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ERGODICITY OF QUEUING NETWORKS

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We consider an open Jackson-type service network with single-channel stations. It is shown that if the load on each station is less than 1, the process defined by the length of the queue satisfies an ergodic theorem in discrete time. If it is additionally supposed that the lengths of the intervals between the arrivals of calls possesses a nonlattice distribution, an ergodic theorem will also hold for the queue length process in continuous time. These results are carried over to the case of multichannel stations. For the special case of "acyclic networks," it is proved that an ergodic theorem will also hold in the most general situation in which the elements of the control sequences form a stationary metrically transitive sequence.

We also consider closed networks for which an ergodic theorem is proved under the condition that the distributions of service times is nonlattice.

1. Introduction and Statement of Basic Results

Suppose that N > 0 is an integer. Let $\{Z(n);\ n\geqslant 0\}$ denote a Markov chain with (N + 2) states 0, 1,...,N, N + 1, initial value Z(0) = 0, and transition matrix $\|p_{ij}\|$, i, j = 0,..., N + 1, where $p_{N+1,N+1} = 1$, $p_{i0} = 0$ for all i. Moreover, we also assume that $p_{ii} = 0$ for all $i\leqslant N$ (as noted in [1, 2] this assumption does not place any limitations on the generality of the subsequent argument). We will assume taht state (N + 1) may be reached from any state $i\geqslant 1$ and, consequently, is absorbing state. Let π_i, $1\leqslant i\leqslant N$ denote the mean number of sojourns of state i by the trajectory $\{Z(n);\ n\geqslant 0\}$.

We will say that the matrix $\|p_{ij}\|$ [and also the Markov chain {Z(n)}] is acyclic if no two states i and j are absorbing states. In other words, a matrix $\|p_{ij}\|$ is acyclic if states {1, 2,...,N} may be renumbered in such a way that for all $1\leqslant i\leqslant N$,

$$\sum_{j=i+1}^{N+1} p_{ij} = 1.$$

Now consider an open service network with N single-channel stations into which a recursive stream of calls arrives, i.e., it is assumed that the times between the arrivals of calls $\{\tau_n\}$ are independent and identically distributed with mean $E\tau_n = 1/\alpha > 0$. Each call from the input stream is directed (independently of all the others) to the station numbered k with probability p_{0k}, $\sum_k p_{0k} = 1$. Servicing times at the k-th station $\{s_i^k\}$ are independent and identically distributed with mean $Es_i^k = a_k$, $0 < a_k < \infty$. Calls are serviced at each station in their order of arrival. After its service at the k-th station has been completed, a call arrives, with probability p_{kj}, at the j-th station and, with probability $p_{k,N+1}$, escapes from the network. These types of networks are called Jackson-type networks.

For $t\geqslant 0$, $i = 1, \ldots, N$, we denote by $q^i(t)$ the number of calls at station i at time t + 0 (i.e., whether in queue or being serviced), while $\chi^i(t)$ denotes the remaining time left to service the first of these calls [we set $\chi^i(t) = 0$ when $q^i(t) = 0$]; $q(t) = (q^1(t),\ldots,q^N(t))$; $\chi(t) = (\chi^1(t),\ldots,\chi^N(t))$. We set $Z_+ = \{0, 1, 2,\ldots\}$; $R_+ = [0, \infty)$.

In [1, 2] the following is proved:

THEOREM 1. If for a Jackson-type network,

$$\alpha\pi_j < 1/a_j \tag{1}$$

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for all $j = 1,\ldots,N$, then when conditions (C1)-(C3) (see below) are satisfied, the service network is ergodic, i.e., there exists a natural distribution $P(\cdot)$ on the set $Z_+^N \times R_+^N$ such that for any initial condition $(q(0), \chi(0))$ $\mathbf{P}((q(t), \chi(t)) \in B) \to P(B)$ as $t \to \infty$ for any Borel set $B \subseteq Z_+^N \times R_+^N$ of the form $B = \{(l_1, \ldots, l_N; x_1, \ldots, x_N): x_i \leqslant u_i;\ i = 1, \ldots, N\};\ l_i \in Z_+;\ u_i \in R_+;\ i = 1, \ldots, N.$

Conditions (C1)-(C3) are as follows:

(C1) There exist numbers $C > 0$ and $\gamma > 0$ such that for all $t, x \geqslant 0$,

$$\mathbf{P}(\tau_n \geqslant t + x/\tau_n \geqslant t) \leqslant Ce^{-\gamma x};\quad \mathbf{P}(s_i^k \geqslant t + x/s_i^k \geqslant t) \leqslant Ce^{-\gamma x}.$$

(C2) The random variables τ_m are not bounded [i.e., $\mathbf{P}(\tau_n > x) > 0$ for all x].

(C3) The random variables τ_n possess a nonlattice distribution.

In the present work we will show that the following assertion holds.

THEOREM 2. Theorem 1 holds under conditions (1) and (C3).

It is known that if either one of the two conditions (1) and (C3) does not hold, Theorem 2 is false (except for the degenerate case). Therefore, Theorem 2 is conclusive in nature and may be stated in the form of a criterion.

Set $t_n = \tau_1 + \ldots + \tau_n$ and denote by $q_n \equiv (q_n^1, \ldots, q_n^N) = q(t_n)$, $\chi_n \equiv (\chi_n^1, \ldots, \chi_n^N) = \chi(t_n)$ the length of a queue and the remaining service time at successive arrival times, respectively. Then

THEOREM 3.* If (1) holds, a stationary sequence $\{(q^n, \chi^n);\ -\infty < n < \infty\}$ may be specified in the same probability space with $\{(q_n, \chi_n)\}$ such that $\mathbf{P}((q_l, \chi_l) = (q^l, \chi^l)$ for all $l \geqslant n) \to 1$ as $n \to \infty$.

An analogous assertion was stated in [4], though under the following two additional assumptions: (1) $p_{k,N+1} > 0$ for all $k = 1,\ldots,N$; (2) for the Markov chain $\{(q_n, \chi_n)\}$ there exists a positive-definite recurrent compactum.

Note that it follows from the results of [5] that Theorem 2 is a direct corollary to Theorem 3. Therefore, we will limit the discussion to a proof of the latter assertion.

Note, too, that under the conditions of Theorem 3 estimates of the rate of convergence of the following form may be stated: if $\mathbf{E}(s_1^k)^\beta < \infty$ [or $\mathbf{E}\exp(\beta s_1^k) < \infty$] for all $k = 1,\ldots,N$ and some $\beta > 1$ ($\beta > 0$), then for $\lambda = \min\{n:\ (q_l, \chi_l) = (q^l, \chi^l)$ for all $l \geqslant n\}$ it is true that $\mathbf{P}(\lambda > n) \leqslant Cn^{-\beta+1}$ [respectively, $\mathbf{P}(\lambda > n) \leqslant C\exp(-\beta' n)$] for some constant $C < \infty$ (and constant $\beta' > 0$). These assertions were also deduced by the present author, though their proof will be published elsewhere.

Consider networks with multichannel stations, where the i-th station constitutes a multichannel system with m_i channels. We suppose that at each station calls are serviced according to the principle of "first in—first out." We will prove the following assertion.

THEOREM 4. Jackson-type networks with multichannel stations satisfy the assertions of Theorems 2 and 3 if condition (1) is replaced by the following:

$$a\pi_j < m_j/a_j \tag{2}$$

for all $j = 1,\ldots N$.

For the case of acyclic networks with single-channel stations we will prove that Theorem 3 remains valid under conditions more general than independence and identically distributed service time. This assertion will be stated in Sec. 2 (Theorem 6).

Let us now consider closed systems. Consider a finite Markov chain with N essential absorbing states and transition matrix $\|p_{ij}\|$, $1 \leqslant i, j \leqslant N$, $\sum_{j=1}^N p_{ij} = 1$ for all i, and by analogy

*The article [3] appeared after the present article had been submitted; the assertion of Theorem 3 was stated in [3]. However, a rigorous proof of the assertion is given in [3] only for the case of limited service time, i.e., even under stronger constraints than in [1, 2], and a proper proof cannot be established for the general case from informal and very abbreviated arguments. Note, too, that the proof proposed in the present article uses entirely different concepts, and for that very reason differs in substantial respects from the proof given in [3].

with the previously introduced concept of an open network, define a Jackson-type closed network with N single-channel stations and M calls. Then

THEOREM 5. If service time distribution in a closed service network is nonlattice, Theorem 2 will hold.

An analogous assertion, though under the assumption of conditions (C1) and (C2), was proved in [1]. I have also learned that a result close to Theorem 5 was obtained independently by E. V. Morozov and is to be published in 1991 in a collection from VNIISI (Moscow).

The present article consists of four sections. In Sec. 2 we study the open acyclic networks. The results obtained in Sec. 2 are applied in Sec. 3 to prove Theorems 3 and 4. In Sec. 4 we prove an ergodic theorem (Theorem 5) for closed networks.

2. Acyclic Open Networks

2.1. Single-Channel Systems. Consider a single-channel waiting system with calls served in their order of arrival. Control of the system is defined in a more complex way than is usually the case, the point of which will become clear in the course of studying service networks.

Suppose that we are given a sequence of pairs $\{(\tau_n, \mu_n);\ n\geqslant 0\}$ of random variables, where $\tau_n > 0$ almost certainly (i.e., with probability one) and the μ_n are integral and nonnegative. Moreover, if $\mu_n\geqslant 1$ we define the random variables $0<u_{n1}\leqslant u_{n2}\leqslant \ldots \leqslant u_{n\mu_n}\leqslant \tau_n$ with probability one. Let $\mathbf{u}_n=(u_{n1}, \ldots, u_{n\mu_n})$ (setting $\mathbf{u}_n=\varnothing$ if $\mu_n = 0$) and $t_n = \tau_1 + \ldots + \tau_n$. An input stream of calls into the system is defined in the following way: if the equality $\mu_n = 0$ holds, this will mean that over the time interval $(t_{n-1}, t_n]$ no calls arrive in the system, and when $k\geqslant 1$ the equality $\mu_n = k$ asserts that over the time interval $(t_{n-1}, t_n]$ k calls arrive in the system at times $t_{n-1} + u_{n1},\ldots,t_{n-1} + u_{nk}$, respectively. We also define the random vectors $\mathbf{s}_n=(s_{n1}, \ldots s_{n\mu_n})$ (here $\mathbf{s}_n=\varnothing$ if $\mu_n = 0$), where s_{ni} is understood as the time required to service a call that arrives in the system at time $t_{n-1} + u_{ni}$ (i.e., the i-th call in the n-th group). We define $S_n=s_{n1}+\ldots+s_{n\mu_n}$ (where $S_n = 0$ if $\mu_n = 0$) and $\xi_n=\{\tau_n, \mu_n, \mathbf{u}_n, \mathbf{s}_n\}$.

Note that if $\mu_n\geqslant 1$ and $u_{ni} = \tau_n$ when $1\leqslant i\leqslant \mu_n$, the incoming stream described above is a grouped-arrival stream, i.e., calls arrive in the system at times t_n in groups of size μ_n. These types of systems are considered in [6].

