APPROXIMATION OF MULTICHANNEL QUEUEING SYSTEMS

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1. Introduction. We consider a queueing system with $m \geq 1$ service channels.

We define two sequences $\{\tau_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ of nonnegative random variables. Service requests arrive at the system one at a time; the first request arrives at time τ_1, and for $i \geq 2$ the i-th request arrives a time interval τ_i after arrival of the (i − 1)-th request. The service time for the i-th request in order (i = 1, 2, . . .) is s_i.

We define a set $\{\omega_{0j}\}_{j=1}^{m}$ of nonnegative random variables as follows. For j = 1, 2, . . . , m we take ω_{0j} to be the time when the j-th channel can start to service requests. Let $\bar{\omega}_0 = (\omega_{01}, \ldots, \omega_{0m})$.

By a service strategy R we mean an algorithm which assigns to each request arriving in the system the number of the channel in which the request must queue up for servicing.

The system operates as follows. In each channel, requests are handled in the order in which they appear. If a request has queued up in some channel which is free, servicing begins immediately. If servicing of a request has been completed in some channel and there are other requests waiting, servicing of the next request starts immediately. After servicing is complete, the requests leave the system.

The "first in—first out" strategy is usually the one considered in queueing theory. We denote this strategy by R°. The following technical approach is used by many workers to prove various results (e.g., ergodic theorems) for service systems with strategy R°: the strategy R° is approximated by a strategy R' which is simpler in some sense, and the corresponding results are proved for R'. In particular, in [1, 2] R' is taken to be the "cyclic" strategy: if the number of a request is equal to $km + l$, where $1 \leq l \leq m$, then the request arrives at the l-th channel. In [3] R' is taken to be the following strategy: Each request arrives at any of the channels with probability 1/m independent of the prior history. Approximation is understood in the following sense: If $\omega_{n,k}$ ($\omega'_{n,k}$) are the time intervals from the moment when the n-th request ($n \geq 1$) arrives for servicing until completion of servicing of the first n requests in the k-th channel ($1 \leq k \leq m$) using strategy R° (R'), then

$$\max_{1 \leqslant k \leqslant m} \omega_{n,k} \leqslant \max_{1 \leqslant k \leqslant m} \omega'_{n,k}. \tag{1}$$

It is asserted in [1, 2] that (1) holds with probability one. However, Stoyan [4] has shown that this is false.

In this paper we prove that under certain conditions, inequality (1) is valid "in distribution," and also

a) the class of strategies {R} is described in which R° minimizes, for every $n \geq 1$, the distribution of the random variable $\max_{1 \leqslant k \leqslant m} \omega_{n,k}$;

b) characteristics of the service system are chosen (such as the virtual waiting time, mean arithmetic waiting time of the first n requests, etc.) whose distribution minimizes the strategy R° in the class of strategies {R}.

In [5] Gittens states analogous results when $\{\tau_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ are two independent sequences of independent identically distributed random variables. However, the proof of these results seems to me to be inaccurate and incomplete.

2. Definitions and Statement of the Main Result. We introduce the following definitions: $E = (-\infty, +\infty)$, $E_+ = [0, \infty)$, $E_+^k = E_+ \times E_+ \times \ldots \times E_+$ (k-fold direct product), $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n, \ldots\}$. If $\bar{x} = (x_1, \ldots, x_m)$ then $(\bar{x})^+ = (x_1^+, \ldots, x_m^+)$, where $x_i^+ = \max(0, x_i)$; if $k \in M$, then $\bar{e}_k = (0, \ldots, 0, \underset{k}{\widehat{1}}, 0, \ldots, 0) \in E_+^m$. Assume we are given Borel functions $g_n : E_+^{n(m+2)-1} \to M$ for n = 1, 2, ...

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Definition 1. A strategy R is a collection of functions $\{g_n\}_{n=1}^{\infty}$. The number R_n of the channel responsible for servicing the n-th request and the vector $\bar{\omega}_n = (\omega_{n,1}, \ldots, \omega_{n,m})$ (where $\omega_{n,k}$ is the time from arrival for servicing of the n-th request to completion of servicing of the first n requests in the k-th channel) are defined inductively:

$$R_n = g_n(\bar{\omega}_{n-1}, \ldots, \bar{\omega}_0, \tau_n, \ldots, \tau_1, s_{n-1}, \ldots, s_1) \tag{2}$$

and

$$\bar{\omega}_n = (\bar{\omega}_{n-1} - \bar{\tau}_n)^+ + s_n \cdot \bar{e}_{R_n}, \tag{3}$$

where $\bar{\tau}_n = (\tau_n, \tau_n, \ldots, \tau_n)$.

Remark 1. The assertions stated in this paper are clearly also true for strategies in which requests may be serviced in an order which differs from the order of arrival.

