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Ergodicity of a Stress Release Point Process Seismic Model with Aftershocks

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Abstract. We prove ergodicity of a point process earthquake model combining the classical stress release model for primary shocks with the Hawkes model for aftershocks.

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1. Introduction

The times of occurrence of earthquakes in a given area of seismic activity form a simple point process N on the real line, where $N((a,b])$ is the number of shocks in the time interval $(a,b]$. In the present model, the dynamics governing the process will be expressed by the stochastic intensity $\lambda(t)$. In intuitive terms (to be precised in the next subsection)

$$\lambda(t) = \lim_{h \downarrow 0} \frac{1}{h} P(N((t,t+h]) = 1 \mid \mathcal{F}_t)$$

where $\mathcal{F}_t$ is the sigma-field summarizing the available information at time t (increasing with t). In the stress release model, for $t \geq 0$,

$$\lambda(t) = \exp\Big\{X_0 + ct - \sum_{n=1}^{N((0,t])} Z_n\Big\}$$

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where $c > 0$ and $\{Z_n\}_{n\geq 1}$ is an i.i.d. sequence of non-negative random variables with finite expectation, whereas X_0 is some real random variable. The process

$$X(t) = X_0 + ct - \sum_{n=1}^{N((0,t])} Z_n$$

is ergodic, and the reader is referred to [9] for a proof and the relevant results concerning a generalization of this particular model.

Another model of interest in seismology is the Hawkes branching process, where the stochastic intensity is

$$\lambda(t) = \nu(t) + \int_{(0,t]} h(t-s)N(ds),$$

where h is a non-negative function, called the fertility rate and ν is a non-negative integrable function. Such point process appears in the specialized literature under the name ETAS (Epidemic Type After-Shock; [12]) and is used to model the aftershocks (see [7], p. 203). It is well known [6] that the corresponding process "dies out" in finite time under the condition

$$\int_0^\infty h(t)\,dt < 1.$$

A model mixing stress release and Hawkes aftershocks is [13]

$$\lambda(t) = \exp\Big\{X_0 + ct - \sum_{n=1}^{N((0,t])} Z_n\Big\} + Y_0 e^{-\alpha t} + k \int_{(0,t]} e^{-\alpha(t-s)}\,N(ds),$$

where $\alpha > 0$. The positive constant c is the rate at which the strain builds up. If there is a shock at time t, then the strain is relieved by the quantity $Z_{N(t)}$. Each shock (primary or secondary) at time t generates aftershocks according to a Poisson process of intensity $a(s) = k\exp\{-\alpha(t-s)\}$. In this article, we give necessary and sufficient conditions of ergodicity for this model. More precisely, we find a necessary condition for the existence and uniqueness of the corresponding stationary process and, for any initial distribution of X_0 and Y_0, of the convergence to that distribution, and we prove formally that it is also sufficient (under a further smoothness condition on the distribution of Z_i).

We shall start with a detailed mathematical description of the model.

2. Description of the model

Let $\varphi : (-\infty, \infty) \to [0, \infty)$ be a non-decreasing function such that

$$\lim_{x\to-\infty} \varphi(x) = 0$$

and

$$\lim_{x\to\infty} \varphi(x) = \infty.$$

We operate under either one of the following assumptions: (a): the function φ may be strictly positive everywhere or (b): it is equal to zero for all x below some level and otherwise strictly increasing.

We are given

(1) a Poisson field Π of intensity 1 in the positive quadrant

(2) and an i.i.d. family of positive random variables $\{Z_n\}_{n\geq 1}$ with a finite mean,

(3) and it is assumed that the Poisson field and the i.i.d. family are independent.

The above Poisson field and i.i.d. family constitute the probabilistic basis of our model.

We consider a simple point process N with the following stochastic intensity:

$$\lambda(t) = \varphi\Big(X_0 + ct - \sum_{n=1}^{N(t)} Z_n\Big) + Y_0 e^{-\alpha t} + k \int\limits_{(0,t]} e^{-\alpha(t-s)}\, dN(s), \tag{2.1}$$

where $N(t) := N((0,t])$, and where X_0, Y_0, c, k, and α are as in the introduction. This means that the point process is constructed recursively as

$$N(t) = \int\limits_{(0,t]} \int\limits_{R} \mathbf{I}(z \leq \lambda(t-))\Pi(dz \times dz).$$

Defining

$$\mathcal{F}_t := \sigma\{X_0; Y_0; N(s), Z_{N(s)}, s \leq t\},$$

the process $\{\lambda(t)\}_{t\geq 0}$ is then the $\mathcal{F}_t$-stochastic intensity of N in the sense of [4] (see also [7, 10]).

In the seismological interpretation,

$$\lambda_1(t) = \varphi\Big(X_0 + ct - \sum_{n=1}^{N(t)} Z_n\Big) \tag{2.2}$$

is the stochastic intensity of the primary shocks, whereas

$$\lambda_2(t) = Y_0 e^{-\alpha t} + k \int\limits_{(0,t]} e^{-\alpha(t-s)}\, dN(s) \tag{2.3}$$

is the stochastic intensity of the aftershocks.

3. On the ergodicity condition

The existence of ergodicity will be proven in the case

$$\frac{k}{\alpha} < 1. \tag{3.1}$$

This section shows that this is indeed a natural (intuitive) condition and moreover that it is necessary if we seek only those solutions for which the steady-state average intensity $\lambda := E[\lambda(t)]$ satisfies $0 < \lambda < \infty$.