Given call c, we let t(c) denote the time it arrives in the system and γ(c) the time its service is concluded. For $t\geqslant 0$ and $u\geqslant 0$, we let Q(t, u) denote the number of calls that have arrived in the system by time t + 0 and have not yet been fully serviced by time t + u + 0, i.e., $Q(t, u)=\#\{c\colon t(c)\leqslant t;\ \gamma(c)>t+u\}$. Note that the function Q(t, u) is continuous with probability one on the right in both arguments. Moreover, Q(t, u) is not increasing in u for any fixed t and Q(t, u) → 0 with probability one as u → ∞, i.e., for any $t\geqslant 0$ there exists (with probability one) a finite random variable w(t) such that Q(t, u(t)) = 0 with probability one.

If $t_n=\tau_1+\ldots+\tau_n,\ u\geqslant 0$, we set $Q_n(u)=Q(t_n, u)$; $Q_n=\{Q_n(u);\ u\geqslant 0\}$; $q_n=Q_n(0)$ is the number of calls in the system by time t_n+0; $w_n=\inf\{u\geqslant 0\colon Q_n(u)=0\}$ is the total time remaining to service these calls.

We introduce the space $D_+^0=D_+^0[0, \infty)$ of nonnegative, integral, nonincreasing, right-continuous functions with finite support on [0, ∞] with metric

$$\rho(f, g)=\inf_{h\in H}\left\{\sup_{x\geqslant 0}|h(x)-x|+\sup_{x\geqslant 0}|f(h(x))-g(x)|\right\},$$

where H is the set of strictly increasing continuous functions h: [0, ∞) → [0, ∞) such that h(0) = 0. Note that the space (D_+^0, ρ) is separable and possesses the property that if the sequence $\{f_n\}$, $f_n\in D_+^0$ is fundamental, a function $f\in D_+^0$ may be found such that $f(x)\geqslant f_n(x)$ for all $x\geqslant 0,\ n\geqslant 1$. It is easily seen that Q(t, u) and $Q_n(u)$ (for any fixed t and n) are random elements with values in this space.

Note that the sequences $\{w_n\}$ and $\{Q_n\}$ are related by the recursive relations $w_n = f(w_{n-1}, \xi_n)$ and $Q_n = F(Q_{n-1}, \xi_n)$, where the functions f and F possess a rather cumbersome form.

LEMMA 1. Suppose that the sequence $\{\xi_n\}$ is stationary and metrically transitive. If $\mathbf{E}\tau_1>\mathbf{E}S_1$, a stationary sequence $\{Q^n\}$, $Q^n\in D_0^+$ with probability one may be found such that $\mathbf{P}\{Q_k=Q^k$ for all $k\geqslant n\}\to 1$ as n → ∞, for any initial distribution Q_0 such that $\mathbf{P}(w_0<\infty)=1$.

Proof. Let us first prove that a stationary sequence $\{w^n\}$, $w^n < \infty$ with probability one may be found such that $\mathbf{P}\ (w_k = w^k \text{ if } k \geqslant n) \to 1$ as $n \to \infty$. We begin with the case $w_0 = 0$.

Consider the following auxiliary system with grouped arrival: calls arrive at times t_n in groups of size μ_n. We denote the characteristics of this system by $\tilde{Q}_n$ and $\tilde{w}_n$. Assuming the sequence $\{\xi_n\}$ to be given for all n, $-\infty < n < \infty$, we define the shift operation U of these random variables in such a way that $U\xi_n = \xi_{n+1}$. We let U^n denote the iteration of this transformation. It has been shown [6] that the sequences $\{U^{-n}\tilde{w}_n\}$ and $\{U^{-n}\tilde{Q}_n\}$ are monotonically nondecreasing and that there exist a random variable $\tilde{w}^0 < \infty$ with probability one and a random element $\tilde{Q}^0 \in D_+^0$ with probability one such that $\mathbf{P}(U^{-k}\tilde{Q}_k = \tilde{Q}^0 \text{ if } k \geqslant n) \geqslant \mathbf{P}(U^{-k}\tilde{w}_k = \tilde{w}^0 \text{ if } k \geqslant n) \to 1$. Note that, first, $\tilde{w}_n \geqslant w_n$ with probability one for all n and, second, the sequence $U^{-n}w_n$ is monotonically nondecreasing. Consequently, it converges, with probability one, to a finite random variable $w^0 \leqslant \tilde{w}^0$ with probability one. Further, by virtue of monotonicity $0 \leqslant U^{-k}w_k - U^{-n}w_n \leqslant U^{-k}\tilde{w}_k - U^{-n}\tilde{w}_n$ with probability one for any $k \geqslant n \geqslant 0$. Taking limits as $k \to \infty$, we obtain the inequalities $0 \leqslant w^0 - U^{-n}w_n \leqslant \tilde{w}^0 - U^{-n}\tilde{w}_n$. Consequently, $\mathbf{P}(w^0 = U^{-k}w_k \text{ if } k \geqslant n) \to 1$ and, thereby, $\mathbf{P}(w_n = w^n) = \mathbf{P}(w_k = w^k \text{ if } k \geqslant n) \to 1$ as $n \to \infty$, where $w^n = U^n w^0$.

Now let $\mathbf{P}(w_0 > 0) > 0$. For any $\varepsilon > 0$, we may find a number c such that $\mathbf{P}(w_0 \leqslant w^0 + c) \geqslant 1 - \varepsilon$.

Consider a system with initial condition $w_0 + w^0 + c$ and introduce the virtual waiting time $w(t) = \inf\{u \geqslant 0\colon Q(t, u) = 0\}$. It is easily seen that $\mathbf{P}(w(t) > 0 \text{ for all } t \geqslant 0) = 0$. Therefore, if a random variable $\eta = \inf\{t \geqslant 0\colon w(t) = 0\}$ is introduced, when $t_n > \eta$ those sequences $\{w_n\}$ with initial condition $w_0 = 0$ and those with initial condition $w_0 = w^0 + c$ may be "glued" together. By virtue of monotonicity, a sequence with any initial condition $0 \leqslant w_0 \leqslant w^0 + c$ may also be "glued" together with these sequences. Consequently, $\liminf_{n\to\infty} \mathbf{P}(w_k = w^k \text{ if } k \geqslant n) \geqslant 1 - \varepsilon$, and since $\varepsilon > 0$ is arbitrarily chosen, $\mathbf{P}(w_k = w^k \text{ if } k \geqslant n) \to 1$ as $n \to \infty$ for any initial state w_0.

To verify the lemma, it remains for us to note that, first, the monotonicity properties stated earlier for $\{w_n\}$ hold also for $\{Q_n\}$ and, second, for n such that $t_n > \eta$, not only the sequences $\{w_n\}$ but also the sequences $\{Q_n\}$ with initial conditions $w_0 = 0$ and $w_0 = w^0 + c$, may be glued together. The lemma is proved.

Now consider the output stream of the system. Let $\tilde{\mu}_n$ denote the number of calls whose servicing concludes within the time interval $(t_{n-1}, t_n]$, and $t_{n-1} + \tilde{u}_{n1} \leqslant \ldots \leqslant t_{n-1} + \tilde{u}_{n\tilde{\mu}_n}$, the corresponding times when service concludes, $\tilde{u}_n = (\tilde{u}_{n1}, \tilde{u}_{n2}, \ldots)$. If $\tilde{\mu}_n = 0$, we set $\tilde{u}_n = \phi$. Note that the pair of random variables $(\tilde{\mu}_n, \tilde{u}_n)$ is uniquely determined from Q_{n-1} and ξ_n, i.e., there exists a function H such that $(\tilde{\mu}_n, \tilde{u}_n) = H(Q_{n-1}, \xi_n)$ with probability one for all n. The function H may be specified constructively, though it is quite cumbersome in form.

COROLLARY. Under the conditions of Lemma 1, there exists a stationary metrically transitive sequence $\{(\tilde{\mu}^n, \tilde{u}^n)\}$, $\tilde{\mu}^n < \infty$ with probability one such that $\mathbf{P}((\tilde{\mu}_k, \tilde{u}_k) = (\tilde{\mu}^k, \tilde{u}^k) \text{ for } k \geqslant n) \to 1$ as $n \to \infty$. Moreover, $\mathbf{E}\tilde{\mu}^n = \mathbf{E}\mu_n$.

In fact, it suffices to determine $(\tilde{\mu}^n, \tilde{u}^n) = H(Q^{n-1}, \xi_n)$. In this case $\{(\tilde{\mu}_n, \tilde{u}_n) = (\tilde{\mu}^n, \tilde{u}^n)\} \supseteq \{Q_{n-1} = Q^{n-1}\}$. Next, assume the contrary, i.e., $\mathbf{E}\tilde{\mu}^1 \geqslant \mathbf{E}\mu_1 + \delta$. Since for any ℓ, $(\tilde{\mu}^{l+1} + \ldots + \tilde{\mu}^{l+n})/n \to \mathbf{E}\tilde{\mu}^1$ with probability one, for every $\varepsilon > 0$ a number n_0 may be found such that $\mathbf{P}\big(\tilde{\mu}^{l+1} + \ldots + \tilde{\mu}^{l+n} \geqslant \frac{n\delta}{3} + \mu_{l+1} + \ldots \mu_{l+n} \text{ for all } n \geqslant n_0\big) \geqslant 1 - \varepsilon$. Let $\gamma < \infty$ be the time at which the sequences $\{Q_n\}$ and $\{Q^n\}$ are glued together. Let us find $\ell = \ell(\varepsilon)$ such that $\mathbf{P}(\gamma < l) \geqslant 1 - \varepsilon$. And since $\tilde{\mu}_{l+1} + \ldots + \tilde{\mu}_{l+n} \leqslant Q_l(0) + \mu_{l+1} + \ldots + \mu_{l+n}$ for all ℓ and n, $\mathbf{P}(Q_l(0) \geqslant n\delta/3 \text{ for all } n \geqslant n_0) \geqslant 1 - 2\varepsilon$, which produces a contradiction.

Let us assume that $\mathbf{E}\mu_1 = \mathbf{E}\tilde{\mu}^1 + \delta$. Since $Q^{l+n}(0) + \tilde{\mu}^{l+1} + \ldots + \tilde{\mu}^{l+n} = Q_{l+n} + \tilde{\mu}_{l+1} + \ldots + \tilde{\mu}_{l+n} \geqslant \mu_{l+1} + \ldots + \mu_{l+n}$ with probability one on the set $\{\gamma < l\}$, then, $\mathbf{P}(Q^{l+n}(0) \geqslant n\delta/3 \text{ for all } n \geqslant n_0) \geqslant 1 - 2\varepsilon$ for any $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$, $l = l(\varepsilon)$. In particular, $\mathbf{P}\left(Q^{l+n}(0) \geqslant \frac{n\delta}{3}\right) = \mathbf{P}(Q^0(0) \geqslant n\delta/3) \geqslant 1 - 2\varepsilon$, which produces a contradiction. The corollary is proved.

Remark 1. In a perfectly analogous way, it is possible to prove that a system with an infinite number of channels is ergodic. For multichannel waiting systems and constrained systems (with failures, with a limited number of waiting places, with limited waiting time, etc.), an analogous result may be obtained only under the additional condition that there exist so-called renovating events.

