \mathbb{R}$.

Since we can always increase M, we may assume that $g(M) \ge 2$.

Recall the convention $(0/0) = (1/0) = \infty$. For all $t > M$ and $x \ge N$, we have

$$\frac{\mathbf{P}(\Delta_x > t)}{\mathbf{P}(\Delta_x < -t)} = \frac{\mathbf{P}(\xi_x > t)}{\mathbf{P}(\xi_x < -t)\mathbf{I}(x > t)} \ge g(t) \ge 2.$$

This inequality guarantees that (3) holds.

If $x \ge N$ and $y > M$ then

$$\mathbf{E}\min\bigl(\Delta_x^+, y\bigr) \ge g(M) \int_M^y \mathbf{P}(\Delta_x^- > u)\, du \ge g(M)(\mathbf{E}\min(\Delta_x^-, y) - M) \equiv g(M) f(x,y).$$

Thus,

$$\frac{\mathbf{E}\min(\Delta_x^+, y)}{\mathbf{E}\min(\Delta_x^-, y)} \ge g(M) \frac{f(x,y)}{f(x,y) + M} \ge 2 \frac{\inf\limits_{x>N} f(x,y)}{\inf\limits_{x>N} f(x,y) + M} \to 2 \quad \text{as } y \to \infty.$$

Hence, condition (2) holds for M sufficiently large. Theorem 3.1 is proved.

Corollary 3.1 follows from Theorem 3.1 on observing that, in the homogeneous case, condition (3.2) follows from $\mathbf{E}\xi_1^+ = \infty$ while the fulfillment of condition (3.1) is guaranteed by the following

Lemma 4.1. *Let* $\mathbf{E}|\xi_1| = \infty$. *Suppose that there exists a number* $M > 0$ *such that for every* $t \geq M$

$$\frac{\mathbf{P}(\xi_1 > t)}{\mathbf{P}(\xi_1 < -t)} \geq 1.$$

Then

$$\overline{\lim} \frac{S_n}{n} = \infty \quad \text{a.s.} \tag{4.1}$$

PROOF. By monotonicity, it is sufficient to consider only the case when $\mathbf{P}(\xi_1 > t) = \mathbf{P}(\xi_1 < -t)$ for all $t \geq M$. Let $\xi_i^{(1)} = \xi_i \mathbf{I}(|\xi_i| \leq M)$, $\xi_i^{(2)} = \xi_i \mathbf{I}(|\xi_i| > M)$, and put $S_n^{(1)} = \xi_1^{(1)} + \cdots + \xi_n^{(1)}$, $S_n^{(2)} = \xi_1^{(2)} + \cdots + \xi_n^{(2)}$. Note that $\liminf \frac{S_n^{(1)}}{n} = \lim \frac{S_n^{(1)}}{n} \geq -M$ a.s. It is well known that, for a symmetric random walk with infinite mean, $\limsup \frac{S_n^{(2)}}{n} = -\liminf \frac{S_n^{(2)}}{n} = \infty$ a.s. Thus, $\limsup \frac{S_n}{n} = \infty$. Lemma 4.1 is proved.

Proofs of Theorems 3.3 and 3.4 require the following

Lemma 4.2 [9]. *Let* $\zeta_1, \zeta_2, \ldots$ *be independent identically distributed random variables such that* $\mathbf{P}(\zeta_1 \geq 0) = 1$ *and* $\mathbf{P}(\zeta_1 > 0) > 0$. *Let*

$$m(t) = \mathbf{E}\min(\zeta_1, t), \quad U(t) = \sum_{n=0}^{\infty} \mathbf{P}(\zeta_1 + \cdots + \zeta_n < t).$$

Then

$$\frac{t}{m(t)} \leq U(t) \leq \frac{2t}{m(t)}.$$

For completeness, we give

PROOF OF LEMMA 4.2. Fix $t > 0$ and let

$$\tilde{\zeta}_i(t) = \min(\zeta_i, t), \quad \tilde{\eta}(t) = \min\left(n : \sum_{i=1}^{n} \tilde{\zeta}_i \geq t\right)$$

and $\widetilde{U}(t) = \sum \mathbf{P}(\tilde{\zeta}_1 + \cdots + \tilde{\zeta}_n < t)$. Then $\widetilde{U}(t) = \mathbf{E}\tilde{\eta}(t)$ and $\widetilde{U}(t) = U(t)$. Note that $t \leq \sum_{i=1}^{\tilde{\eta}(t)} \tilde{\zeta}_i(t) \leq 2t$ a.s. and, by Wald's identity,

$$\mathbf{E}\left(\sum_{i=1}^{\tilde{\eta}(t)} \tilde{\zeta}_i\right) = \mathbf{E}\tilde{\eta}(t) m(t).$$

Corollary 4.1. *Let* $U^1(x)$ *be the renewal function in the case where the* ζ_n*'s are distributed as* ξ_1^+ *and let* $U^2(x)$ *be the renewal function in the case where the* ζ_n*'s are distributed as* $\mathbf{P}(\zeta_n \in \cdot) = \mathbf{P}(\xi_1 \in \cdot \mid \xi_1 \geq 0)$. *For* $r = 1, 2$, *let*

$$J_-^r = \int_0^{\infty} U^r(x) \mathbf{P}(\xi_1 \in -dx).$$

Then all integrals J_-, J_-^1, *and* J_-^2 *are simultaneously finite or infinite.*

We proceed with the proofs of Theorems 3.3 and 3.4. Clearly, it suffices to consider only the case where $\mathbf{P}(\xi_1 > 0) > 0$ and $\mathbf{P}(\xi_1 < 0) > 0$. We only prove assertions (a) and (b) of Theorem 3.4; then all other assertions of Theorems 3.3 and 3.4 follow directly. The proof is divided into four steps.