We therefore henceforth assume ergodicity. From now on we use the notation

$$X(t) = X_0 + ct - \sum_{n=1}^{N(t)} Z_n$$

and

$$Y(t) = Y_0 e^{-\alpha t} + k \int_{(0,t]} e^{-\alpha(t-s)}\, dN(s).$$

The process $(X(t), Y(t)), t \geq 0$, is a time-homogeneous Markov process with initial value (X_0, Y_0), and

$$\lambda(t) = \varphi(X(t)) + Y(t).$$

Further, ergodicity means, in particular, that there exists a stationary version of the process $(X(t), Y(t))$. For such a stationary version, let $\lambda_1 = E[\varphi(X(t))]$ and $\lambda_2 = EY(t)$. Then $\lambda = \lambda_1 + \lambda_2$, so the finiteness of λ implies that of λ_1 and of λ_2. Observe that

$$\begin{aligned} E[Y(t)] &= E[Y(0)]e^{-\alpha t} + kE\left[\int_{(0,t]} e^{-\alpha(t-s)} N(ds)\right] \\ &= E[Y(0)]e^{-\alpha t} + k\int_0^t e^{-\alpha(t-s)} \lambda ds \\ &= E[Y(0)]e^{-\alpha t} + \lambda \frac{k}{\alpha}(1 - e^{-\alpha t}) \end{aligned}$$

where we used Campbell's formula. Therefore, from the stationarity,

$$E[Y(0)] = \lambda \frac{k}{\alpha} = E[Y(t)].$$

Then

$$\lambda = \lambda_1 + \lambda \frac{k}{\alpha} \equiv \lambda_1 + \lambda_2.$$

The supercritical case. Suppose, in view of contradiction, that $k/\alpha > 1$. The last equality then implies that $\lambda = \infty$, which we excluded, or that $\lambda = 0$, and then $\lambda_1 = E[\varphi(X(t))] = 0$. Since $\varphi(X(t)) \geq 0$, this implies $P(\varphi(X(t)) = 0) = 1$, that is $P(X(t) = -\infty) = 1$. Similarly $P(Y(t) = 0) = 1$.

The critical case. Suppose now, again in view of contradiction, that $k/\alpha = 1$. The last displayed equality implies then that $\lambda = \infty$ (excluded) or $\lambda_1 = 0$ and therefore $P(\varphi(X(t)) = 0) = 1$. Then

$$\lambda(t) = Y(0)e^{-\alpha t} + k \int_{(0,t]} e^{-\alpha(t-s)} N(ds).$$

We show that any point process N with this stochastic intensity and with finite average intensity is necessarily null (with intensity equal to 0). Suppose that $\lambda > 0$. Clearly,

$$\begin{aligned} P(N(R_+) = 0) &= E[P(N(R_+) = 0 \mid Y(0))] \\ &= E\left[\exp\left\{-\int_0^\infty Y(0)e^{-\alpha t} dt\right\}\right] \\ &\geq \exp\left\{-E[Y(0)]\frac{1}{\alpha}\right\} = \exp\left\{-\lambda\frac{1}{\alpha}\right\} > 0 \end{aligned}$$

and therefore, since we assumed $\lambda < \infty$, we have that $P(N(R_+) = 0) > 0$. Now,

$$\{N(R_+) = 0\} \subseteq \theta_t\{N(R_+) = 0\} = \{N([t, \infty)) = 0\}.$$

That is, $\{N(R_+) = 0\}$ is expanded by the (ergodic) shift, and therefore it has probability 0 or 1. By the above, this probability must be 1. We conclude that $\lambda = 0$, a contradiction.

Therefore in the critical case there is no solution except the trivial one (no earthquakes).

4. Explicit expressions for the average rates

In this section, we exhibit an interesting feature of the model. We assume here again ergodicity and the condition $0 < \lambda < \infty$. We continue to consider the model in the stationary regime. Writing

$$\begin{aligned} \varphi(X(t)) &= \varphi\Big(X(0) + t\Big(c - \frac{N(t)}{t}\frac{1}{N(t)}\sum_{n=1}^{N(t)} Z_n\Big)\Big) \\ &= \varphi(X(0) + t(c - \lambda E[Z_1] + \varepsilon(t))), \end{aligned}$$

where $\lim_{t\uparrow\infty}\varepsilon(t) = 0$ a.s. Let $\Delta := c-\lambda E[Z_1]$. Let τ be the (a.s. finite) random time such that $t \geq \tau$ implies $|\varepsilon(t)| \leq (1/2)|\Delta|$.

Suppose that $c - \lambda E[Z_1] > 0$. We have

$$E[\varphi(X(t))] \geq E[\varphi(X(t))1_{\{t\geq\tau\}}] \geq E\Big[\varphi(X(0) + t\frac{1}{2}\Delta)1_{\{t\geq\tau\}}\Big]$$

But

$$\varphi(X(0) + t\frac{1}{2}\Delta)1_{\{t\geq\tau\}} \uparrow \infty$$

as $t \to \infty$ and therefore

$$\lambda_1 = E[\varphi(X(t))] \to \infty$$

implying that $\lambda = \infty$ which is excluded.

Suppose that $c - \lambda E[Z_1] < 0$. We show that this is impossible. Here $\lim_{t\uparrow\infty}\varphi(X(t)) = 0$ by a similar argument. We prove that $\lim_{t\uparrow\infty} E[\varphi(X(t))] = 0$, $\lambda_1 = 0$ and therefore $\lambda = 0$ which is impossible.