Definition 2. Let $R^{(1)}$ and $R^{(2)}$ be two strategies. We define an order relation between them as follows: let $n \in N$; then

a) $R^{(1)} \leq {}_nR^{(2)}$ if for every $x \in E_+$

$$P\left\{\max_{1 \leqslant k \leqslant m} \omega_{n,k}^{(1)} < x\right\} \geqslant P\left\{\max_{1 \leqslant k \leqslant m} \omega_{n,k}^{(2)} < x\right\}. \tag{4}$$

b) $R^{(1)} = {}_nR^{(2)}$ if $R^{(1)} \leq {}_nR^{(2)}$ and $R^{(2)} \leq {}_nR^{(1)}$,

c) $R^{(1)} \leq R^{(2)}$ if for all $n \in N$ $R^{(1)} \leqslant {}_nR^{(2)}$, (5)

d) $R^{(1)} = R^{(2)}$ if $R^{(1)} \leq R^{(2)}$ and $R^{(2)} \leq R^{(1)}$.

Definition 3. We define a set of strategies $\{K\}$ as follows. For any number $n \in N$, strategy $K \in \{K\}$, and vector $(y_1, \ldots, y_{n(m+2)-1}) \in E_+^{n(m+2)-1}$, the value of the function $g_n(y_1, \ldots, y_{n(m+2)-1})$ is equal to the index of some minimal coordinate of the vector $\bar{y} = (y_1, \ldots, y_m) \in E_+^m$.

Remark 2. The "first in–first out" principle is used for all the strategies in the set $\{K\}$.

The definition of $\{K\}$ implies the following result. Let $K^{(1)}, K^{(2)} \in \{K\}$ be two strategies and $n \in N$, and let $\omega_{n,k}^{(1)}$ $(\omega_{n,k}^{(2)})$ be the time from arrival of the n-th service request to completion of servicing of the first n requests by the k-th channel for the strategies $K^{(1)}$ $(K^{(2)})$. Let $\bar{u}_n^{(1)}$ $(\bar{u}_n^{(2)})$ be a permutation of the vector $\bar{\omega}_n^{(1)}$ $(\bar{\omega}_n^{(2)})$ so that the components are nondecreasing. Then $\bar{u}_n^{(1)} = \bar{u}_n^{(2)}$ a.s. and hence $K^{(1)} = K^{(2)}$ in the sense of Definition 2.

Definition 4. In what follows, R° denotes the following strategy in the class $\{K\}$: let $n \in N$ and $1 \leq k \leq m$. Let the time at which servicing of the first $n-1$ requests in channel k is complete be $t_{n-1,k}$ $(t_{0,k} = \omega_{0k})$, $t_{n-1} = \min\{t_{n-1,1}, \ldots, t_{n-1,m}\}$, $r = \min\{l \mid 1 \leq l \leq m,\ t_{n-1,l} = t_{n-1}\}$. Then $g_n^0(y_1, \ldots, y_m, \ldots, y_{n(m+2)-1}) = r$, i.e., we choose the channel having smallest index from among those channels which processed the previous requests faster than the others.

Definition 5. Let $R^{(1)}$ and $R^{(2)}$ be two strategies. We say that $R^{(1)}$ and $R^{(2)}$ coincide at step $k \in N$ $(R_k^{(1)} = R_k^{(2)})$ if for every vector $\bar{y} \in E_+^{k(m+2)-1}$ we have the equality $g_k^{(1)}(\bar{y}) = g_k^{(2)}(\bar{y})$. We say that $R^{(1)}$ and $R^{(2)}$ coincide from step k to step l $(k, l \in N,\ k \leqslant l)$ and write $R_{(k,l)}^{(1)} = R_{(k,l)}^{(2)}$ if for every $i \in N$, $k \leq i \leq l$ we have $R_i^{(1)} = R_i^{(2)}$. Assume we are given three strategies $R^{(1)}$, $R^{(2)}$, $R^{(3)}$. If for some $k < l \leqslant r$ $(k, l, r \in N)$ we have $R_{(k,l-1)}^{(1)} = R_{(k,l-1)}^{(2)}$ and $R_{(l,r)}^{(1)} = R_{(l,r)}^{(3)}$ then we write

$$R_{(k,r)}^{(1)} = R_{(k,l-1)}^{(2)} \cup R_{(l,r)}^{(3)}. \tag{6}$$

Remark 3. Equality (6) is defined for $k < l \leq r$. We will also use Eq. (6) formally when $l = k$ or $l = r + 1$, in the first case assuming that $R_{(k,k-1)}^{(2)} \cup R_{(k,r)}^{(3)} = R_{(k,r)}^{(3)}$, and in the second taking $R_{(k,r)}^{(2)} \cup R_{(r+1,r)}^{(3)} = R_{(k,r)}^{(2)}$.