STEP 1. If $\lim S_n = \infty$ a.s. then $J_- < \infty$.

STEP 2. If $J_- < \infty$ then $\lim S_n = \infty$ a.s.

STEP 3. If $\lim S_n = \infty$ then $\lim \frac{S_n}{n} = \infty$ a.s.

STEP 4. If $\limsup S_n = \infty = -\liminf S_n$ then $\limsup \frac{S_n}{n} = \infty = -\liminf \frac{S_n}{n}$ (and $J_+ = J_- = \infty$ of necessity).

STEP 1. If $\lim S_n = \infty$ then

$$\mathbf{E}(\tau_+ \mathbf{I}\{\xi_1 < 0\}) \le \mathbf{E}\tau_+ < \infty.$$

In this case

$$\mathbf{E}(\tau_+ \mathbf{I}\{\xi_1 < 0\}) = \int_0^\infty (1 + U_\xi(x)) \mathbf{P}(\xi_1 \in -dx),$$

where $U_\xi(x) = \mathbf{E}\eta_\xi(x)$ and

$$\eta_\xi(x) = \min\{n : S_n \ge x\} \ge \min\left\{n : \sum_1^n \xi_i^+ \ge x\right\}.$$

Therefore, $U_\xi(x) \ge U^{(1)}(x)$ and $J_-^1 < \infty$. Hence, the integral J_- is finite as well.

STEP 2. Let $j_0 = 0$ and, for $n \ge 0$,

$$j_{n+1} = \min\{i > j_n : \xi_i < 0\}.$$

Let also $\nu_0 = 0$, $\nu_{n+1} = j_{n+1} - j_n$, $\psi_{n+1} = -\xi_{j_{n+1}}$,

$$\varphi_{n+1} = S_{j_{n+1}-1} - S_{j_n} = \sum_{i=j_n+1}^{j_{n+1}-1} \xi_i$$

and let $\varphi_{n+1} = 0$ if $j_{n+1} = j_n + 1$.

Note that

(1) the r.v.'s $\{\psi_n\}$ are independent and distributed as

$$\mathbf{P}(\psi_n \in \cdot) = \mathbf{P}(-\xi_1 \in \cdot \mid \xi_1 < 0);$$

(2) the r.v.'s $\{\nu_n\}$ are independent and identically distributed, do not depend on $\{\psi_n\}$ and are geometrically distributed with the parameter $p = \mathbf{P}(\xi_1 < 0) \in (0,1)$. In particular, $\mathbf{E}\nu_1 = 1/p > 1$;

(3) the r.v.'s $\{\varphi_n\}$ are independent and may be represented as

$$\varphi_n = \sum_{i=\nu_{n-1}-n+1}^{\nu_n - n} \zeta_i, \tag{4.2}$$

where $\{\zeta_i\}_{i=1}^\infty$ are mutually independent, do not depend on ν_n and $\{\psi_n\}$, and are distributed as $\mathbf{P}(\zeta_n \in \cdot) = \mathbf{P}(\xi_1 \in \cdot \mid \xi_1 \ge 0)$.

We have to show that $\mathbf{P}(\inf_{n\ge1} S_n = -\infty) = 0$. Since $S_{j_n} = \sum_{i=1}^n (\varphi_i - \psi_i)$; therefore, $\{\inf S_n = -\infty\} \subseteq A$, where

$$A = \{S_{j_n} \le 0 \ \text{ infinitely often}\} = \left\{ \sum_1^n \psi_i \ge \sum_1^n \varphi_i \text{ infinitely often}\right\}.$$

Now $\mathbf{P}(A) \le \mathbf{P}(B)$, where $B = \{\eta\big(\sum_1^n \psi_i\big) \ge \eta\big(\sum_1^n \varphi_i\big) \text{ infinitely often}\}$ and $\eta(t) = \min\{n \ge 1 : \zeta_1 + \cdots + \zeta_n > t\}$. By (4.2)

$$\eta\left(\sum_{i=1}^n \varphi_i\right) \equiv \sum_{i=1}^n (\nu_i - 1) + 1$$

and, consequently,

$$\mathbf{P}(B) \le \mathbf{P}\left\{\eta\left(\sum_1^n \psi_i\right) \ge \sum_1^n (\nu_i - 1) \ \text{ infinitely often}\right\}.$$

By the strong law of large numbers, $\frac{\sum_{i=1}^n \nu_i}{n} \to \frac{1}{p} > 1$ a.s. Thus, if we show that

$$\frac{\eta\big(\sum_{i=1}^n \psi_i\big)}{n} \to 0 \quad \text{a.s.,} \tag{4.3}$$

then $\mathbf{P}(A) = \mathbf{P}(B) = 0$ will follow.