For the proof that $\lim_{t\uparrow\infty} E[\varphi(X(t))] = 0$ we can make use of the following lemma (in fact taking care of both situations when $c - \lambda E[Z_1] \neq 0$).

Lemma 4.1. *If the stationary stochastic process $\{Z(t)\}_{t\geq 0}$ is such that it tends almost surely to a deterministic constant c as $t \uparrow \infty$, then it is almost surely equal to this constant.*

Proof. Fix $\varepsilon > 0$, and consider the set

$$C = \{\omega; Z(t,\omega) \in [c-\varepsilon, c+\varepsilon] \text{ for all } t \geq 0\}.$$

Then for all $a > 0$,

$$\theta_a C = \{\omega; Z(t,\omega) \in [c-\varepsilon, c+\varepsilon] \text{ for all } t \geq a\}.$$

But $\theta_a C \uparrow \Omega$, and therefore $P(C) = P(\theta_a C) \uparrow 1$. So that $P(C) = 1$. Since this is true for all $\varepsilon > 0$,

$$P\{Z(t) = c\} = 1, \quad \text{for all} \quad t \geq 0.$$

Therefore, necessarily

$$\lambda = \frac{c}{E[Z_1]}.$$

□

Therefore, in this model, the rate of occurrences of earthquakes is given by the physics of stress build up (the constant c) and stress release ($E[Z_1]$), whereas the global rate is shared among primary and secondary earthquakes according to the physics of the aftershocks (α and k).

5. Two embeddings

We now turn to the technical core of the paper, namely the proof of existence of a unique ergodic solution of the model, under the condition $k/\alpha < 1$ and a further condition on the distribution of Z_i (see Condition (CZ) in Section 7). The technique used is that of Harris chains, and we start as usual by studying a natural embedded process. More precisely, let $\{(t_n)\}_{n\geq 0}$, with $t_0 = 0$, be the sequence of time events of N, and let for each $n \geq 0$, $T_{n+1} := t_{n+1} - t_n$, $X_n := X(t_n)$, $Y_n := Y(t_n)$. We then have the recurrence equations that exactly reflect the dynamics described in the previous section:

$$X_{n+1} = X_n + cT_{n+1} - Z_{n+1}$$

and

$$Y_{n+1} = Y_n \exp\{-\alpha T_{n+1}\} + k$$

where S_{n+1} is a positive random variable whose hazard rate is, conditionally to $X_0, \dots, X_n$, $Y_0, \dots, Y_n$, $T_1, \dots, T_n$ and $Z_1, \dots, Z_n$

$$\varphi(X_n + cs) + Y_n e^{-\alpha s}.$$

It is clear that $\{(X_n, Y_n)\}_{n\geq 0}$ is a homogeneous Markov chain. Its transition mechanism is fully described by the first transition, which can be implemented as follows.

Let $X_0 = x$ and $Y_0 = y \geq 0$. On the positive half-plane with the time t running on the horizontal coordinate axis, draw two curves:

(a) a curve with graph $(t, \varphi(x + ct))$ (that starts from $(0, \varphi(x))$);

(b) a curve with graph $(t, -ye^{-\alpha t})$ (that starts from $(0, -y)$).

Consider the projection on the time axis of the Poisson field between the above two curves and let T_1 be the point of this projection with the smallest t-coordinate. It has, as the notation anticipated, the required hazard rate $\varphi(x + cs) + y\exp\{-\alpha s\}$. In particular,

$$\begin{aligned}\mathbf{P}_{x,y}(T > t) &\equiv \mathbf{P}(T_1 > t \mid X_0 = x, Y_0 = y) \\ &= \exp\Big\{-\frac{y}{\alpha}\Big(1 - e^{-\alpha t}\Big)\Big\} \cdot \exp\Big\{-\int_0^t \varphi(x + cv)dv\Big\}.\end{aligned}$$

and

$$\begin{aligned} X_1 &= x + cT_1 - Z_1, \\ Y_1 &= ye^{-\alpha T_1} + k. \end{aligned}$$

Two lemmas concerning this particular realization of the transition kernel will be useful.

Let $T_{x,y}$ be a "generic" random variable with distribution

$$\mathbf{P}(T_{x,y} \in \cdot) = \mathbf{P}(T_1 \in \cdot \mid X_0 = x, Y_0 = y).$$

Following the comments from above, one can represent $T_{x,y}$ as

$$T_{x,y} = \min\bigl(T^{(1,x)}, T^{(2,y)}\bigr) \tag{5.1}$$

where $T^{(1,x)}$ and $T^{(2,y)}$ are independent and

$$\mathbf{P}(T^{(1,x)} > t) = \exp\Bigl\{-\int_0^t \varphi(x+cv)dv\Bigr\},$$

$$\mathbf{P}(T^{(2,y)} > t) = \exp\Bigl\{-\frac{y}{\alpha}\Bigl(1-e^{-\alpha t}\Bigr)\Bigr\}.$$

Clearly,

$$\mathbf{P}(T^{(2,y))} = \infty) = e^{-y/\alpha} > 0,$$

for any $y \geq 0$.