2.2. Acyclic Networks. Consider an open acyclic network and for k = 0, 1,...,N define sequences of random variables $\{\mathbf{v}_n^k=(v_{n1}^k, v_{n2}^k, \ldots)\}$, and for k = 1,...,N, sequences $\{\mathbf{s}_n^k=(s_{n1}^k, s_{n2}^k, \ldots)\}$. Let us describe how the network functions. At its input there arrives the stream of calls $\{\tau_n, \mu_n, \mathbf{u}_n\}$ described in Sec. 2.1. For arbitrary n, a call that has arrived at time $t_{n-1} + u_{ni}$ is directed towards the station numbered v_{ni}^0. If $j \geqslant 1$ calls have arrived at station k within the time interval $(t_{n-1}, t_n]$, they are "assigned" service times $s_{n1}^k \ldots s_{nj}^k$. If servicing of $j \geqslant 1$ calls concludes within the time interval $(t_{n-1}, t_n]$ at station k, the first of these calls is directed towards station v_{n1}^k, the second towards station v_{n2}^k, and so on. We then set $\xi_n = \{\tau_n, \mathbf{u}_n, \mu_n, \{\mathbf{v}_n^k\}, \{\mathbf{s}_n^k\}\}$.

THEOREM 6. Suppose that an open acyclic network satisfies the following conditions:

(1) the sequence $\{\xi_n\}$ is stationary and metrically transitive;

(2) the sequences $\{(\tau_n, \mathbf{u}_n)\}$; $\{\mathbf{v}_n^0\}$; $\{\mathbf{s}_n^1\}$; $\{\mathbf{v}_n^1\}$; ...; $\{\mathbf{s}_n^N\}$; $\{\mathbf{v}_n^N\}$ are mutually independent;

(3) for fixed k = 0, 1, 2,...,N, the random variables $\{v_{nj}^k\}$ are identically distributed; $\mathbf{P}(v_{nj}^k = i) = p_{ki}$;

(4) for fixed k = 1, 2,...,N, the random variables $\{s_{nj}^k\}$ all process the same mean $\mathbf{E}s_{nj}^k = a_k > 0$.

Then if for all k,

$$\alpha\pi_k\mathbf{E}\mu_1 < 1/a_k, \tag{3}$$

the assertion of Theorem 3 will hold.

The proof may be conducted by means of induction. We partition the set of stations $\{1, 2, \ldots, N\}$ into subclasses $D_1 \cup D_2 \cup \ldots \cup D_l$, where $D_1 = \{i \geqslant 1\colon p_{ji} = 0$ for all $1 \leqslant j \leqslant N\}$, and for $1 \leqslant r \leqslant l-1$ $D_{r+1} = \{i \geqslant 1\colon p_{ji} = 0$ for all $j \notin D_1, \ldots, j \notin D_r\}$.

Consider $k \in D_1$. Over the time interval $(t_{n-1}, t_n]$ a total of $\tilde{\mu}_n^k \equiv \sum_{i \leqslant \mu_n} I(v_{ni}^0 = k)$ calls arrive at the k-th station, and $\mathbf{E}\tilde{\mu}_n^k = \mathbf{E}\mu_n \times \mathbf{P}(v_{11}^0 = k) = \mathbf{E}\mu_1 p_{0k}$; $\mathbf{E}\sum_{i \leqslant \tilde{\mu}_n^k} s_{ni}^k = a_k \mathbf{E}\tilde{\mu}_n^k = a_k \mathbf{E}\mu_1 p_{0k}$. Therefore, Lemma 1 holds, and the output stream from the k-th station may be glued to the stationary stream.

Consider $k \in D_2$. At the input of this station a stationary input stream arrives from outside with the set of output streams from stations $i \in D_1$ which, beginning with some random station number, are glued to the stationary stream. And since the union of stationary streams again forms a stationary stream, we may apply Lemma 1. It is then only necessary to verify that the conditions of the lemma are satisfied. In fact, if, for $j \in D_1$, we let $\mu_n^j(k)$ denote the number of calls that arrive at the k-th station from the j-th station over the time interval $(t_{n-1}, t_n]$, then (in the steady state) $\mathbf{E}\mu_n^j(k) = \mathbf{E}\tilde{\mu}_n^j \; p_{jk} = \mathbf{E}\mu_1 p_{0j} p_{jk}$. And since $\tilde{\mu}_n^k = \sum_{j \in D_1} \mu_n^j(k) + \sum_{i \leqslant \mu_n} I(v_{ni}^0 = k)$ we find that $\mathbf{E}\tilde{\mu}_n^k = \mathbf{E}\mu_1\left(p_{0k} + \sum_j p_{0j} \cdot p_{jk}\right) = \mathbf{E}\mu_1\pi_k$, and the conditions of Lemma 1 follow from (3). If $i = 3, \ldots, l$, $k \in D_i$, the reasoning is entirely analogous. The theorem is proved.

2.3. Acyclic Networks with Multichannel Stations. Let us show how to prove Theorem 4 for an acyclic network.

We begin by considering the m-channel controlled system described in Sec. 2.1, operating by the principle "first in—first out." As before, we set $w_n = \inf\{u \geqslant 0\colon Q_n(u) = 0\}$. The following, weaker version of Lemma 1 holds.

LEMMA 2. Suppose that the sequence $\{\xi_n\}$ is stationary and metrically transitive. If $m\mathbf{E}\tau_1 < \mathbf{E}S_1$ and $\mathbf{P}(w_0 \leqslant c) = 1$ for some c < ∞, a stationary sequence $\{Q^n\}$, $Q^n \in D_+^0$ with probability one may be found such that $\mathbf{P}(Q_n \leqslant Q^n$ for all $n \geqslant 0) = 1$.

The proof follows directly from the properties of monotonicity of multichannel systems and Lemma 2 from [7, p. 356].

Let us now consider acyclic Jackson-type networks and prove Theorem 4 for the following two special cases:

(1) so-called multiphase-multichannel systems;

(2) a single network with three stations.

We first remark that the proof may be carried out in analogous fashion for an arbitrary acyclic network.

We begin with case 1. A service network is called a multiphase-multichannel system if $p_{i,i+1} = 1$ for all $i = 0, 1, \ldots, N$ [i.e., the output stream from the i-th station forms the input stream of the (i + 1)-th station]. Note that a different type of proof (for other characteristics of systems) was given in [8].

Let us conduct the proof by means of induction on N. If N = 1, the network constitutes a single multichannel system GI/GI/m for which the assertion of Theorem 4 holds (cf. [7, p. 361]). Consider the case of N = 2 stations with m_1 and m_2 channels, respectively. We then require the concept of a renovating event, the definition of which may be found in [7, p. 340], for example. Since the output stream from the 1st station is glued to the stationary stream, beginning from some random time $\lambda < \infty$ with probability one, a stationary stream of calls will arrive at the second station. For any $\varepsilon > 0$, a number n_ε may be found such that $\mathbf{P}(\lambda \leqslant n_\varepsilon) \geqslant 1-\varepsilon$, and for this n_ε a function $f_\varepsilon \in D_+^0$ may be found such that $\mathbf{P}(Q_{n_\varepsilon}^2 \leqslant f_\varepsilon) \geqslant 1-\varepsilon$. Consider for $n \geqslant n_\varepsilon$ a network in which a stationary input stream arrives at the second station, and denote the characteristics of such a network by $\{\tilde{Q}_n^1, \tilde{Q}_n^2\}$. By Lemma 2 there exists a stationary sequence $\{\tilde{Q}^{ni};\ n \geqslant n_\varepsilon\}$ such that $\tilde{Q}_n^i \leqslant \tilde{Q}^{ni}$ with probability one for all $n \geqslant n_\varepsilon$, $i = 1, 2$.

Consider for $n \geqslant n_\varepsilon$ the events

$$B_n = \{\tilde{Q}^{ni}(0) \leqslant r;\quad \tilde{w}^{ni} \leqslant x;\ i = 1, 2\}.$$

These events form a stationary sequence and possess positive probability for sufficiently large r and x. Consider, also, the events

$$C_n = C_n(r, x, \varepsilon, \delta) = \bigcap_{i=0}^{L} \{\tau_{n+i} \geqslant d;\ m_1 d - \varepsilon - \delta \leqslant s_{n+i}^1 \leqslant m_1 d - \varepsilon;\ s_{n+i,j}^2 \leqslant m_2 d - \varepsilon;\ j = 1, 2, \ldots, r+i\}.$$

From the conditions of the theorem, it follows that numbers d and ε may be found such that event C_n possesses positive probability for any $\delta > 0$ and $L > 0$. Moreover, the events $\{C_n\}$ are stationary and L-dependent, while the events B_n and C_n are independent for any n.

If $L \gg 1$, $\delta \ll 1$, in the event $A_n = B_n \cap C_n$ all calls beginning with some call with number $n_1 = n_1(r, x, d, \varepsilon, \delta)$ arrive at a free device at the first station. We let γ_n denote time when servicing of the n-th call concludes at the first station and, for $n+i \geqslant n_1$, compare the times when servicing of two neighboring calls terminates: $\gamma_{n+i+1} = t_{n+i+1} + s_{n+i+1}^1 \geqslant \gamma_{n+i} + \tau_{n+i+1} - \delta$, and analogously, $\gamma_{n+i+1} \leqslant \gamma_{n+i} + \tau_{n+i+1} + \delta$. Consequently, for $d > \delta$, $n+i \geqslant n_1$ in the event A_n, calls exit from the first system in their order of arrival at the system over time intervals τ'_{n+i}, $\tau_{n+i} - \delta \leqslant \tau'_{n+i} \leqslant \tau_{n+i} + \delta$. Therefore, if $\delta < \varepsilon$, in the case of event A_n all calls beginning with some number n_2 arrive at a free device at the second system. Consequently, a number $L \gg 1$ may be specified such that event A_n will be a renovating event on (n, n + L) for the entire network. From [7, p. 341] it follows that there exists a stationary sequence $\{Q^{n1}, Q^{n2}\}$ such that $\mathbf{P}((\tilde{Q}_k^1, \tilde{Q}_k^2) = (Q^{k1}, Q^{k2})$ for all $k \geqslant n) \to 1$. Therefore, the original network satisfies the inequality

$$\liminf_{n\to\infty} \mathbf{P}\left((Q_k^1, Q_k^2) = (Q^{k1}, Q^{k2}) \text{ for all } k \geqslant n\right) \geqslant 1 - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily chosen, we obtain the desired assertion. The proof for N > 2 may be carried out by induction in an entirely analogous fashion.

Consider case 2. More specifically, consider a network of three stations (systems) with m_1, m_2, and m_3 channels, respectively, such that $p_{01} > 0$, $p_{02} > 0$, $p_{01} + p_{02} = 1$, $p_{13} = p_{23} = 1$.