Remark 4. Let $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $R^{(4)}$ be arbitrary strategies and $k, n \in N$, $1 \leqslant k \leqslant n$. If the relations $R_{(1,n)}^{(4)} = R_{(1,k-1)}^{(2)} \cup R_{(k,n)}^{(3)}$ and $R^{(1)} \geq {}_nR^{(4)}$ hold, then we will write: $R_{(1,n)}^{(1)} \geqslant {}_nR_{(1,k-1)}^{(2)} \cup R_{(k,n)}^{(3)}$ and $R_{(1,n)}^{(1)} \geqslant {}_nR_{(1,n)}^{(4)}$. If the relations $R_k^{(1)} = R_k^{(3)}$, $R_{(1,k-1)}^{(1)} = R_{(1,k-1)}^{(2)}$, $R_{(k+1,n)}^{(1)} = R_{(k+1,n)}^{(2)}$ hold, then we will use the notation $R_{(1,n)}^{(1)} = R_{(1,k-1)}^{(2)} \cup R_k^{(3)} \cup R_{(k+1,n)}^{(4)}$.

THEOREM 1. Assume the sets of random variables $\{S_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty}$, $\{\omega_{0j}\}_{j=1}^{m}$ satisfy the following condition. For every $n \in N$, and any Borel sets $B_1, B_2, \ldots, B_n \in E_+$, we have

$$P\{s_1 \in B_1, \ldots, s_n \in B_n \mid \bar{\omega}_0, \tau_1, \ldots, \tau_n\} = P\{s_1 \in C_1, \ldots, s_n \in C_n \mid \bar{\omega}_0, \tau_1, \ldots, \tau_n\} \text{ a.s.}, \tag{7}$$

where $(C_1, \ldots, C_n)$ is an arbitrary permutation of $(B_1, \ldots, B_n)$. Then the strategy R° (cf. Definition 4) is minimal among strategies of the form (2) with respect to the order (5).

3. Proof of Theorem 1. Fix $n \in N$.

Remark 4. Let $\bar\psi = (\omega_{01}, \ldots, \omega_{0m}, \tau_1, \ldots, \tau_n)$ be a random vector with values in E_+^{n+m}, $\bar y = (y_1, \ldots, y_{n+m}) \in E_+^{n+m}$. Then it follows from

$$P\left\{\max_{1\leqslant k\leqslant m} \omega_{n,k} < x\right\} = \int_{E_+^{n+m}} P\{\max \omega_{n,k} < x \mid \bar\psi = \bar y\}\, dP\{\bar\psi < \bar y\}$$

that if Theorem 1 is valid when $\omega_{01}, \ldots, \omega_{0m}$, $\tau_1, \ldots, \tau_n$ are arbitrary fixed numbers, then Theorem 1 is true in the general case.

Remark 5. Let $\bar a = (a_1, \ldots, a_n) \in E_+^n$ be an arbitrary vector and $\Pi = \{\pi\}$ the set of permutations $\pi(\bar a) = (a_{\pi_1}, \ldots, a_{\pi_n})$ of the set $(a_1, \ldots, a_n)$. Let the random variable s_1 be such that $P\{s_1 \in \{a_i\}_{i=1}^n\} = 1$. Then condition (7) implies that

$$\begin{gathered} P\left\{(s_1, \ldots, s_n) \in \bigcup_{\pi\in\Pi} \{(a_{\pi_1}, \ldots, a_{\pi_n})\}\right\} = 1 \\ P\{(s_1, \ldots, s_n) = (a_{\pi_1}, \ldots, a_{\pi_n})\} = P\{(s_1, \ldots, s_n) = (a_1, \ldots, a_n)\} \end{gathered} \tag{8}$$

for every permutation $\pi \in \Pi$.

Remark 6. Let $\bar\eta_n = (\eta_{n,1}, \ldots, \eta_{n,n})$ be a random vector with values in E_+^n which is a nondecreasing permutation of the vector $(s_1, \ldots, s_n)$ (the inequalities $\eta_{n,1} \leq \eta_{n,2} \leq \ldots \leq \eta_{n,n}$ are satisfied with probability one). Let $D_n = \{\bar z = (z_1, \ldots, z_n) \in E_+^n,\ z_1 \leqslant z_2 \leqslant \ldots \leqslant z_n\}$. Then we have the equality

$$P\left\{\max_{1\leqslant k\leqslant m} \omega_{n,k} < x\right\} = \int_{E_+^{n+m}} \int_{D_n} P\left\{\max_{1\leqslant k\leqslant m} \omega_{n,k} < x \mid \bar\psi = \bar y, \bar\eta_n = \bar z\right\} dP\{\bar\eta_n < \bar z \mid \bar\psi = \bar y\}\, dP\{\bar\psi < \bar y\}.$$

Consequently, if Theorem 1 is valid under the conditions: 1) $\omega_{01}, \ldots, \omega_{0m}$, $\tau_1, \ldots, \tau_n$ are arbitrarily fixed numbers and 2) we are given an arbitrary vector $\bar a = (a_1, \ldots, a_n) \in E_n^+$ and random variables $s_1, \ldots, s_n$ such that Eq. (8) holds, then Theorem 1 is true in general.