Lemma 5.1. (1) *For any* $0 \leq y_1 \leq y_2$,

$$T^{(2,y_1)} \geq_{st} T^{(2,y_2)}. \tag{5.2}$$

(2) *For any* $x_1 < x_2$,

$$T^{(1,x_1)} \geq_{st} T^{(1,x_2)} \tag{5.3}$$

while

$$x_1 + cT^{(1,x_1)} \leq_{st} x_2 + cT^{(1,x_2)}. \tag{5.4}$$

Also, for any x, $\mathbf{P}(T^{(1,x)} < \infty) = 1$ *and, moreover, for any* $a > 0$,

$$\mathbf{P}(T^{(1,x)} > t)e^{at} \to 0, \quad \text{as} \quad t \to \infty.$$

Proof. Inequality (5.2) is straightforward. Inequality (5.3) follows from the monotonicity of φ while inequality (5.4) follows from the following coupling construction. Let $t_0 > 0$ be such that $x_1 + ct_0 = x_2$. If there is a point of the Poisson field in $\{(t,u) : 0 \leq t \leq t_0, 0 \leq u \leq \varphi(x_1 + ct)\}$, then $T^{(1,x_1)} < t_0$ and $x_1 + cT^{(1,x_1)} \leq x_2$. If however there is no such a point, then

$$x_1 + cT^{(1,x_1)} = x_2 + cT^{(1,x_2)} \cdot \theta^{t_1}$$

where $\{\theta^t\}_{t\geq 0}$ is a family of measure-preserving shift transformations. So, with probability 1

$$x_1 + cT^{(1,x_1)} \leq x_2 + cT^{(1,x_2)} \cdot \theta^t =_{st} x_2 + cT^{(1,x_2)}.$$

□

The remaining results follow from inequality (5.4) and from the fact that $\varphi(x) \to \infty$ as $x \to \infty$.

Corollary 5.1. *For any* x_0,

$$\sup_{x \geq x_0} \mathbf{E}T_{x,y} \to 0 \quad \text{as} \ \ y \to \infty. \tag{5.5}$$

Also

$$\sup_{y \geq 0} \mathbf{E}T_{x,y} \to 0 \quad \text{as} \ \ x \to \infty. \tag{5.6}$$

Proof. By Lemma 5.1, $0 \leq T_{x,y} \leq_{st} T_{x_0,y}$ for any $x \geq x_0$. Clearly, $T^{(2,y)} \to 0$ in probability as $y \to \infty$. Since $\mathbf{E}T^{(1,x_0)}$ is finite, the family of random variables $\{T_{x_0,y}\}_{y \geq 0}$ is uniformly integrable, and therefore

$$\sup_{x \geq x_0} \mathbf{E}T_{x,y} = \mathbf{E}T_{x_0,y} \to 0 \quad \text{as} \ \ y \to \infty.$$

Further, from inequality (5.1), $T_{x,y} \leq_{st} T^{(1,x)}$ where $T^{(1,x)} \to 0$ in probability. By (5.3) and since $\mathbf{E}T^{(1,0)}$ is finite, the family $\{T^{(1,x)}\}_{x \geq 0}$ is uniformly integrable, and therefore $\mathbf{E}T^{(1,x)} \to 0$ as $x \to \infty$, and then (5.6) follows. □

Lemma 5.2. *As* $y \to \infty$,

$$y\mathbf{E}\big(\exp\{-\alpha T^{(2,y)}\} - 1\big) \to -\alpha.$$

Proof. Indeed,

$$\begin{aligned}
y\mathbf{E}\big(1 - \exp\{-\alpha T^{(2,y)}\}\big) &= y \int_0^1 \mathbf{P}\big(\exp\{-\alpha T^{(2,y)}\} < v\big) dv \\
&= y \int_0^1 \mathbf{P}(T^{(2,y)} > \ln v/(-\alpha)) dv \\
&= y \int_0^1 \exp\Big\{-\frac{y}{\alpha}(1-v)\Big\} dv \\
&\qquad \text{(change of variables: } u = 1 - v) \\
&= y \int_0^1 \exp\Big\{-\frac{yu}{\alpha}\Big\} du \quad \text{(change of variables: } r = yu/\alpha) \\
&= \alpha \int_0^{y/\alpha} e^{-r} dr \to \alpha,
\end{aligned}$$

as $y \to \infty$. □

Remark 5.1. As follows from (5.1) and Lemma 5.1, if $x \geq 0$, then

$$T_{x,y} \leq_{st} T^{(1,0)}$$

and, therefore,

$$\sup_{x>0, y\geq 0} \mathbf{E}T_{x,y} \leq \mathbf{E}T^{(1,0)} < \infty. \tag{5.7}$$

One may also deduce from Lemma 5.1 that if $x \leq 0$, then

$$cT_{x,y} \leq_{st} |x| + cT^{(1,0)} + Z$$

where the random variables in the right-hand side are integrable. So one can find a universal constant $C > 0$ such that

$$\mathbf{E}T_{x,y} \leq C(|x| + 1), \quad \text{for all} \;\; x \leq 0 \;\; \text{and} \;\; y \geq 0. \tag{5.8}$$

Then it follows from (5.7) and (5.8) that, for any negative x_1,

$$\sup_{x>x_1, y\geq 0} \mathbf{E}T_{x,y} < \infty. \tag{5.9}$$

However the supremum in (5.9) becomes infinite if one replaces x_1 by $-\infty$.

To keep the supremum finite, we consider a slightly different embedding. Again we describe the first transition only. We fix a sufficiently large positive v_0 and a sufficiently large negative x_1 (to be chosen in the next section) and define the new embedding $\{\widetilde{T}_{x,y}\}$ as follows:

(a) if $x \leq x_1$, then $\widetilde{T}_{x,y} = \min(T_{x,y}, v_0)$ while

(b) if $x > x_1$, then $\widetilde{T}_{x,y} = T_{x,y}$.