Repeating the arguments presented in case 1, it may be verified that the output streams from the first and second stations may be glued to corresponding stationary streams, and, consequently, the input stream to the third station, understood as a superposition of these output streams, also may be glued to a stationary stream. Denoting the time this occurs by λ, we find for ε a number n_ε and function f_ε such that $\mathbf{P}(\lambda \leqslant n_\varepsilon) \geqslant 1-\varepsilon$ and $\mathbf{P}(Q_{n_\varepsilon}^3 \leqslant f_\varepsilon) \geqslant 1-\varepsilon$. Consider a network in which when $n \geqslant n_\varepsilon$, a stationary input stream arrives at the third station, and denote its characteristics by $(\tilde{Q}_n^1, \tilde{Q}_n^2, \tilde{Q}_n^3)$. We introduce the events $B_n = \{\tilde{Q}^{ni}(0) \leqslant r;\ \tilde{w}^{ni} \leqslant x;\ i = 1, 2, 3\}$ and C_n, which are defined in somewhat different fashion. Take numbers

k_1, $k_2 \gg 1$ such that $p_{01} \approx k_1/(k_1+k_2)$. Recall that ν_n^0 denotes the number of the station to which the n-th call from the input stream is directed, $\mathbf{P}(\nu_n^0 = i) = p_{0i}$, $i = 1, 2$. We set

$$C_n = C_n(r, x, d, \varepsilon, \delta) = \bigcap_{i=0}^{L} \{\tau_{n+i} \geqslant d;\ \nu_{n+i}^0 = h_i;$$
$$(m_k d - \varepsilon - \delta)/p_{0k} \leqslant s_{n+i}^k \leqslant (m_k d - \varepsilon)/p_{0k};\quad k = 1,2;$$
$$s_{n+i,j}^3 \leqslant m_3 d - \varepsilon;\ j = 1, 2, \ldots, 2r + i\}.$$

Here $h_i = 1$ whenever $l(k_1+k_2)+1 \leqslant i \leqslant l(k_1+k_2)+k_1$ and $h_i = 2$ whenever $l(k_1+k_2)+k_1+1 \leqslant i \leqslant (l+1)(k_1+k_2)$, $l = 0, 1, 2, \ldots$. In other words, in the case of event C_n the first (beginning with n) k_1 calls arrive at the first station, the next k_2 calls at the second station, and the next k_1 at the first station, and so on.

By means of direct computations, it is easily shown that if $k_1 \gg 1$ and $k_2 \gg 1$, the events $A_n = B_n \cap C_n$ are renovating events for the construction of the network. Now we only have to repeat the corresponding arguments presented in case 1.

The proof of Theorem 4 for an arbitrary acyclic Jackson-type network may be obtained by means of induction using the required number of iterations of the constructions presented above.

3. Proof of Theorems 3 and 4

3.1. Monotonicity Properties. We will require an elementary arithmetic assertion.

LEMMA 3. Suppose that (1) holds for all $j = 1,\ldots,N$. Then numbers $b_j > a_j$ may be found such that

$$\alpha\pi_j < 1/b_j, \tag{4}$$
$$\sum_{k=1}^{N} p_{kj}/b_k + \alpha p_{0j} < 1/b_j \tag{5}$$

for all $j = 1,\ldots,N$.

Proof. It is known that the numbers $\{\pi_j\}$ satisfy the relations

$$\pi_j = p_{0j} + \sum_{k=1}^{N} \pi_k p_{kj}.$$

If $k, j = 1,\ldots,N$, we set $p'_{kj} = p_{kj} + p_{k,N+1}/N$. Consider a Markov chain with N states $\{1, 2,\ldots, N\}$ and transition matrix $P' = \|p'_{kj}\|$. By definition there exists at least one k such that $p_{k,N+1} > 0$ and, consequently, $p'_{kj} > 0$ for all $j = 1, 2,\ldots,N$. Therefore, all the states of the particular chain are absorbing and essential, and the chain is aperiodic. Consequently, there exists a unique ordered sequence of numbers $\{\beta_k, k = 1,\ldots,N\}$ (called an invariant measure) such that $\sum \beta_k = 1$ and $\beta_j = \sum_{k=1}^{N} \beta_k p'_{kj}$ for all j. And since for any fixed j the strict inequality $p_{kj} < p'_{kj}$ holds for at least one k, we have $\beta_j > \sum_k \beta_k p_{kj}$ for all j.

Take $\varepsilon > 0$ so small that $\alpha\pi_j + \varepsilon\beta_j < 1/a_j$ holds for all j, and define the numbers $\{b_j\}$ from the equalities $b_j = (\alpha\pi_j + \varepsilon\beta_j)^{-1}$. By definition $b_j > a_j$, and (4) holds. Moreover,

$$\sum_k p_{kj}/b_k + \alpha p_{0j} = \alpha \sum_k \pi_k p_{kj} + \varepsilon \sum_k \beta_k p_{kj} + \alpha p_{0j} = \alpha\pi_j + \sum_k \varepsilon\beta_k p_{kj} < \alpha\pi_j + \varepsilon\beta_j = b_j^{-1}.$$

The lemma is proved.

Now consider an open network with nonrandom characteristics which may be conceived as a realization of a Jackson-type network with a single elementary outcome. As before, denote by $t_n = \tau_1 + \ldots + \tau_n$ the time the n-th call arrives, by ν_n the number of the station to which this call is directed, and by T_i^k the time the i-th call from the input stream $T_i^k = t_{\delta_i^k}$ arrives at the k-th station, where $\delta_1^k = \min\{n \geqslant 1\colon \nu_n = k\}$ and $\delta_{i+1}^k = \min\{n > \delta_i^k\colon \nu_n = k\}$ for $i = 1, 2, \ldots$. By s_j^k we denote the time it takes to service the j-th call at the k-th station, and by ν_j^k the number of the station to which the call is directed after being serviced ($\nu_j^k = N + 1$ if this call exits the network).

The behavior of the service network Σ over the entire time interval $[0, \infty)$ may be uniquely determined by the sequence $\{(T_i^k, s_j^k, \nu_j^k);\ k = 1, \ldots, N;\ i, j = 1, 2, \ldots\}$ introduced above as long as

$$\sum_{j=1}^{\infty} s_j^k = \infty \tag{6}$$

for any k = 1,...,N. Note that (6) holds with probability one for Jackson-type networks.

We will use the notation $(x_i^k) \leqslant (y_i^k)$ for two number sequences (x_i^k) and (y_i^k) if $x_i^k \leqslant y_i^k$ for all i, k, and $(x_i^k) = (y_i^k)$ if $x_i^k = y_i^k$ for all i, k

We let V_j^k denote the time when servicing of the j-th call at the k-th station concludes. If for fixed k, we consider only times when calls exit the network after having been serviced at the k-th station, then V_j^{k0} will be understood as the time the j-th call exits.

In [2] the following assertion was proved.

<u>LEMMA 4.</u> Consider two service networks

$$\Sigma = \{(T_i^k)\ (s_j^k), (\nu_j^k)\} \text{and } \tilde{\Sigma} = \{(\tilde{T}_i^k);\ (\tilde{s}_j^k);\ (\tilde{\nu}_j^k)\},$$

that satisfy (6). If

$$\text{a) } (T_i^k) \leqslant (\tilde{T}_i^k), \text{ b) } (s_j^k) \leqslant (\tilde{s}_j^k), \text{ c) } (\nu_j^k) = (\tilde{\nu}_j^k),$$

then $(V_j^k) \leqslant (\tilde{V}_j^k)$ and $(V_j^{k0}) \leqslant (\tilde{V}_j^{k0})$.

We will require a number of corollaries from this lemma. We introduce certain additional characteristics. For a network Σ and number t we introduce an auxiliary network Σ(t) by the rule

$$(s_j^k(t)) = (s_j^k), \quad (\nu_j^k(t)) = (\nu_j^k),$$

$$T_i^k(t) = \begin{cases} T_i^k, & \text{if } \quad T_i^k \leqslant t, \\ \infty & \text{otherwise}. \end{cases}$$

In other words, in the auxiliary network we "terminate" the process in which calls arrive at time t.

If $u \geqslant 0$ we let Q(t, u) denote the number of calls into the network Σ(t) at time t + u + 0. Note that $\{Q(t, u): u \geqslant 0\} \in D_+^0$ for any fixed t.

<u>COROLLARY 1.</u> Consider two service networks Σ and $\tilde{\Sigma}$. If

$$\text{a) } (T_i^k) = (\tilde{T}_i^k), \quad \text{b) } (s_j^k) \leqslant (\tilde{s}_j^k), \quad \text{c) } (\nu_j^k) = (\tilde{\nu}_j^k),$$

then $Q(t, u) \leqslant \tilde{Q}(t, u)$ for all t and u.

In fact, consider the networks Σ(t) and $\tilde{\Sigma}$(t) and note that

$$Q(t, u) = \sum_{k,i} I(T_i^k \leqslant t) - \sum_{k,j} I(V_j^{k0}(t) \leqslant t + u).$$

Since $(T_i^k) = (\tilde{T}_i^k)$, and by Lemma 4, $(V_j^{k0}(t)) \leqslant (\tilde{V}_j^{k0}(t))$, we have that $Q(t, u) \leqslant \tilde{Q}(t, u)$.

As before, let $Q_n(u) = Q(t_n, u)$.

<u>COROLLARY 2.</u> Consider two service networks Σ and $\tilde{\Sigma}$. If

$$\text{a) } (\tau_i) \leqslant (\tilde{\tau}_i), \quad \text{b) } (\nu_i) = (\tilde{\nu}_i), \quad \text{c) } (s_j^k) = (\tilde{s}_j^k), \quad \text{d) } (\nu_j^k) = (\tilde{\nu}_j^k),$$

then $Q_n(u) \geqslant \tilde{Q}_n(u)$ for all n and u.

In fact, it suffices to note that "contraction" of the intervals between the times that cells arrive is equivalent to "dilatation" of the service times.

Finally, there is one more corollary that will be useful to us. Consider a network Σ and "color" all the calls arriving in this network in white. Consider another stream of calls $\{(\hat{T}_i^k), (\hat{s}_j^k), (\hat{\nu}_j^k)\}$, which we call the "red" stream. We form a new network $\tilde{\Sigma}$ which both streams of calls enter. Network $\tilde{\Sigma}$ functions in the following way. Calls are serviced in their order of arrival, independently of color. The number of service actions is computed separately for each color, and if a red (white) call arrives at the k-th station as the j-th of all red (white) calls, it "receives" service time $\hat{s}_j^k$ (respectively, s_j^k), and following the conclusion of the service is directed to station $\hat{\nu}_j^k$ (respectively, ν_j^k). Let $\tilde{Q}$(t, u) denote the total number of calls in $\tilde{\Sigma}(t)$ at time t + u + 0, and $\tilde{Q}_1(t, u)$ denote the number of white calls at the same time in the network.

COROLLARY 3. For the networks Σ and $\tilde{\Sigma}$ just introduced, $Q(t, u) \leqslant \tilde{Q}(t, u)$ for all t, u $\geq$ 0.

In fact, we need only note that, by Lemma 4, $Q(t, u) \leqslant \tilde{Q}_1(t, u)$ for all t and u.

3.2. Proof of Theorem 3. Note that the sequence (q_n, χ_n) forms a homogeneous Markov chain. Therefore, as follows from [9], to prove Theorem 3 it suffices to verify that there exists a natural probability distribution $\mathbf{P}(\cdot)$ on $Z_+^N \times R_+^N$ such that for any initial state (q_0, χ_0),

$$\sup |\mathbf{P}((q_n, \chi_n) \in B) - \mathbf{P}(B)| \to 0 \text{ as } n \to \infty,$$

where the supremum is taken over all Borel sets $B \subseteq Z_+^N \times R_+^N$.