We will prove Theorem 1 in this special case, assuming condition (8) to be satisfied. The proof of (8) follows from Lemma 2 with k = 0.

LEMMA 1. Consider an arbitrary strategy $R^{(2)}$ and numbers $k, n \in N$, $1 \leqslant k \leqslant n$. There exists a strategy $R^{(4)}$ such that for all strategies $R^{(1)}$ and $R^{(3)}$ which satisfy

$$R^{(1)}_{(1,n)} = R^{(2)}_{(1,k)} \cup R^0_{(k+1,n)} \quad \text{and} \quad R^{(3)}_{(1,n)} = R^{(2)}_{(1,k-1)} \cup R^0_k \cup R^{(4)}_{(k+1,n)}, \tag{9}$$

we have the relation

$$R^{(3)} \leqslant_n R^{(1)}. \tag{10}$$

Proof of Lemma 1. Assume that $s_1 = c_1$, $s_2 = c_2, \ldots, s_{k-1} = c_{k-1}$, where $(c_1, \ldots, c_n)$ is some permutation of the set $(a_1, \ldots, a_n)$; we prove the lemma for fixed values of $c_1, \ldots, c_{k-1}$. We denote by $\omega^{(1)}_{n,k}$, $\omega^{(3)}_{n,k}$ the waiting times under the strategies $R^{(1)}$, $R^{(3)}$, respectively.

We construct $R^{(4)}$. Let $R^{(1)}_k = R^{(2)}_k = r \in M$ and $g^0_k(\bar\omega^{(1)}_{k-1}, \ldots, \bar\omega^{(1)}_0, \tau_k, \ldots, \tau_1, s_{k-1}, \ldots, s_1) = l \in M$. We set $R^{(4)}_k = l$ and remark that if $r = l$ then the statement in Lemma 1 is obvious. Let $r \neq l$. By (9)

$$R^{(1)}_{k+1} = g^0_{k+1}(\bar\omega^{(1)}_k, \ldots, \bar\omega^{(1)}_0, \tau_{k+1}, \ldots, \tau_1, s_k, \ldots, s_1) = l.$$

We consider the two cases

$$\alpha)\ \omega^{(1)}_{k-1} - \tau_k + \tau_{k+1} \geqslant 0;$$

$$\beta)\ \omega^{(1)}_{k-1} - \tau_k + \tau_{k+1} < 0.$$

We put $R^{(4)}_{k+1} = r$ and in case α) for $i = k+2, k+3, \ldots, n$ we put

$$R^{(4)}_i = R^{(1)}_i = g^0_i(\bar\omega^1_{i-j}, \ldots, \bar\omega^{(1)}_0, \tau_i, \ldots, \tau_1, s_{i-1}, \ldots, s_1), \tag{11}$$

while in case β) we set:

$$R_i^{(4)} = \begin{cases} r, & \text{if} \quad R_i^{(1)} = l, \\ l, & \text{if} \quad R_i^{(1)} = r, \\ R_i^{(1)} & \text{otherwise}. \end{cases} \tag{12}$$

We define $R^{(4)}$ arbitrarily from step 1 to step $(k-1)$ and from step $(n+1)$ on in accordance with (2). It is clear that $R^{(4)}$ has the form (2).

We verify that (10) holds for $R^{(3)}$. In accordance with Remark 5, for any two permutations $(c_k', c_{k+1}', \ldots, c_n')$ and $(c_k'', c_{k+1}'', \ldots, c_n'')$ of the set $(c_k, c_{k+1}, \ldots, c_n)$, the events $\{s_k = c_k', \ldots, s_n = c_n'\}$ and $\{s_k = c_k'', \ldots, s_n = c_n''\}$ have the same probability. It therefore suffices to prove relation (10) by checking the following assertion.

Let $\pi = \{(c_k', \ldots, c_n')\}$ be the set of permutations of the set $(c_k, \ldots, c_n)$. In both cases α) and β) there exists a one-to-one correspondence $\varphi: \pi \to \pi$ $(\varphi(c_k', \ldots, c_n') = (\varphi_k(c_k', \ldots, c_n'), \ldots, \varphi_n(c_k', \ldots, c_n'))$ such that the value $\max\limits_{i \leqslant j \leqslant m} \omega_{n,j}^{(1)}$ for $R^{(1)}$ on the set $\{s_k = c_k', \ldots, s_n = c_n'\}$ is not less than the value of $\max\limits_{1 \leqslant j \leqslant m} \omega_{n,j}^{(3)}$ for $R^{(3)}$ on the set $\{s_k = \varphi_k(c_k', \ldots, c_n'), \ldots, s_n = \varphi_n(c_k', \ldots, c_n')\}$ for every set $(c_k', \ldots, c_n') \in \pi$.