Then clearly

$$\sup_{x\in(-\infty,\infty), y\geq 0} \mathbf{E}\widetilde{T}_{x,y} < \infty. \tag{5.10}$$

Denote by $(\widetilde{X}_n, \widetilde{Y}_n)$ a new time-homogeneous Markov chain obtained by the new embedding. It satisfies the relations: given $\widetilde{X}_0 = x, \widetilde{Y}_0 = y$, if $x > x_1$, then

$$\widetilde{X}_1 =_{st} x + cT_{x,y} - Z$$

where $T_{x,y}$ and Z are mutually independent, and

$$\widetilde{Y}_1 =_{st} ye^{-\alpha T_{x,y}} + k,$$

and if $x \leq x_1$, then

$$\widetilde{X}_1 =_{st} x + c\widetilde{T}_{x,y} - Z\,\mathbf{I}(T_{x,y} \leq v_0)$$

where $\widetilde{T}_{x,y}$ and Z are mutually independent, and

$$\widetilde{Y}_1 =_{st} ye^{-\alpha \widetilde{T}_{x,y}} + k\,\mathbf{I}(T_{x,y} \leq v_0).$$

6. Positive recurrence of the embedded process

In this section, we show positive recurrence of the Markov chain obtained by the second embedding $(\widetilde{X}_n, \widetilde{Y}_n)$, see the end of the previous section. We recall some known facts.

Definition 6.1. Consider a discrete-time and time-homogeneous Markov chain $W_n, n \geq 0$, on a measurable state space $(\mathcal{W}, \mathcal{B}_{\mathcal{W}})$. A measurable set $V \subseteq \mathcal{Z}$ is *positive recurrent* if the following two conditions hold:

(a) A random variable

$$\tau_w(V) := \min\{n \geq 1 : W_n \in V \mid \ W_0 = w\}$$

is a.s. finite, for all $w \in \mathcal{W}$;

(b) Moreover, $\sup_{w \in V} E\tau_w(V) < \infty$.

For a Markov chain W_n, the following result is known as *Foster's criterion*:

Proposition 6.1. *Let $L : \mathcal{W} \to [0, \infty)$ be a measurable function, and let $\widehat{L}$ be a non-negative number. The set $V = \{w \in \mathcal{W} : L(w) \leq \widehat{L}\}$ is positive recurrent if*

(i) $\sup_{w \in V} E(L(W_1) \mid W_0 = w)$ *is finite;*

(ii) *there exists $\varepsilon > 0$ such that $E(L(W_1) \mid W_0 = w) - L(w) \leq -\varepsilon$, for any $w \in \mathcal{W} \setminus V$.*

Theorem 6.1. *Let $k < \alpha$. For suitably chosen v_0 and x_1, there exists a compact set V in $(-\infty, \infty) \times [0, \infty)$ which is positive recurrent for the Markov chain $(\widetilde{X}_n, \widetilde{Y}_n)$.*

Remark 6.1. A sequence (X_n, Y_n) is a subsequence of $(\widetilde{X}_n, \widetilde{Y}_n)$. With arguments similar to those of Theorem 6.1, one can also prove that the same set V is positive recurrent for the Markov chain (X_n, Y_n).

Proof of Theorem 6.1. We use Foster's criterion, with the following choice of the test function:

$$L(x, y) \equiv L_1(x) + L_2(y) = \begin{cases} r_1 x + r_2 y, & \text{if } x \geq 0, \\ r_3|x| + r_2 y, & \text{if } x < 0, \end{cases}$$

where r_1, r_2, r_3 are strictly positive (to be chosen later).

First of all, for any $C_1, C_2 > 0$,

$$\sup_{|x| \leq C_1} \sup_{y \leq C_2} \mathbf{E}(L(\widetilde{X}_1, \widetilde{Y}_1) \mid \widetilde{X}_0 = x, \widetilde{Y}_0 = y) < \infty. \tag{6.1}$$

Indeed, let $r = \max(r_1, r_2, r_3)$. Then, for any (x, y) from the set above,

$$\widetilde{T}_{x,y} \leq_{st} T^{(1,-C_1)} := \widehat{T},$$

and $\widehat{T}$ has a finite mean. Therefore, for all (x, y) from this set,

$$\mathbf{E}(L(\widetilde{X}_1, \widetilde{Y}_1) \mid \widetilde{X}_0 = x, \widetilde{Y}_0 = y) \leq r(C_1 + C_2 + \mathbf{E}\widehat{T} + \mathbf{E}Z + k) < \infty,$$

and (6.1) follows.

Now we impose several constraints on the coefficients r_1, r_2, r_3. First, we assume that

$$r_3 < r_1 \tag{6.2}$$

and

$$r_1 \mathbf{E}Z > r_2 k. \tag{6.3}$$

Let $\alpha - k = 2\Delta > 0$. We also assume that

$$r_2 \Delta > r_3 \mathbf{E}Z. \tag{6.4}$$

In the proof, we use only conditions (6.2)–(6.4) which are, in particular, satisfied if $r_1 >> r_2 >> r_3 > 0$.

Now we proceed to show that all the differences

$$E(L(\widetilde{X}_1, \widetilde{Y}_1) \mid (\widetilde{X}_0, \widetilde{Y}_0) = (x, y)) - L(x, y)$$

are bounded above by some negative constant if (x, y) is outside the set $[x_1, x_0] \times [0, y_0]$ where x_1, x_0 and y_0 will be chosen in the proof. For this, consider separately two cases: (a) $x > 0$ and (b) $x \leq 0$.