Let $Q_n^k(u)$ denote the number of calls in $\Sigma(t)$ at the k-th device at time t_n + u + 0, so that $Q_n(u) = \sum\limits_k Q_n^k(u)$.

Consider the following method of specifying a family of networks in a common probability space. We will assume that calls may possess either positive (n > 0) or nonpositive $(n \leqslant 0)$ number. To each call numbered n that has arrived in the network, we assign a set $G_n = \{\nu_n; \{s_{nj}^k\}; \{\nu_{nj}^k\}\}$ of independent random variables where, as before, ν_n denotes the number of the device to which a call is directed at the time it arrives; s_{nj}^k, the time it takes to service the j-th call at the k-th device; and ν_{nj}^k, the number of the device to which the call is directed after this servicing event. Denote as well by δ_n^k the total number of times the n-th call is serviced at the k-th device, $\mathbf{E}\delta_n^k = \pi_k$. We also introduce a sequence of independent identically distributed variables $\{\tau_n, -\infty < n < \infty\}$.

For given N, denote by $\mathscr{F}_n^N$ a σ-algebra of sets generated by the random variables $\{(\tau_i, G_i), i \leqslant n\}$; $\mathscr{F}^N = \mathscr{F}_\infty^N$; and by U a metrically transitive measure-preserving shift transformation of $\mathscr{F}^N$-measurable random variables, so that $(\tau_{n+1}, G_{n+1}) = U(\tau_n, G_n)$, finally let T denote the corresponding transformation of sets from $\mathscr{F}^N$. The symbols T^n and U^n, $-\infty < n < \infty$, will denote the n-th iteration of T and U, respectively.

If $m \geqslant 0$, $-\infty < n < \infty$, we construct a set of service networks Σ_m^n in a common probability space, letting $\hat{Q}_{m,n}^k(u)$ denote the number of calls at the k-th station in Σ_m^n at time t_n + u + 0. Let $\hat{Q}_{m,n}(u) = (\hat{Q}_{m,n}^1(u), \ldots, \hat{Q}_{m,n}^N(u))$ and $Q_{m,n}(u) = \sum\limits_k \hat{Q}_{m,n}^k(u)$.

We begin with case n = 0, and define the network Σ_m^0 by induction.

Into Σ_0^0 there arrives a single call numbered 0 at time t_0 = 0, and into Σ_1^0 two calls numbered 0 and (−1), respectively, at times 0 and $t_{-1} = -\tau_0$. From Corollary 3 it is easily deduced that $Q_{10}(u) \geqslant Q_{00}(u)$ with probability one for all $u \geqslant 0$. In the network Σ_1^0 we introduce certain new notation, i.e., denoting by $\tilde{s}_{1j}^k$ the time when the k-th station services its j-th call (where the number of the call is arbitrary), and by $\tilde{\nu}_{1j}^k$ the number of the device to which the call is directed after having been serviced there. Formally speaking, the random variables $\tilde{s}_{1j}^k$ and $\tilde{\nu}_{1j}^k$ are defined only when $1 \leqslant j \leqslant \delta_0^k + \delta_{-1}^k$. Without any loss in generality, however, we may extend their definitions for $j > \delta_0^k + \delta_{-1}^k$ by means of two arbitrary ordered sequences of independent random variables. Let $\tilde{G}_{-1} = \{\{\nu_0, \nu_{-1}\}, \{\tilde{s}_{1j}^k\}, \{\tilde{\nu}_{1j}^k\}\}$ denote the control in Σ_1^0. Here, as before, $\{\tilde{s}_{1j}^k\}$ and $\{\tilde{\nu}_{1j}^k\}$ are sequences of independent identically distributed random variables.

Consider a network Σ_2^0 into which there arrive two ordered sequences of calls, calls numbered {0, −1} with control $\tilde{G}_{-1}$ and a call numbered (−2) with control G_{-2} which arrives at time $t_{-2} = -\tau_{-1} - \tau_0$. From Corollary 3 it follows that $Q_{20}(u) \geqslant Q_{10}(u)$ with probability one for all u $\geq$ 0. In Σ_2^0 we introduce, as before, a new service enumeration $\{\tilde{s}_{2j}^k\}$ and enumeration of transitions $\{\tilde{\nu}_{2j}^k\}$, i.e., we specify the control $\tilde{G}_{-2}$, after which we define the network Σ_2^0 in which the two ordered sequences of calls arrive, i.e., calls numbered {0, −1, −2} with control $\tilde{G}_{-2}$ and a call numbered (−3) with control $\tilde{G}_{-3}$, which arrives at time $t_{-3} = -\tau_{-2} - \tau_{-1} - \tau_0$, and so on.

Thus, for every $m \geqslant 0$ we have defined a service network Σ_m^0 into which there arrive (m + 1) calls numbered −m, −m + 1,...,−1, 0 with arrival times $t_{-i} = -\tau_{-i+1} - \ldots - \tau_0$, while $Q_{m+1,0}(u) \geqslant Q_{m0}(u)$ with probability one for all $u \geqslant 0$ and m = 0, 1,... .

We repeat the above construction for every n, $-\infty < n < \infty$, assuming that for any fixed n there arrive in Σ_m^n exactly $(m+1)$ calls numbered $i = n-m, n-m+1, \ldots, n$ at times t_i, where for $n < 0$

$$t_i = -\sum_{j=0}^{n-i+1} \tau_j$$

while for $n > 0$

$$t_i = \begin{cases} \sum_{j=1}^{i} \tau_j & \text{if} \quad i > 0, \\ 0 & \text{if} \quad i = 0, \\ -\sum_{j=i+1}^{0} \tau_j & \text{if} \quad i = 0 \end{cases}$$

From the construction it is clear that the random elements $\hat{Q}_{m,n}^k$ satisfy the relations $\widehat{Q}_{m,n+l}^k = U^l \widehat{Q}_{m,n}^k$ for all k, $1 \leqslant k \leqslant N$, $m \geqslant 0$, $-\infty < n, l < \infty$. We have the following assertion:

LEMMA 5. Suppose that (1) holds. Then there exists a random element $\widetilde{Q}^0 \in D_+^0$ with probability one such that $Q_{m,0} \leqslant \widetilde{Q}^0$ with probability one for all $m \geqslant 0$.

Let us present one corollary. Given a network Σ_m^n we denote by $q_{m,n}^k$ the number of calls at the k-th device at time $t_n + 0$, and by $\chi_{m,n}^k$ the time left to service the call being serviced at this time at the k-th device, with $\widetilde{Q}^n = U^n \widetilde{Q}^0$, $\tilde{w}^n = \inf\{u \geqslant 0\colon \widetilde{Q}^n(u) = 0\}$, $w_{m,n} = \inf\{u \geqslant 0\colon Q_{m,n}(u) = 0\}$. From Lemma 5 it follows that since $\sum_k q_{m,n}^k = Q_{m,n}(0) \leqslant \widetilde{Q}^n(0)$ with probability one, $q^{k,n} \equiv \sup_m q_{m,n}^k < \infty$ with probability one, and since $\chi_{m,n}^k \leqslant w_{m,n} \leqslant \widetilde{w}^n$ with probability one, $\widetilde{\chi}^{k,n} \equiv \sup_m \chi_{m,n}^k < \infty$ with probability one. Moreover, if $\gamma_{m,n}$ denotes the number of service events in Σ_m^n which begin following time t_n, it follows from the preceding relations and the strong law of large numbers that $\gamma^n \equiv \sup_m \gamma_{m,n} < \infty$ with probability one.

Further, note that if in the original network the initial conditions are null (i.e., $q_0 = \chi_0 = 0$), then the joint distribution of the random variables $(\chi_{m,0}^k, \widehat{Q}_{m,0}^k; 1 \leqslant k \leqslant N)$ and $(\chi_{m+1}^k, Q_{m+1}^k, 1 \leqslant k \leqslant N)$ are the same for any m. Therefore, speaking not very rigorously, however, the assertion of Lemma 5 may be restated as follows: For networks with null initial conditions there exists a stationary majorant. Analogously, a stationary majorant may be constructed given initial conditions lying, with probability one, in some compactum. Consider, for example, the case $q_0^i \equiv 1$, $q_0^j = 0$, $j \neq i$.

For $m \geqslant 0$ we will consider a network Σ_m^0 into which there arrives at time $t_{-m} - 0$ an additional single call with control

$$G_{-m-1}^1 = \{\nu_{-m-1}^0; \{s_{-m-1,j}^k\}, \{\nu_{-m-1,j}^k\}, k \neq i; \{\chi_0^i, s_{-m-1,j}^i\}_{j \geqslant 2}; \{\nu_{-m-1,j}^i\}\},$$

where $\nu_{-m-1}^0 \equiv i$, $\chi_0^i \leqslant c$ with probability one is the first service time and, as before $\{s_{-m-1,j}^k\}$ and $\{\nu_{-m-1,j}^k\}$ are sequences of independent identically distributed random variables. Denote the new network by $\Sigma_{m,1}^0$.

Together with Σ_m^0 we consider for ε, $0 < \varepsilon < 1$, a network $\Sigma_m^{0,\varepsilon}$ that differs from Σ_m^0 only by the fact that in $\Sigma_m^{0,\varepsilon}$ calls numbered $i = 0, -1, \ldots, -m$ arrive at times $t_i^\varepsilon = (1-\varepsilon)t_i$. We select $\varepsilon > 0$ to be so small that condition (1) remains valid when $\alpha = (\mathrm{E}\tau_1)^{-1}$ is replaced by $\alpha^\varepsilon = [(1-\varepsilon)\mathrm{E}\tau_1]^{-1}$.

Note that by Corollary 2 the network $\Sigma_{m,1}^{0,\varepsilon}$ (which is obtained from $\Sigma_m^{0,\varepsilon}$ by adding at time $t_{-m} - 0$ the call with control G_{-m-1}^1) majorizes, in a certain sense, the network $\Sigma_{m,1}^0$. Now select m to be so large that $c < \varepsilon|t_{-m-1}|$. We define the network $\Sigma_{m,2}^{0,\varepsilon}$ obtained from $\Sigma_m^{0,\varepsilon}$ by the addition at time $(1-\varepsilon)t_{-m-1}$ of the call with control $G_{-m-1}^2 = \{\nu_{-m-1}^0; \{s_{-m-1,j}^k\}; \{\nu_{-m-1,j}^k\}; j \geqslant 1\}$. By Corollary 2 the network $\Sigma_{m,2}^{0,\varepsilon}$ majorizes $\Sigma_{m,1}^{0,\varepsilon}$ if $c < \varepsilon|t_{-m-1}|$.