Case α). We take φ to be the map

$$\varphi_k(c_k, c_{k+1}, \ldots, c_n) = c_{k+1}, \ \varphi_{k+1}(c_k, c_{k+1}, \ldots, c_n) = c_k,$$
$$\text{and for} \quad i \in N, \ k+2 \leqslant i \leqslant n, \ \varphi_i(c_k, c_{k+1}, \ldots, c_n) = c_i.$$

We prove that for every i, $k+1 \le i \le n$

a) $\omega_{i,r}^{(1)} = \omega_{i,r}^{(1)}(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_i) \geqslant \tilde{\omega}_{i,r}^{(3)} = \omega_{i,r}^{(3)}(c_1, \ldots, c_{k-1}, c_{k+1}, c_k, c_{k+2}, \ldots, c_i)$;

b) $\omega_{il}^{(1)} \geqslant \tilde{\omega}_{i,l}^{(3)}$;

c) for all $j \in M$ such that $j \neq r$, $j \neq l$, we have the equality $\omega_{i,j}^{(1)} = \tilde{\omega}_{i,j}^{(3)}$.

Assertion c) is obvious since the values of $R_i^{(1)}$ on the set $\{s_1 = c_1, \ldots, s_{k-1} = c_{k-1}, s_k = c_k, \ldots, s_n = c_n\}$ and $R_i^{(3)}$ on $\{s_1 = c_1, \ldots, s_{k-1} = c_{k-1}, s_k = c_{k+1}, s_{k+1} = c_k, \ldots, s_n = c_n\}$ coincide if they are not equal to r or l. We prove a) and b).

First of all, $\omega_{k-1,r}^{(1)} = \tilde{\omega}_{k-1,r}^{(3)}$ and $\omega_{k-1,l}^{(1)} = \tilde{\omega}_{k-1,e}^{(3)}$.

Further,

$$\omega_{k,r}^{(1)} = \omega_{k-1,r}^{(1)} - \tau_k + c_k; \ \omega_{k+1,r}^{(1)} = \omega_{k-1,r}^{(1)} - \tau_k - \tau_{k+1} + c_k;$$
$$\tilde{\omega}_{k,r}^{(3)} = \tilde{\omega}_{k-1,r}^{(3)} - \tau_k; \ \tilde{\omega}_{k+1,r}^{(3)} = \tilde{\omega}_{k-1,r}^{(3)} - \tau_k - \tau_{k+1} + c_k;$$
$$\omega_{k,l}^{(1)} = \left(\omega_{k-1,l}^{(1)} - \tau_k\right)^+; \ \omega_{k+1,l}^{(1)} = \left(\left(\omega_{k-1,l}^{(1)} - \tau_k\right)^+ - \tau_{k+1}\right)^+ + c_{k+1};$$
$$\tilde{\omega}_{k,l}^{(3)} = \left(\tilde{\omega}_{k-1,l}^{(3)} - \tau_k\right)^+ + c_{k+1}; \ \tilde{\omega}_{k+1,l}^{(3)} = \left(\left(\tilde{\omega}_{k-1,e}^{(3)} - \tau_k\right)^+ + c_{k+1} - \tau_{k+1}\right)^+.$$

Thus, inequalities a) and b) are satisfied for $i = k+1$. By (11), these inequalities remain valid for $i > k+1$, since the values of $R_i^{(1)}$ on the set $\{s_1 = c_1, \ldots, s_{k-1} = c_{k-1}, s_k = c_k, \ldots, s_n = c_n\}$ and $R_i^{(3)}$ on the set $\{s_1 = c_1, \ldots, s_{k-1} = c_{k-1}, s_k = c_{k+1}, s_{k+1} = c_k, \ldots, s_n = c_n\}$ coincide for $i \ge k+2$. In particular, a), b), and c) imply that $\max\limits_{1 \leqslant j \leqslant m} \omega_{n,j}^{(1)} \geqslant \max\limits_{1 \leqslant j \leqslant m} \tilde{\omega}_{n,j}^{(3)}$.

Case β). We take φ to be the identity map: $\varphi(c_k, \ldots, c_n) = (c_k, \ldots, c_n)$ and prove that for every i, $k+1 \leqslant i \leqslant n$, the following inequalities hold: $\omega_{i,r}^{(1)} = \omega_{i,r}^{(1)}(c_1, \ldots, c_i) \geqslant \omega_{i,l}^{(3)} = \omega_{i,l}^{(3)}(c_1, \ldots, c_i)$, $\omega_{i,e}^{(1)} \geqslant \omega_{i,r}^{(3)}$, and for all $j \in M$ such that $j \neq r$, $j \neq l$ we have $\omega_{i,j}^{(1)} = \omega_{i,j}^{(3)}$. Using arguments similar to the ones in case α), we get that it suffices to show the following:

a) $\omega_{k+1,r}^{(1)} \geqslant \omega_{k+1,l}^{(3)}$, b) $\omega_{k+1,l}^{(1)} \geqslant \omega_{k+1,r}^{(3)}$.