Case $x > 0$

In this case, the one-step embedding is the natural one, so we may write (X_1, Y_1) instead of $(\widetilde{X}_1, \widetilde{Y}_1)$.

Let

$$3\gamma = \min\{r_2 \Delta - r_3 \mathbf{E}Z, r_1 \mathbf{E}Z - r_2 k\} > 0. \tag{6.5}$$

Choose $x_0 > 0$ so big that

$$r_1 c \mathbf{E} T^{(1,x_0)} \leq \gamma, \tag{6.6}$$

and

$$(r_1 + r_3)\mathbf{E}(Z - x_0)^+ \leq \gamma. \tag{6.7}$$

By Lemma 3, we may choose $y_0 > 0$ so large that

$$y\mathbf{E}\big(1 - \exp\{-\alpha T^{(2,y)}\}\big) - k \geq \frac{5}{3}\Delta, \quad \text{for all} \;\; y \geq y_0 \tag{6.8}$$

and that

$$r_1 c\mathbf{E}T_{0,y_0} \le \gamma. \tag{6.9}$$

Then choose $v_0 > 0$ so large that

$$y\mathbf{E}\big(1 - \exp\{-\alpha \min(v_0, T^{(2,y)})\}\big) - k \ge \frac{4}{3}\Delta, \quad \text{for all} \;\; y \ge y_0 \tag{6.10}$$

and that the following inequality holds:

$$\frac{r_3 c v_0}{2} \exp\Big\{-\frac{y_0}{\alpha}\Big\} > \gamma + r_3\mathbf{E}Z + r_2 k. \tag{6.11}$$

Write for short

$$\mathbf{E}_{x,y}L(X_1, Y_1) = \mathbf{E}(L(X_1, Y_1) \mid X_0 = x, Y_0 = y) = \mathbf{E}_{x,y}L_1(X_1) + \mathbf{E}_{x,y}L_2(Y_1).$$

If $x > 0$, then

$$\begin{aligned}
\mathbf{E}_{x,y}L_1(X_1) - L_1(x) &= r_1\mathbf{E}\left((x + cT_{x,y} - Z)\,\mathbf{I}(x + cT_{x,y} - Z > 0)\right) \\
&\qquad + r_3\mathbf{E}\left((-x - cT_{x,y} + Z)\,\mathbf{I}(x + cT_{x,y} - Z \le 0)\right) - r_1 x \\
&= c\,\mathbf{E}(T_{x,y}(r_1\,\mathbf{I}(x + cT_{x,y} - Z > 0) \\
&\qquad - r_3\,\mathbf{I}(x + cT_{x,y} - Z \le 0))) + r_1\mathbf{E}(x - Z) \\
&\qquad + (r_1 + r_3)\mathbf{E}((-x + Z)\,\mathbf{I}(x + cT_{x,y} - Z \le 0)) - r_1 x \\
&\le r_1 c\,\mathbf{E}T_{x,y} - r_1\mathbf{E}Z + (r_1 + r_3)\,\mathbf{E}(Z - x)^+
\end{aligned}$$

and

$$\mathbf{E}L_2(Y_1) - L_2(y) = r_2(y\mathbf{E}\exp\{-\alpha T_{x,y}\} + k) - r_2 y.$$

In particular, if $x \ge x_0$,

$$\mathbf{E}_{x,y}L_1(X_1) - L_1(x) \le r_1 c\,\mathbf{E}T^{(1,x_0)} - r_1\mathbf{E}Z + (r_1 + r_3)\,m\mathbf{E}(Z - x_0)^+$$

(where we used representation (5.1) and Lemma 5.1) and

$$\mathbf{E}L_2(Y_1) - L_2(y) \le r_2 k,$$

so in view of (6.5), (6.6), and (6.7),

$$\mathbf{E}_{x,y}L(X_1, Y_1) - L(x, y) \le \gamma + \gamma - 3\gamma = -\gamma.$$

Furthermore, if $y \ge y_0$ and $0 \le x \le x_0$, then, by Lemma 5.1,

$$\mathbf{E}_{x,y}L_1(X_1) - L_1(x) \le r_1 c\mathbf{E}T_{0,y_0} + r_3\mathbf{E}Z$$

and, by inequality (6.8),

$$\mathbf{E}L_2(Y_1) - L_2(y) \le -r_2\Delta,$$

so

$$\mathbf{E}_{x,y}L(X_1,X_2) - L(x,y) \le r_1 c\,\mathbf{E}T_{0,y_0} + r_3\mathbf{E}Z - r_2\Delta \le \gamma - 3\gamma \le -\gamma,$$

by (6.5) and (6.9). So if $x > 0$ and if either $x \ge x_0$ or $y \ge y_0$, then

$$E_{x,y}L(X_1,Y_1) - L(x,y) \le -\gamma.$$

<u>*Case* $x \le 0$</u>

For the time being, fix any value of $x_1 < 0$. First, we observe that if $y \ge y_0$ where y_0 satisfies inequalities (6.8) and (6.9), then again the increments $E_{x,y}L(X_1,Y_1) - L(x,y)$ have a "uniformly" negative drift in all $x_1 < x \le 0$. Indeed, if $\widetilde{X}_0 = x \in (x_1, 0]$, then, for any $y \ge 0$, $\widetilde{X}_1 = x + T_{x,y} - Z_1$ admits the following bounds:

$$|\widetilde{X}_1|\,\mathbf{I}(\widetilde{X}_1 \le 0) \le_{a.s.} (|x| + Z_1)\,\mathbf{I}(\widetilde{X}_1 \le 0) \tag{6.12}$$

and

$$\widetilde{X}_1\mathbf{I}(\widetilde{X}_1 > 0) = \max(0, \widetilde{X}_1) = \max(0, \min(x + cT^{(1,x)}, x + T^{(2,y)})). \tag{6.13}$$