Finally, consider the network $\Sigma_{m,3}^{0,\varepsilon}$ in which an additional call arrives at time $t = 0$ and which possess control $G_0^3 = \{i; \{s_{0,j}^k\}; \{\nu_{0,j}^k\}\}$, while the remaining calls possess controls $G_l^3 = \{\nu_{-l-1}; \{s_{-l-1,j}^k\}; \{\nu_{-l-1,j}^k\}\}$ for $-m \leqslant l \leqslant 0$. Then, by Corollary 2, the network $\Sigma_{m,3}^{0,\varepsilon}$ majorizes the network $\Sigma_{m,2}^{0,\varepsilon}$. We let $Q_{m,0,\varepsilon}$, $q_{m,0,\varepsilon}$, $w_{m,0,\varepsilon}$ denote the characteristics of $\Sigma_{m,3}^{0,\varepsilon}$. Since $c \geqslant \varepsilon|t_m|$

holds, with probability one, for a finite ordered sequence of numbers m, it remains for us to show that there exists an element $\tilde{Q}^{0,\varepsilon} \in D^0_+$ with probability one such that $Q_{m,0,\varepsilon} \leqslant \tilde{Q}^{0,\varepsilon}$ for all m. By Corollary 3, we have $Q_{m,0,\varepsilon} \leqslant Q_{m+1,0,\varepsilon}$ with probability one for all m. Therefore, it suffices to show that $\sup_n \mathbf{P}(q_{m,0,\varepsilon} > x) \to 0$ and $\sup_n \mathbf{P}(w_{m,0,\varepsilon} > x) \to 0$ as $x \to \infty$. But this result follows at once from the proof of Lemma 5, which we will now present.

Proof of Lemma 5. By the remarks given above, it suffices to show that $\sup_n \mathbf{P}(q_n > x) \to 0$ and $\sup_n \mathbf{P}(w_n > x) \to 0$ as $x \to \infty$, where q_n and w_n are characteristics of the original service network introduced in Sec. 1. The latter result will, in turn, hold if we can prove that

$$\sup_n \mathbf{P}(q_n > x) \to 0 \text{and} \sup_n \max_{1 \leqslant k \leqslant N} \mathbf{P}(\chi^k_n > x) \to 0$$

as $x \to \infty$. Let us introduce, as in Sec. 3.1, a sequence of processes $\{Q_n(u)\}$ and let $q_n = Q_n(0)$; $w_n = \inf\{u \geqslant 0\colon Q_n(u) = 0\}$. Consider the network ${}_1\Sigma$ which differs from the original network only in service times: ${}_1s^k_j = s^k_j + (\alpha\pi_k + \varepsilon\beta_k)^{-1} - a_k > s^k_j$ with probability one, where the numbers ε and β_k are defined in Lemma 3. From Lemma 3 it follows that $b_k = \mathbf{E}_1 s^k_j$ satisfies (4) and (5). Moreover, by Corollary 1, ${}_1Q_n(u) \geqslant Q_n(u)$ with probability one for all n, u.

Take numbers $h > 0$ and $N \geqslant 1$, and define the random variables $\{{}_2\tau_i\}$ and $\{{}_2s^k_j\}$ by the rule

$$ {}_2\tau_i = \begin{cases} lh & \text{if} \quad nh \leqslant \tau_i \leqslant (n+1)h, \quad n < N, \\ Nh & \text{if} \quad \tau_i \geqslant Nh, \end{cases} $$

$$ {}_2s^k_j = (l + 1 + \eta^k_j)h \quad \text{if} \quad lh < {}_1s^k_j \leqslant (l+1)h, $$

where $\{\eta^k_j\}$ is a sequence of independent identically distributed random variables that do not together depend upon $\{\{\tau_n\}, \{s^k_j\}, \{\nu_n\}, \{\nu^k_j\}\}$; $\mathbf{P}(\eta^k_j = 1) = \mathbf{P}(\eta^k_j = 0) = 1/2$.

Consider the network ${}_2\Sigma = \{({}_2\tau_n); (\nu_n); ({}_2s^k_j); (\nu^k_j)\}$.

It is clear that by choosing a sufficiently small number $h > 0$ and sufficiently large $N \geqslant 1$, (4) and (5) will be satisfied for $a = \mathbf{E}_2\tau_n$ and $b_k = \mathbf{E}_2 s^k_j$. Moreover,

$$\text{G.C.D.}\{n \geqslant 1\colon \mathbf{P}({}_2s^k_1 = nh) > 0\} = 1$$

for any $k = 1,\ldots,N$. Applying Corollaries 1 and 2 of Lemma 4 in succession, we find that ${}_2Q_n(u) \geqslant {}_1Q_n(u)$ for all $n \geqslant 1$ and $u \geqslant 0$.

Finally, construct a network ${}_0\Sigma$ which majorizes ${}_2\Sigma$ by means of the construction set forth below. The essence of the construction is as follows. At every time ${}_2t_n = {}_2\tau_1 + \ldots + {}_2\tau_n$ a certain random number of calls is added from outside to each station of the network so that within the time interval $({}_2t_n, {}_2t_{n+1}]$ there are no down times in the corresponding station, i.e., so that there is always at least one call at that station.

Formally speaking, the construction has the following form. For n = 1 and k = 1,...,N, we let ${}_2w^k_1$ denote the time remaining to service calls that have arrived at the k-th device up until time ${}_2t_1 + 0$. Consider for every k a sequence of independent identically distributed random variables $\{\hat{s}^k_j\}$ that do not depend upon those already introduced and such that $\hat{s}^k_j \overset{D}{=} {}_2s^k_1$. Set $j^k_1 = \min\{n \geqslant 1\colon \hat{s}^k_1 + \ldots + \hat{s}^k_n \geqslant {}_2\tau_2\}$.

If ${}_2w^k_1 < {}_2\tau_2$, at time ${}_2t_1$ an additional γ^k_1 calls are directed to station k to which service times $\hat{s}^k_1, \hat{s}^k_2, \ldots$, respectively, are assigned. All the additional calls will be assumed to be "red" (cf. Corollary 3). If a sequence of random variables that describes the transitions from station to station and the lengths of subsequent servicing of red calls is specified in a common probability space with the previously introduced random variables, we find from Corollary 3 that the new network "majorizes" the preceding network. In particular, we may assume that the probability distribution of the transitions and the service lengths for the red calls is the same as for the while calls.

In the new service network, ${}_2w^k_2$ denotes the remaining service time for calls that have arrived at the k-th device prior to time ${}_2t_2 + 0$, and we add to those stations for which ${}_2w^k_2 < {}_2\tau_3$ new "batches" of red calls in accordance with the procedure described above. Analogous additions are made for n = 3, 4,... . The network thus constructed is denoted ${}_0\Sigma$. Meanwhile ${}_0Q_n(u) \geqslant Q_n(u)$ with probability one for all $n \geqslant 1$ and $u \geqslant 0$.

We let $\{{}_0w_n, {}_0q_n, {}_0q_n^k, {}_0\chi_n^k\}$ denote the characteristics of ${}_0\Sigma$. It suffices to show that $\sup_n \mathbf{P}({}_0q_n^k > x) \to 0$ and $\sup_n \mathbf{P}({}_0\chi_n^k > x) \to 0$ as $x \to \infty$. From here it follows that $\sup_n \mathbf{P}({}_0q_n > x) \to 0$ and $\sup_n \mathbf{P}({}_0w_n > x) \to 0$ as $x \to \infty$. The latter relations together with the fact that ${}_0q_n \geqslant q_n$ with probability one and ${}_0w_n \geqslant w_n$ with probability one prove the lemma.

Let us fix $1 \leqslant k \leqslant N$ and consider the operation of each k-th station in ${}_0\Sigma$ taken separately. We let $\psi_1^k, \psi_2^k, \ldots$ denote the successive service times for calls at this station. The sequence $\{\psi_i^k\}$ consists of independent identically distributed random variables $\psi_1^k \overset{D}{=} {}_2s_1^k$ and is independent of $\{{}_2\tau_i\}$. Moreover, the random variables $\{\psi_i^k\}$ and $\{{}_2\tau_i\}$ are lattice variables with step h. Therefore (cf. [10]), there exists a stationary sequence $\{\chi^{nk}\}$ such that $\mathbf{P}({}_0\chi_l^k = \chi^{lk} \text{ for all } l \geqslant n) \to 1$ as $n \to \infty$ that is specified in a common probability space with $\{\psi_i^k\}$ and $\{{}_2\tau_i\}$. The random variables χ^{nk} have distribution

$$\mathbf{P}(\chi^{nk} \geqslant jh) = \left(\sum_j^\infty \mathbf{P}(\psi_1^k \geqslant jh)\right)\Big/\mathbf{E}\psi_1^k,$$

and $\chi^{n1}, \ldots, \chi^{nN}$ are independent for any $n = 1, 2, \ldots$. In fact, if the initial conditions $\chi^{11}, \ldots, \chi^{1N}$ are specified to be independent, then

$$\mathbf{P}(\chi^{21} \in B_1, \ldots, \chi^{2N} \in B_N) = \sum \mathbf{P}({}_2\tau_2 = lh)\, \mathbf{P}(\chi^{21} \in B_1, \ldots$$

$$\ldots, \chi^{2N} \in B_N / {}_2\tau_2 = lh) = \sum \mathbf{P}({}_2\tau_2 = lh)\, \mathbf{P}(\chi^{11} \in B_1, \ldots, \chi^{1N} \in B_N) = \mathbf{P}(\chi^{11} \in B_1) \cdot \ldots \cdot \mathbf{P}(\chi^{1N} \in B_N).$$

We let $\lambda = \min\{n \geqslant 1 : {}_0\chi_n^k = \chi^{nk} \text{ for all } k = 1, \ldots, N\}$. Since $\lambda < \infty$ with probability one, $\sup_{n \leqslant \lambda} {}_0Q_n \in D_+^0$ also. For any $\varepsilon > 0$, a number n_ε may be found such that $\mathbf{P}(\lambda < n_\varepsilon) \geqslant 1 - \varepsilon$. We consider for $n \geqslant n_\varepsilon$ the network ${}_0\Sigma$ with initial (at time t_{n_ε}) state $\{({}_0q_n^k, \chi^{nk});\ 1 \leqslant k \leqslant N\}$. By simplifying the notation, we perform a left time shift by n_ε units, i.e., we set $n_\varepsilon = 0$. Let μ_l^k denote the number of service events that are completed over the time interval $(t_{l-1}, t_l]$ at the k-th station, $\mu_l^k(i)$ the number of calls that have arrived at the i-th station from the k-th station during this time, and $\{\psi_{i,l}^k;\ 1 \leqslant i \leqslant \mu_l^k\}$ the corresponding service lengths. Note that the family $\{\psi_{i,l}^k;\ 1 \leqslant i < \infty, l \geqslant 1\}$ consists of independent identically distributed random variables and that the sequences $\{\mu_l^k, l \geqslant 1\}$ and $\{\mu_l^k(i);\ l \geqslant 1\}$ are stationary and metrically transitive for any k and i. Therefore,

$$\mathbf{E}\mu_l^k = \frac{\mathbf{E}_2\tau_2}{\mathbf{E}_2 s_1^k} = \frac{1}{\alpha b_k}, \quad \mathbf{E}\mu_l^k(i) = \mathbf{E}\mu_l^k p_{ki} = \frac{p_{ki}}{\alpha b_k},$$

where the numbers α and b_k satisfy (4) and (5). Without any loss of generality, it may be assumed that the random variables $\mu_l^k, \mu_l^k(i), \nu_l$ are defined for all ℓ, $-\infty < \ell < \infty$.