In addition, $\omega_{k-1,r}^{(1)} = \omega_{k-1,r}^{3}$ and $\omega_{k-1,l}^{(1)} = \omega_{k-1,l}^{(3)}$. Moreover,

$$\omega_{k,r}^{(1)} = \left(\omega_{k-1,r}^{(1)} - \tau_k\right)^+ + c_k; \ \omega_{k+1,r}^{(1)} = \left(\left(\omega_{k-1,r}^{(1)} - \tau_k\right)^+ + c_k - \tau_{k+1}\right)^+;$$
$$\omega_{k,r}^{(3)} = \left(\omega_{k-1,r}^{(3)} - \tau_k\right)^+; \ \omega_{k+1,r}^{(3)} = \left(\left(\omega_{k-1,r}^{(3)} - \tau_k\right)^+ - \tau_{k+1}\right)^+ + c_{k+1} = c_{k+1};$$
$$\omega_{k,l}^{(1)} = \left(\omega_{k-1,l}^{(1)} - \tau_k\right)^+; \ \omega_{k+1,l}^{(1)} = \left(\left(\omega_{k-1,l}^{(1)} - \tau_k\right)^+ - \tau_{k+1}\right)^+ + c_{k+1} = c_{k+1};$$
$$\omega_{k,l}^{(3)} = \left(\omega_{k-1,l}^{(3)} - \tau_k\right)^+ + c_k; \ \omega_{k+1,l}^{(3)} = \left(\left(\omega_{k-1,l}^{(3)} - \tau_k\right)^+ + c_k - \tau_{k+1}\right)^+.$$

We obtain the required result using the inequality $\omega_{k-1,r}^{(1)} \geqslant \omega_{k-1,l}^{(3)}$. In particular, the previous arguments imply that

$$\max_{1\leqslant j\leqslant m} \omega_{n,j}^{(1)} \geqslant \max_{1\leqslant j\leqslant m} \omega_{n,j}^{(3)}.$$

Lemma 1 is proved.

LEMMA 2. Let $n \in N$. Then for every strategy $R^{(2)}$ and integer k ($0 \leq k \leq n$) we have the relation

$$R_{1,n}^{(2)} \geqslant_n R_{(1,k)}^{(2)} \cup R_{(k+1,n)}^{0}$$

Proof of Lemma 2. For $k = n$ and $k = n - 1$ the assertion is obvious. Assume Lemma 2 has been proved for all k greater than or equal to k_0 ($1 \leq k_0 \leq n - 1$). We prove it for $k = k_0 - 1$.

By the induction assumption, $R_{(1,n)}^{(2)} \geqslant_n R_{(1,k_0)}^{(2)} \cup R_{(k_0+1,n)}^{0}$. We define a strategy $R^{(1)}$ so that the equality

$$R_{(1,n)}^{(1)} = R_{(1,k_0)}^{(2)} \cup R_{(k_0+1,n)}^{0} \tag{13}$$

holds. Consequently, $R_{(1,n)}^{(2)} \geqslant_n R_{(1,n)}^{(1)}$. By Lemma 1 there exist strategies $R^{(3)}$ and $R^{(4)}$ such that

$$R_{(1,n)}^{(3)} = R_{(1,k_0-1)}^{(2)} \cup R_{k_0}^{0} \cup R_{(k_0+1,n)}^{(4)} \tag{14}$$

and

$$R_{(1,n)}^{(1)} \geqslant_n R_{(1,n)}^{(3)}. \tag{15}$$

It follows from (14) that $R_{(1,k_0)}^{(3)} = R_{(1,k_0-1)}^{(2)} \cup R_{k_0}^{0}$. By induction hypothesis, strategy $R^{(3)}$ also satisfies the relation

$$R_{(1,n)}^{(3)} \geqslant_n R_{(1,k_0)}^{(3)} \cup R_{(k_0+1,n)}^{0}. \tag{16}$$

By definition, we have $R_{(k_0,n)}^{0} = R_{k_0}^{0} \cup R_{(k_0+1,n)}^{0}$. Hence,

$$R_{(1,k_0)}^{(3)} \cup R_{(k_0+1,n)}^{0} = R_{(1,k_0-1)}^{(2)} \cup R_{k_0}^{0} \cup R_{(k_0+1,n)}^{(0)} = R_{(1,k_0-1)}^{(2)} \cup R_{(k_0,n)}^{0}. \tag{17}$$

It follows from (16) and (17) that

$$R_{(1,n)}^{(3)} \geqslant_n R_{(1,k_0-1)}^{(2)} \cup R_{(k_0,n)}^{0}. \tag{18}$$

Using (13), (15), (18), and the transitivity of the relation ($\geq_n$), we obtain

$$R_{(1,n)}^{(2)} \geqslant_n R_{(1,k_0-1)}^{(2)} \cup R_{(k_0,n)}^{0}.$$

Since $n \in N$ is finite, the assertion in Lemma 2 is proved.