From Lemma 5.1 and from independence of $T^{(1,x)}$ and $T^{(2,y)}$, we obtain

$$\widetilde{X}_1\,\mathbf{I}(\widetilde{X}_1 > 0) \le_{st} \min(cT^{(1,0)}, cT^{(2,y)}) = cT_{0,y}. \tag{6.14}$$

Therefore, for any $x_1 < x \le 0$

$$\begin{aligned}\mathbf{E}_{x,y}|L_1(\widetilde{X}_1)| &= r_1\mathbf{E}_{x,y}(\widetilde{X}_1\,\mathbf{I}(\widetilde{X}_1 > 0)) + r_3\mathbf{E}_{x,y}(|\widetilde{X}_1|\,\mathbf{I}(\widetilde{X}_1 \le 0)) \\ &\le r_1\mathbf{E}_{x,y}(|\widetilde{X}_1|\cdot\mathbf{I}(\widetilde{X}_1 > 0)) + r_3\mathbf{E}_{x,y}((|x| + \mathbf{E}Z_1)\mathbf{I}(\widetilde{X}_1 \le 0)) \\ &\le r_1 c\mathbf{E}T_{0,y} + r_3|x| + r_3\mathbf{E}Z.\end{aligned}$$

Since y_0 satisfies inequalities (6.8) and (6.9), we have, for all $x_1 < x \le 0$ and $y \ge y_0$,

$$\mathbf{E}_{x,y}L(\widetilde{X}_1, \widetilde{Y}_1) - L(x,y) \le r_1 c\mathbf{E}T_{0,y_0} + r_3\mathbf{E}Z - r_2\Delta \le -\gamma.$$

We now choose $x_1 << -1$ so large that the increment of $L(\widetilde{X}, \widetilde{Y})$ has a uniformly negative drift for all $x \le x_1$. We start with the assumption that

$$x_1 \le -cv_0. \tag{6.15}$$

Therefore, if $\widetilde{X}_0 = x \le x_1$ and $\widetilde{Y}_0 = y$, then

$$L_1(\widetilde{X}_1) = r_3(-x - c\widetilde{T}_{x,y} + Z\,\mathbf{I}(T_{x,y} \le v_0)) \le r_3(-x - c\widetilde{T}_{x,y} + Z)$$

and

$$L_2(\widetilde{Y}_1) = r_2\big(y\exp\{-\alpha\widetilde{T}_{x,y}\} + k\,\mathbf{I}(T_{x,y} \le v_0)\big) \le r_2\big(y\exp\{-\alpha\widetilde{T}_{x,y}\} + k\big).$$

We impose two additional constraints on x_1 making it even more negatively large. Since $\varphi(x) \to 0$ as $x \to -\infty$, one can choose $x_1 < -cv_0$, $x_1 << -1$ such that

$$\exp\Big\{-\int_0^{v_0} \varphi(x_1 + cv)dv\Big\} \ge \frac{1}{2}. \tag{6.16}$$

Secondly, it follows that $T^{(1,x)} \to \infty$ in probability as $x \to -\infty$, and therefore, from (6.10), one can choose $x_1 \le -cv_0$ such that

$$y\,\mathbf{E}(1 - \exp\{-\alpha\widetilde{T}_{x,y}\}) - k \ge \Delta, \quad \text{for all } \ y \ge y_0 \ \text{ and } \ x \le x_1. \tag{6.17}$$

Assume that x_1 satisfies all of the three conditions (6.15)–(6.17). If $y \ge y_0$ then, for any $x \le x_1$,

$$\begin{aligned}\mathbf{E}_{x,y}L(\widetilde{X}_1, \widetilde{Y}_1) - L(x,y) &\le r_3(-c\mathbf{E}\widetilde{T}_{x,y} + \mathbf{E}Z) + r_2(y\mathbf{E}\exp\{-\alpha\widetilde{T}_{x,y}\} + k) - r_2 y\\ &\le -r_2\Delta + r_3\mathbf{E}Z \le -\gamma,\end{aligned}$$

by (6.17).

If instead $y \le y_0$ and $x \le x_1$, then

$$\mathbf{P}(T_{x,y} > v_0) \ge \mathbf{P}(T_{x,y_0} > v_0) \ge \frac{1}{2}\exp\{-y_0/\alpha\},$$

since the random variables $T_{x,y}$ are stochastically decreasing in y (again by Lemma 5.1). Therefore, for $\widetilde{T}_{x,y} = \min(v_0, T_{x,y})$,

$$\mathbf{E}T_{x,y} \ge \mathbf{E}\widetilde{T}_{x,y} \ge v_0 \cdot \mathbf{P}(T_{x,y} \ge v_0) \ge \frac{v_0}{2}\exp\{-y_0/\alpha\}, \tag{6.18}$$

and

$$\begin{aligned}\mathbf{E}_{x,y}L(\widetilde{X}_1, \widetilde{Y}_1) - L(x,y) &\le r_3(-c\mathbf{E}\widetilde{T}_{x,y} + \mathbf{E}Z) + r_2(y\mathbf{E}\exp\{-\alpha\widetilde{T}_{x,y}\} + k) - r_2 y\\ &\le -\frac{r_3 c v_0}{2}\exp\{-y_0/\alpha\} + r_3\mathbf{E}Z + r_2 k < -\gamma,\end{aligned}$$

due to (6.11).