For our ε we find a number $x_\varepsilon > 0$ such that $\mathbf{P}({}_0q_0^k \leqslant x_\varepsilon \text{ for all } k) \geqslant 1 - \varepsilon$. We let $\eta_n^k = \sum_j \mu_n^j(k) + I(\nu_n = k) - \mu_n^k$. Note that for any k, the $\{\eta_n^k\}$ form a stationary metrically transitive sequence and $\mathbf{E}\eta_n^k < 0$ by (5). Further, for any k the numbers ${}_0q_n^k$ are related by the formulas

$${}_0q_{n+1}^k = {}_0q_n^k + \eta_n^k + j_{n+1}^k I({}_0w_{n+1}^k < {}_2\tau_{n+1}),$$

where ${}_0w_{n+1}^k$ is the total service time remaining for all ${}_0q_n^k + \eta_n^k$ calls at time ${}_2t_{n+1}$. Since by construction ${}_2s_j^k \geqslant h > 0$ with probability one, ${}_0w_{n+1}^k \geqslant h({}_0q_n^k + \eta_n^k - 1)$ with probability one. Consequently

$${}_0q_{n+1}^k \leqslant {}_0q_n^k + \eta_n^k + \gamma_{n+1}^k I({}_0q_n^k + \eta_n^k < \varphi_{n+1}) \text{ a.s.},$$

where $\varphi_{n+1} = {}_2\tau_{n+1}/h + 1$, $\mathbf{E}\varphi_{n+1} < \infty$. Writing down the latter inequality by induction, we find that on the set $\{{}_0q_0^k \leqslant x_\varepsilon\}$

$${}_0q_{n+1}^k \leqslant \max\left\{x_\varepsilon + \sum_{i=0}^n \eta_i^k;\ \max_{0 < j \leqslant n}\left(\varphi_{j+1} + \gamma_{j+1}^k + \sum_{i=j}^n \eta_i^k\right)\right\} \leqslant x_\varepsilon + \sup_{-\infty < j \leqslant n}\left(\sum_{i=j}^n \eta_i^k + \varphi_{j+1} + \gamma_{j+1}^k\right) \equiv q^{nk}.$$

The sequence $\{q^{nk};\ -\infty < n < \infty\}$ is stationary and metrically transitive and, as noted in [7, p. 358], from the two relations $\mathbf{E}\eta_i^k < 0$ and $\mathbf{E}(\varphi_1 + \gamma_1^k) < \infty$ it follows that $q^{nk} < \infty$ with probability one.

Let us summarize the above arguments. First,

$$\sup_n \mathbf{P}\left({}_0\chi_n^k > x\right) \leqslant \varepsilon + \max_{n \leqslant n_\varepsilon} \mathbf{P}\left({}_0\chi_n^k > x\right) + \sup_{n > n_\varepsilon} \mathbf{P}\left(\chi^{nk} > x\right) = \varepsilon + \max_{n \leqslant n_\varepsilon} \mathbf{P}\left({}_0\chi_n^k > x\right) + \mathbf{P}\left(\chi^{0k} > x\right)$$

that is $\lim_{x\to\infty} \sup_n \mathbf{P}\left({}_0\chi_n^k > x\right) \leqslant \varepsilon$ for any $\varepsilon > 0$. Thus, $\sup_n \mathbf{P}\left({}_0\chi_n^k > x\right) \to 0$ as $x \to \infty$. Second, if $x > x_\varepsilon$,

$$\sup_n \mathbf{P}\left({}_0q_n^k > x\right) \leqslant 2\varepsilon + \max_{n \leqslant n_\varepsilon} \mathbf{P}\left({}_0q_n^k > x\right) + \mathbf{P}\left(q^{0k} > x\right)$$

for any $\varepsilon > 0$ and, consequently, $\sup_n \mathbf{P}\left({}_0q_n^k > x\right) \to 0$ as $x \to \infty$. In light of the above remarks, Lemma 5 is proved.

Let us now move on to the proof of Theorem 3. We will require the following fact from arithmetic. If a matrix $\|p_{ij}\|$ satisfies conditions (1), a matrix of transition probabilities $\|p'_{ij}\|$ may be found such that

(1) $\|p'_{ij}\|$ satisfies (1) for the same α and a_k;

(2) for any i and j, if $p_{ij} = 0$, then $p'_{ij} = 0$;

(3) the numbers $\{1, 2, \ldots, N\}$ may be renumbered so that in the new enumeration $p'_{ij} = 0$ for any $i \geq j$.

In other words, $\|p'_{ij}\|$ defines an acyclic network. We will not prove this assertion, noting only that it may be easily deduced by means of induction.

Let us specify some initial condition Q and consider the family of networks $\Sigma_m^n(Q)$ with this initial condition. We denote the characteristics of these networks by $\hat{Q}^k_{m,n}(Q)$, $\chi^k_{m,n}(Q)$, and so on. We will show that there exists a stationary sequence of events $B_n \in \mathscr{F}_n^N$ such that for $n \geqslant L$ in the event B_n $\widehat{Q}^k_{m,n}(Q) = \widehat{Q}^k_{L,n}$ for all $m \geqslant L$, $1 \leqslant k \leqslant N$.

Let us denote by $\{\tilde{Q}^n\}$ the stationary majorant for the family $\Sigma_m^n(Q)$. We will determine large integers K_{ij} and M such that the numbers $p''_{ij} = K_{ij}/M$ are close to p'_{ij} ($K_{ij} = 0$ if $p'_{ij} = 0$) and, consequently, $\|p''_{ij}\|$ satisfies (1).

For n such that $-\infty < n < \infty$, $d \geqslant 1$, c_1, c_2, $c_3 > 0$, we determine the events

$$A_n = \left\{\nu^i_{n+r,1} = j \text{ if } \sum_{l=1}^{j-1} K_{il} < r \leqslant \sum_{l=1}^{j} K_{il},\ 0 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N,\ 1 \leqslant r \leqslant M\right\};$$

$$D_n \equiv D_n(c_1, c_2, c_3) = \{\tilde{q}^n \leqslant c_1;\ \tilde{w}^n \leqslant c_2;\ \tilde{j}^n \leqslant c_3\};$$

$$E_n = \{\tau_{n+i} \geqslant 1/\alpha;\ s^k_{n+i,1} \leqslant a_k \text{ if } 1 \leqslant i \leqslant M;\ 1 \leqslant k \leqslant N\};$$

$$H_{n,d} = \left(\bigcap_{j=0}^{d-1} E_{n+jM}\right) \cap D_n.$$

It is easily seen that a number $d \gg 1$ may be found such that in the event $H_{n,d}$ in any of the networks Σ_{dM+l}^{n+dM}, $l \geqslant 0$ every call, after the next time it is serviced, either enters a station with higher number beginning with time t_{n+dM}, or exits the network. Now take numbers c_1, c_2, c_3 and c'_1, c'_2, c'_3 so that the event $H_{n,d} \cap D_{n+dM}(c'_1, c'_2, c'_3)$ has positive probability, and consider the event

$$H_{n,d,R} \equiv H_{n,d} \cap D_{n+dM} \cap \left(\bigcap_{j=d}^{R} E_{n+jM}\right),$$

where $R \gg d$. If the networks are considered only at times t_{n+iM}, $R \geqslant i \geqslant 0$, then in the event $H_{n,d,M}$ the behavior of these networks will be the same as the behavior of an acyclic network with stationary control. Therefore, Theorem 6 may be applied, and, consequently, a random variable α_n may be found and from this variable a large number R such that in the event of positive probability $B_{n+RM} \equiv H_{n,d,r} \cap \{\alpha_n \leqslant RM\}$ the desired relations are satisfied with probability one for L = RM. It remains for us to note that if stationary sequences $\hat{Q}^{n,k}$ are defined for $1 \leqslant k \leqslant N$ from the equalities

$$\widehat{Q}^{n,k} = \sum_{i=0}^{\infty} \widehat{Q}^k_{L,n-i} I\left(B_{n-i} \setminus \bigcup_{j=0}^{i-1} B_{n-j}\right),$$

then it follows from what has already been poroved that

$$\mathbf{P}\left(\widehat{Q}^k_{n,n}(Q) = \widehat{Q}^{n,k}\right) \to 1 \quad \text{as} \quad n \to \infty.$$

Therefore, in particular, for $B \subseteq Z_+^N \times R_+^N$,

$$|\mathbf{P}((q_n, \chi_n) \in B) - \mathbf{P}((\widehat{q}^n, \widehat{\chi}^n) \in B)| \leqslant \mathbf{P}(\widehat{Q}_{n,n} \neq \widehat{Q}^n) \to 0$$

as $n \to \infty$. Theorem 3 is proved.

3.3. Proof of Theorem 4. The proof of this assertion is analogous to that of Theorem 3, and therefore, we will limit ourself to an outline. First, under conditions (2) a natural analog of Lemma 3 holds if $1/b_j$ is replaced in (4) and (5) by m_j/b_j, $j = 1,\dots,N$. Second, for networks with multichannel stations the monotonicity properties (Lemma 4 and its corollaries) remain valid. Third, Lemma 5 may be proved in virtually the same way if in the course of the proof additional batches of calls in each channel are directed towards the stations, independently of the other channels. Finally, at the concluding step we construct an event B_n in which there exists an identity with an acyclic network over a lengthy time interval, where the loads on all the stations satisfy condition (2). Consequently, the arguments of Sec. 2.3 may be applied.

4. Proof of Theorem 5

So as not to burden the proof with extraneous technical details, we further suppose that $s_1^i > 0$ with probability one for all $i = 1,\dots,N$.

Let us specify the initial condition $\{(q_0^k, \chi_0^k);\ 1 \leqslant k \leqslant N\}$, where $\sum_k q_0^k = M$ and $\chi_0^k = 0$ if $q_0^k = 0$. We denote by $0 \leqslant t_1 < t_2 < \dots$ the successive times at which servicing terminates at the stations, i.e., t_1 is the first moment of time at which the servicing of at least one station terminates, $t_2 > t_1$ is the next moment of time, and so on. We introduce the control $\{(s_n^k, \nu_n^k);\ n \geqslant 1\}$ which has the following meaning. If at time t_n servicing of some call terminates at the k-th station, this call is directed to station ν_n^k; if at this time, servicing of some call begins at the k-th station, the length of this servicing is set equal to s_n^k. For all k, we set $q_n^k = q^k(t_n + 0)$, $\chi_n^k = \chi^k(t_n - 0)$, $\tau_n = t_n - t_{n-1}$. Note that if $n \geqslant 2$, the inequality $\tau_n \leqslant \max_k s_n^k \equiv \varphi_n < \infty$ holds with probability one and, consequently, $\mathbf{E}\tau_n \leqslant \sum_k \mathbf{E}s_n^k < \infty$.

We first present two proofs of Theorem 5 under certain special assumptions, and then the proof in the general case.

Case (a). Suppose that there exists a number k such that $p_{kk} > 0$.