4. Generalizations of Theorem 1. Theorem 1 asserts that the strategy $R°$ minimizes the distribution of the random variable $\max\limits_{1\leqslant j\leqslant m} \omega_{n,j}$. We now describe fairly completely a set of characteristics of a service system whose distribution minimizes $R°$.

Let $n \in N$. We define the following classes of functions: $F_1^{(n)} = \{f \mid f: E_+^n \to E;\ f \text{ Borel}\}$; $F_2^{(n)} = \{f \mid f: E_+^n \to E;\ f$ Borel, and for any permutation $(c_1, \ldots, c_n)$ of $(a_1, \ldots, a_n) \in E_+^n$ we have $f(a_1, \ldots, a_n) = f(c_1, \ldots, c_n)\}$; $F_3^{(n)} = \{f \mid f \in F_2^{(n)}$; if $b_1 \leqslant a_1;\ b_2 \leqslant a_2; \ldots;\ b_n \leqslant a_n$, then $f(b_1, \ldots, b_n) \leqslant f(a_1, \ldots, a_n)\}$; $F^{(n)} = \{f \mid f: E_+^{3n} \to E$; there exist $f_1 \in F_1^{(n)}$, $f_2 \in F_2^{(n)}$, $f_3 \in F_3^{(n)}$ such that for every vector $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) \in E_+^{3n}$ the equality $f(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) = f_1(x_1, \ldots, x_n) + f_2(y_1, \ldots, y_n) + f_3(z_1, \ldots, z_n)\}$ holds.

Now assume that a set G of measurable functions is such that $G = \{g \mid g: \Omega \times E_+^\infty \times E_+^\infty \times E_+^\infty \to E$; there exists a decomposition of the space of elementary outcomes Ω into disjoint sets A_n belonging to the σ-algebra generated by the random variables $\tau_1, \tau_2, \ldots, \tau_n, \tau_{n+1}$, and for all $n \in N$ there exists an $f^{(n)} \in F^{(n)}$ such that for almost all $\omega \in A_n$ the equality $g(\omega; x_1, \ldots, x_n, \ldots; y_1, \ldots, y_n, \ldots; z_1, \ldots, z_n, \ldots) = f^n(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)\}$ holds.

Let $j_k^{(1)}$, $j_k^{(2)}$ be the time intervals from time zero to completion of servicing the k-th request for strategies $R^{(1)}$, $R^{(2)}$, respectively.

Definition 6. Assume we are given two strategies $R^{(1)}$ and $R^{(2)}$. Then

$$R^{(1)} \geqslant_2 R^{(2)}, \tag{19}$$

if for every function $g \in G$ and every number $x \in E$ we have the equality

$$P\{g(\omega; \tau_1, \ldots, \tau_n, \ldots; s_1, \ldots, s_n, \ldots; j_1^{(1)}, \ldots, j_n^{(1)}, \ldots) < x\} \leqslant P\{g(\omega; \tau_1, \ldots, \tau_n, \ldots; s_1, \ldots, s_n, \ldots; j_1^{(2)}, \ldots, j_n^{(2)}, \ldots) < x\}.$$

THEOREM 2. Assume the hypotheses of Theorem 1 hold. Then the strategy R° is minimal among strategies of the form (2) with respect to the order (19).

Remark 7. The proof of Theorem 2 is a repetition of the proof of Theorem 1. The only change is that Eq. (7) is replaced by an equivalent equality $P\{s_1 \in B_1, \ldots, s_n \in B_n | \bar{\omega}_0, \tau_1, \ldots, \tau_{k+1}\} = P\{s_1 \in C_1, \ldots, s_k \in C_k | \bar{\omega}_0, \tau_1, \ldots, \tau_{k+1}\}$ a. s., which is obtained from (7) for $k = n - 1$ and $B_n = C_n = E_+$.

COROLLARY 1. Let δ_k', δ_k^0 be the waiting times for the k-th request from the time of arrival to the start of servicing for strategies R', R°, respectively, and let σ_k', σ_k^0 be the waiting times for the k-th request from the moment of arrival to completion of servicing. Let $w'(t)$, $w°(t)$ $(t > 0)$ be the times from time t to completion of servicing of requests which arrived prior to time t for strategies R', R°, respectively. Then the following three inequalities hold:

1) for every $n \in N$ and $x \in E$

$$P\left\{\frac{1}{n}\sum_{k=1}^{n}\delta_k' < x\right\} \leqslant P\left\{\frac{1}{n}\sum_{k=0}^{n}\delta_k^0 < x\right\};$$

2) for every $n \in N$ and $x \in E$

$$P\left\{\frac{1}{n}\sum_{k=1}^{n}\sigma_k' < x\right\} \leqslant P\left\{\frac{1}{n}\sum_{k=1}^{n}\sigma_k^0 < x\right\};$$