Define

$$X_{[1,n]} = \eta\left(\sum_{i=1}^n \psi_i\right)$$

and, for $j \ge i > 1$,

$$X_{[i,j]} = \min\left\{n : \sum_{l=1}^n \zeta_{X_{[1,i-1]}+l} > \sum_{k=i}^j \psi_k\right\}.$$

It is not difficult to see that the family of random variables $X_{[i,j]}$ is stationary and subadditive; i.e.,

(a) the families $\{X_{[i,j]}\}_{1\le i\le j<\infty}$ and $\{X_{[i+1,j+1]}\}_{1\le i\le j<\infty}$ are identically distributed;

(b) for $i < j < k$

$$X_{[i,k]} \le X_{[i,j]} + X_{[j+1,k]} \quad \text{a.s.}$$

Furthermore, the tail σ-algebra is degenerate and $\mathbf{E}X_{[1,1]} = J_-^2 < \infty$. Thus, Kingman's subadditive theorem implies

$$\lim_{n\to\infty} \frac{X_{[1,n]}}{n} = \lim_{n\to\infty} \frac{\mathbf{E}X_{[1,n]}}{n} = \lim\inf_{n\to\infty} \frac{\mathbf{E}X_{[1,n]}}{n} \quad \text{a.s.} \tag{4.4}$$

Next,

$$\mathbf{E}X_{[1,n+1]} = \mathbf{E}X_{[1,n]} + \int_0^\infty \mathbf{P}(\zeta_1 \in dt) \int_0^t dV_n(u) U^2(t-u) \equiv \mathbf{E}X_{[1,n]} + I_{n+1},$$

where $U^2(t) = \mathbf{E}\eta(t)$ and

$$V_n(t) = \mathbf{P}\left(\sum_{i=1}^{\eta(\psi_1+\cdots+\psi_n)} \zeta_i - (\psi_1 + \cdots + \psi_n) < t\right)$$

is the distribution function of the first overshoot of the random level $\psi_1 + \cdots + \psi_n$ for the partial sums $\sum_1^n \zeta_i$. Estimate I_{n+1} as

$$I_{n+1} \le \int_0^\infty \mathbf{P}(\zeta_1 \in dt) \int_0^t dV_n(u) U^2(t) = \int_0^\infty U^2(t) V_n(t) \mathbf{P}(\zeta_1 \in dt).$$

Since $J_- < \infty$, Corollary 4.1 yields

$$J_-^2 = \int_0^\infty U^2(t) \mathbf{P}(\zeta_1 \in dt) < \infty.$$

Fix $\varepsilon > 0$ and choose T such that

$$\int_T^\infty U^2(t)\mathbf{P}(\zeta_1 \in dt) \le \varepsilon.$$

Observe now that $V_n(T) \to 0$ as $n \to \infty$, since $\psi_1 + \dots + \psi_n \to \infty$ a.s. Therefore,

$$\limsup_{n\to\infty} I_{n+1} \le \lim_{n\to\infty} U^2(T)V_n(T)\mathbf{P}(\zeta_1 \le T) + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim \frac{\mathbf{E}X_{[1,n]}}{n} = 0,$$

and in view of (4.4) this implies (4.3).

Step 3. Fix some c, introduce random variables $\tilde{\xi}_n = \xi_n - c$, and define the integral

$$\widetilde{J}_- = \int_0^\infty \frac{x}{\mathbf{E}\min(\tilde{\xi}_1, x)}\mathbf{P}(\tilde{\xi}_1 \in -dx).$$

Clearly, both integrals J_- and $\widetilde{J}_-$ converge or diverge simultaneously.

Thus, if $\lim S_n = \infty$ then $\widetilde{J}_- < \infty$. But then $\lim \widetilde{S}_n = \lim_{n\to\infty}(\lim S_n - nc) = \infty$ a.s. for every c. Therefore, $\liminf \frac{S_n}{n} \ge c$. Tending c to $+\infty$, we arrive at the required assertion.

Step 4. Suppose that $\limsup \frac{S_n}{n}$ or $\liminf \frac{S_n}{n}$ is finite. Assume for definiteness that $\liminf \frac{S_n}{n} = d > -\infty$ (note that $d = \text{const}$ of necessity). Introduce the random variables $\tilde{\xi}_n = \xi_n + |d| + 1$, $\widetilde{S}_n = \sum_1^n \tilde{\xi}_i$. Then $\liminf \frac{\widetilde{S}_n}{n} \ge 1$ and so $\widetilde{S}_n \to \infty$ a.s. Using the arguments of Steps 1 and 3 consecutively (for $c = -|d| - 1$), we obtain $J_- < \infty$ and therefore $\lim S_n = \lim \frac{S_n}{n} = \infty$ a.s. We thus arrive at a contradiction.

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