For i, j = 1,...,N, let p_{ij}^n denote the probability of a transition from state i to state j in n steps. The set of states {1, 2,...,N} is partitioned into a certain number of classes $D_0 \cup D_1 \cup \dots \cup D_l$ by the rule $D_0 = \{k\}$; $D_1 = \{i\colon i \neq k,\ p_{ik} > 0\}$ and for $1 < r \leqslant l$, $D_r = \{i\colon i \neq k,\ p_{ik} = 0,\ p_{ik}^2 = 0, \dots, p_{ik}^{r-1} = 0,\ p_{ik}^r > 0\}$. If $r = 1, 2,\dots,\ell$ and $i \in D_r$ we let $j = j(i) \in D_{r-1}$ denote a number such that $p_{ij} > 0$. If there are several such numbers, we select one of them.

We fix a number $L \geqslant 1$ and consider for $n \geqslant 1$ the events

$$A_n = \{\nu_{n+v}^k = k;\ \nu_{n+v}^i = j(i) \text{ if } i \neq k;\ 1 \leqslant v \leqslant L\}.$$

It is easily seen that by virtue of our choice of the number $L \gg 1$, the equalities $q_{n+L}^k = M$ and $\chi_{n+L}^k = 0$ will hold with probability one in the event A_n. Moreover, $\mathbf{P}(A_n) > 0$ and $\{A_n\}$ form a stationary sequence of events which are L-dependent. That is, for the sequence $\{q_n^i, \chi_n^i;\ 1 \leqslant i \leqslant N\}$, which forms a Markov chain, the state $\{q_n^k = M;\ \chi_n^k = 0;\ q_n^i = 0;\ \chi_n^i = 0$ for $i \neq k\}$ is a regenerating state where the length of the regenerating cycle possesses finite mathematical expectation.

Let μ denote the length of, say, the first regenerating cycle for the sequence $\{(q_n^i, \chi_n^i)\}$, and γ the length of the first regenerating cycle for the process $\{q^i(t), \chi^i(t)\}$ in continuous time. Note that, first, the random variable γ has nonlattice distribution, since $\gamma \overset{D}{=} s_1^k + \psi$, where ψ is independent of s_1^k, and, second,

$$\mathbf{E}\gamma = \mathbf{E}\sum_1^{\mu} \tau_l \leqslant \mathbf{E}\sum_1^{\mu} \varphi_l = \mathbf{E}\mu \cdot \mathbf{E}\varphi_1 < \infty$$

by Wald's identity. Therefore, Smith's theorem for regenerating processes may be applied, and Theorem 6 is proved.

Case (b). Now assume that $p_{ii} = 0$ for all i, but that there exists a number k such that

$$\mathbf{P}\left(s_1^k > \sum_{i \neq k} s_1^i\right) > 0. \tag{7}$$

An ordered sequence of distinct indices $i_1, i_2, \ldots, i_m$ may be found such that $p_{ki_1} \cdot p_{i_1 i_2} \cdot \ldots \cdot p_{i_m k} > 0$ and the stations renumbered such that in the new enumeration $i_1 = 1, \ldots, i_m = m$, $k = m + 1$. From condition (7), it follows that there exist numbers d, $b_1, \ldots, b_m$ such that $d > b_1 + \ldots + b_m$ and the event $\{d + \delta_0 \geqslant s_1^{m+1} \geqslant d;\ b_i \geqslant s_1^i \geqslant b_i - \delta_i;\ 1 \leqslant i \leqslant m\}$ has positive probability for any $\delta_0, \delta_1, \ldots, \delta_m$.

Let us partition the set $\{1, 2, \ldots, N\}$ into classes $D_0 \cup D_1 \cup \ldots \cup D_l$ by the rule

$$D_0 = \{1, 2, \ldots, m + 1\};\ D_1 = \left\{i\colon\ i \notin D_0;\ \sum_{j \in D_0} p_{ij} > 0\right\};$$

$$D_2 = \left\{i\colon\ i \notin D_0 \cup D_1;\ \sum_{j \in D_1} p_{ij} > 0\right\} \text{ etc.}$$

and for $r = 1, \ldots, \ell$ and $i \in D_r$ denote by $j = j(i) \in D_{r-1}$ a number such that $p_{ij} > 0$. If there are several such numbers, we select one of them.

We fix $L \geqslant 1$ and consider for the events $A_n = \{\nu_{n+v}^{m+1} = 1;\ \nu_{n+v}^i = i + 1$ if $1 \leqslant i \leqslant m;$ $\nu_{n+v} = j(i)$ if $i \notin D_0;$ $d + \delta_0 \geqslant s_{n+v}^{m+1} \geqslant d;\ b_i \geqslant s_{n+v} \geqslant b_i - \delta_i$ if $1 \leqslant i \leqslant m;\ 1 \leqslant v \leqslant L\}$. It is easily seen that a number $L \gg 1$ may be found such that for any initial (at time t_n) condition, the relation $\{q_{n+v}^k = M,\ \chi_{n+v}^k = 0\}$ holds for at least one number v, $1 \leqslant v \leqslant L$, i.e., regeneration of the sequence $\{q_n^i, \chi_n^i\}$ occurs.

As in case (a), the $\{A_n\}$ form a stationary sequence of L-dependent random events, $\mathbf{P}(A_n) > 0$. Therefore, setting $\lambda = \lambda(\omega) = \min\{n \geqslant 0\colon\ \omega \in A_{nL+1}\}$, we have $E\lambda < \infty$. Consequently, as before, if the length μ of the first regeneration cycle of the sequence $\{q_n^i, \chi_n^i\}$ is introduced, it follows from the relation $\mu \leqslant L(\lambda + 1)$ with probability one that $\mathbf{E}\mu < \infty$. Therefore, as in case (a), Smith's theorem may be applied.

General Case. Suppose the assumptions of cases (a) and (b) do not hold. From the conditions of the theorem a number k may be found such that

$$\mathbf{P}\left(s_1^k > \max_{i \neq k} s_1^i\right) > 0. \tag{8}$$

We carry out the same constructions as in case (b), with the natural substitution of condition $d > \max_{i \leqslant m} b_i$ for condition $d > b_1 + \ldots + b_m$.

Consider, first, the case $M \leqslant m + 1$. Suppose that $L \gg 1$. Then, as may be easily shown, for sufficiently large $L' < L$, in the event A_n the following set of conditions may be realized: at any time t, $t_{n+L'} \leqslant t \leqslant t_{n+L}$,

(1) all calls are found at stations numbered D_0;

(2) at any station numbered $i \leqslant m$, there is at most one call, that is, when any call that has arrived at this station leaves, the station is empty.

Thus, there is at least one call at the (m + 1)-th station at any time.

Further, by choosing sufficiently small $\delta_0, \delta_1, \ldots, \delta_m$ it becomes possible to make the sequence of service terminations at the stations cyclic with length of cycle (m + 1). By the above reasoning, it is possible to determine a stationary sequence $\{(q_n(i), \chi_n(i));\ 1 \leqslant i \leqslant m + 1;\ n \geqslant 1\}$ such that for every n the ordered sequence $\{q_n(i), \chi_n(i);\ 1 \leqslant i \leqslant m + 1\}$ is uniquely and uniformly determined from the set of random variables $\{s_{n-j}^i;\ 1 \leqslant i \leqslant m + 1;\ 1 \leqslant j \leqslant M(m+1)\}$ and for v such that servicing at the (m + 1)-th station, $L' \leqslant v \leqslant L$, terminates at time t_{n+v}, it is true that $q_{n+v}^i = q_{n+v}(i);\ \chi_{n+v}^i = \chi_{n+v}(i)$, $1 \leqslant i \leqslant m + 1$ in the event A (where from the form of the conditions $\chi_{n+v}^{m+1} = 0$ and $\chi_{n+v}^1 = q_{n+v}^1 = 0$).

Let us introduce the random variable

$$\mu = \min\{n \geqslant M(m + 1)\colon\ (q_n^i, \chi_n^i) = (q_n(i), \chi_n(i))\}.$$

Note that $\mu \leqslant L(\lambda + 1)$, where the variable λ is defined in case (b). Therefore, $\mathbf{E}\mu < \infty$.

Consider for $L \gg 1$, $L > M(m + 1)$ the sequence of random variables $\mu_1 = \mu$ and for $j \geqslant 1$ $\mu_{j+1} = \min\{n > \mu_j + L\colon\ (q_n^i, \chi_n^i) = (q_n(i),\ \chi_n(i))\}$. It was noted in [11] that the random variables

$\{(\mu_{j+1}-\mu_j); j\geqslant 1\}$ are independent and identically distributed with finite mean. Therefore, for R = GCD $\{i: \mathbf{P}(\mu_{j+1}-\mu_j=i)>0\}$ there exists a stationary sequence $\{(\tilde{q}_n^k, \tilde{\chi}_n^k), n\geqslant 1\}$ such that $\mathbf{P}((q_{jR}^k, \chi_{jR}^k)=(\tilde{q}_j^k, \tilde{\chi}_j^k)$ for all $j\geqslant n)\to 1$. Let us now consider processes with continuous time and note that the length γ between renovating times has a nonlattice distribution and that Theorem 6 follows from the results of [5].

Let us now consider the case M > m + 1. Here the computations become even lengthier, and, therefore, we will not present a rigorous proof, but instead limit ourself to an outline.

Consider event A_n defined earlier, and enumerate the calls numbered 1, 2,...,M in such a way that they circulate along the route {1, 2,...,m + 1} in accordance with the numeration. Note that by the conditions of the theorem at least one more number d' may be found such that for sufficiently small δ_0 the intervals (d, d + δ_0) and (d', d' + δ_0) are disjoint, and $\mathbf{P}(s_1^k\in(d', d'+\delta_0))>0$. To simplify the discussion, consider only the case $\mathbf{P}(s_1^k=d)>0$; $\mathbf{P}(s_1^k=d')>0$; $d'<d$. We determine an event B_n that describes the operation of the network when v > L in the following way: Calls continue to circulate along the route {1, 2,...,m + 1}; the service times at stations from the route {1, 2,...,m} are the same, but at the (m + 1)-th station we assign to call M service time of d', and to all the other calls service times of d. Then if v ≫ L, upon arriving at the (m + 1)-th station call M will always take the place of call (M − 1). Set $L^1 \gg 1$ and if $v > L+L^1$ carry out the following analogous procedure: At the (m + 1)-th station we assign to all calls numbered 1, 2,...,M − 2 service times of d, and to calls numbered (M − 1) and M, service times of d'. If $v\gg L+L^1$, both types of calls, i.e., those numbered (M − 1) and those numbered M, will always arrive at a busy station. Performing similar constructions by induction, we find that for large v, in the event $A_n\cap B_n$, upon arriving at the (m + 1)-th station all calls numbered 2, 3,...,M will take the place of the preceding call. We now consider the times when servicing of the first call terminates at the (m + 1)-th station, and show that at these times the sequence we are studying is "renovated," and we may reason as in the case $M\leqslant m+1$. The proof of the theorem is concluded.

Remark 2. By means of slight improvements it is possible to deduce the assertion of Theorem 6 in the case of multichannel stations as well.

In conclusion, the author would like to note the great influence upon the development of his personal mathematical interests which has been the result of many discussions with Aleksandr Alekseevich Borovkov and the study of his works, and for this I am deeply grateful.

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