3) for every $t > 0$ and $x \in E$

$$P\{w'(t) < x\} \leqslant P\{w°(t) < x\}.$$

Proof. We prove the first assertion. Since $\frac{1}{n}\sum_{k=1}^{n}\delta_k' = \frac{1}{n}\sum_{k=1}^{n} j_k' - \frac{1}{n}\sum_{k=1}^{n}\tau_k - \frac{1}{n}\sum_{k=j}^{n} s_k$, it suffices to define the functions $f_1(x_1, \ldots, x_n) = -\frac{1}{n}\sum_{k=j}^{n} x_k$; $f_2(y_1, \ldots, y_n) = -\frac{1}{n}\sum_{k=1}^{n} y_k$; $f_3(z_1, \ldots, z_n) = \frac{1}{n}\sum_{k=1}^{n} z_k$, $g \equiv f = f_1 + f_2 + f_3$.

We prove 2). Since the equality

$$\frac{1}{n}\sum_{k=1}^{n}\sigma_k' = \frac{1}{n}\sum_{k=1}^{n} j_k' - \frac{1}{n}\sum_{k=1}^{n}\tau_k$$

holds, it suffices to define the functions f_1, f_2, f_3, g by

$$f_1(x_1, \ldots, x_n) = -\frac{1}{n}\sum_{k=j}^{n} x_k; \quad f_2(y_1, \ldots, y_n) \equiv 0; \quad f_3(z_1, \ldots, z_n) = \frac{1}{n}\sum_{k=1}^{n} z_k;$$
$$g \equiv f_1 + f_2 + f_3.$$

We prove 3). Let the event A_n consist in the truth of the inequalities

$$\sum_{k=1}^{n}\tau_k \leqslant t \text{ and } \sum_{k=j}^{n+1}\tau_k > t.$$

Then the equality $W'(t) = \left(\max_{1 \leqslant j \leqslant n} j_j' - t\right)^+$ holds on the set A_n. We take the function $f^n \in F^{(n)}$ to be the sum $f^{(n)} = f_1 + f_2 + f_3$, where $f_1(x_1, \ldots, x_n) \equiv 0$, $f_2(y_1, \ldots, y_n) \equiv 0$, $f_3(z_1, \ldots, z_n) = \left(\max_{1 \leqslant j \leqslant m} z_j - t\right)^+$. We now define the function g on A_n by $g = f^{(n)}$.

Corollary 1 is proved.

We state another generalization of Theorem 1.

THEOREM 3. Let the functions g_k appearing in definition (2) be random. Then if for every $n \in N$ the sets $\{g_1, \ldots, g_n\}$ and $\{s_n, s_{n+1}, \ldots\}$ of random variables are mutually independent, Theorems 1 and 2 are true under the same assumptions.

The proof of Theorem 3 reduces in a natural way to the proof of Theorem 1.

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FINITE GROUPS WITH A NORMALIZER CONDITION

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Let G be a finite group. A subgroup H of G will be called normally independent in G if for any nonunit normal subgroup F of H we have the inclusion $N_G(F) \leq N_G(H)$. If the validity of this inclusion does not require the normality of F in H, the subgroup H is called simply independent in G. The structure of a finite group depends essentially on the degree of its saturation with normally independent subgroups.

It can be required that normalizer conditions should be satisfied by one or another system of subgroups with given properties.

If all the subgroups of the group G that have some property σ are normally independent in G, we shall call G a σ-normalizable group. A group which is normalizable by all its subgroups will be called a universally normalizable group.

The aim of the present note is the study of the structure of finite σ-normalizable groups. Here, by the property σ we shall understand, apart from universality, nilpotence and, in particular, the primary nature of subgroups.

We shall state some properties which follow easily from the definitions.

Property 1. Any σ-subgroup of a σ-normalizable group G containing a nonunit subgroup that is normal in G is itself normal in G.

Property 2. A finite universally normalizable group G is nilpotent if its Frattini group $\Phi(G)$ does not reduce to a unit group.

Property 3. Any σ-subgroup of a σ-normalizable group is itself σ-normalizable.

Proof. Let G be a group satisfying this condition, and K a certain σ-subgroup of G. Furthermore, let H be a subgroup of K, and F a normal nonunit subgroup of H. According to our assumption, $N_G(F) \leq N_G(H)$. Since $N_K(F) \leq N_G(F)$, we have $N_K(F) \leqslant K \cap N_G(F) \leqslant K \cap N_G(H) = N_K(H)$, from which it follows that K is universally normalizable.

It follows from Property 3 that in particular the intersection of two σ-subgroups of a σ-normalizable group is itself a σ-normalizable group.

Property 4. The factor group of a universally normalizable group by any normal subgroup is universally normalizable.

Proof. Let G be a universally normalizable group, and A a normal subgroup of G. Consider the factor group $\overline{G} = G/A$. Let $\overline{H}$ be a subgroup of $\overline{G}$, and $\overline{F}$ a normal subgroup of $\overline{H}$. If H and F are the complete inverse

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