As an outcome, we have that if y_0 satisfies conditions (6.8)–(6.9), if v_0 satisfies (6.10)–(6.11), and if x_1 satisfies (6.15)–(6.17), then the increments of $\mathbf{E}L(\widetilde{X}_1, \widetilde{Y}_1)$ have a drift bounded above by $-\gamma$ for all initial values such that either $x \le x_1$, or $x \le 0$ and $y \ge y_0$.

The set

$$V = [x_1, x_0] \times [0, y_0]$$

is therefore positive recurrent for the Markov chain $(\widetilde{X}_n, \widetilde{Y}_n)$.

Also, as follows from the classical proof of Foster's criterion, for any initial value $(\widetilde{X}_0, \widetilde{Y}_0) = (x, y)$, a random variable

$$\tau_{x,y}(V) = \min\{n \geq 1 : (\widetilde{X}_n, \widetilde{Y}_n) \in V \mid (X_0, Y_0) = (x, y)\}$$

is almost-surely finite and, moreover, there exists an absolute constant $C > 0$ such that

$$\mathbf{E}\tau_{x,y}(V) \leq C(L(x, y) + 1),$$

for all (x, y) (see, e.g., [11] or [8]). The proof of Theorem 6.1 is complete. □

7. Harris ergodicity

We recall the following classical result (see for instance [11]).

Proposition 7.1. *Assume that a Markov chain $W_n, n \geq 0$, taking values in a measurable space $(\mathcal{W}, \mathcal{B}_{\mathcal{W}})$ is aperiodic and that there exists a positive recurrent set V that admits a minorant measure, i.e. there exist a positive integer m, a positive $p \leq 1$ and a probability measure μ such that*

$$\mathbf{P}(W_m \in \cdot \mid W_0 = w \in V) \geq p\mu(\cdot). \tag{7.1}$$

Then the Markov chain is Harris ergodic, *which means that there exists a unique stationary distribution (say π) and that, for any initial value $W_0 = w$, there is a convergence of the distributions of W_n to the stationary one in the total variation norm,*

$$\sup_{B \in \mathcal{B}_{\mathcal{W}}} |P(W_n \in B) - \pi(B)| \to 0, \quad n \to \infty.$$

In practice, the most technical part in applying this criterion is to verify the aperiodicity. There are a number of sufficient conditions available for the Markov chain to be aperiodic and Harris ergodic.

We mention two of them. The most common is the following condition.

Sufficient condition 1 (SC1). A Markov chain W_n is Harris ergodic if there exists a positive recurrent set V such that condition (7.1) holds with $m = 1$ and with μ such that $\mu(V) > 0$.

However, in our proof, it seems to be easier to verify another — slightly more general — sufficient condition.

Sufficient condition 2 (SC2). A Markov chain W_n is Harris ergodic if there exists a positive recurrent set V such that condition (7.1) holds with a finite number of different values of m, say m_i, $i = 1, 2, \ldots, k$ which are such that

$$g.c.d.\{m_i, 1 \leq i \leq k\} = 1.$$

We will apply condition (SC2) with $k = 2$ and with $m_1 = 2$ and $m_2 = 3$. For that, we introduce a condition on the distribution of Z which leads to (SC2).

Condition (CZ). There exist $0 \le z_1 < z_2 < \infty$ such that, for some $h > 0$ and for any $[u_1, u_2] \subseteq [z_1, z_2]$,

$$\mathbf{P}(Z \in [u_1, u_2]) \ge h(u_2 - u_1).$$

In other words, the distribution of Z has an absolutely continuous (with respect to Lebesgue measure) component whose density function is above level h everywhere in the interval $[z_1, z_2]$.

Theorem 7.1. *Assume condition* (*CZ*) *holds. Then the Markov chain* (X_n, Y_n) *is Harris ergodic.*

Proof. We may assume without loss of generality that $z_2 - z_1 \le x_0 - x_1$.

Let $\widetilde{y}_1 = y_1 + k$ and

$$\widetilde{x}_0 = \inf\{x \ge x_0 : \varphi(x) > 0\} + z_2 + 2,$$

and let

$$V_1 = [x_1, \widetilde{x}_0] \times [0, \widetilde{y}_0],$$

so $V \subset V_1$. Then, for any $(x, y) \in V$, $\mathbf{P}_{x,y}(Y_1 \le \widetilde{y}_0) = 1$, so, by Lemmas 5.1–5.2,

$$\begin{aligned}\mathbf{P}_{x,y}((X_1, Y_1) \in V_1) &= \mathbf{P}_{x,y}(X_1 \in [x_1, \widetilde{x}_0]) \\ &\ge \mathbf{P}(x + T_{x,y} \in [\widetilde{x}_0 + z_1 - 1, \widetilde{x}_0 + z_1], Z_1 \in [z_1, z_2]) \\ &\ge \mathbf{P}(x_1 + cT^{(1,x_1)} \in [\widetilde{x}_0 + z_1 - 1, \widetilde{x}_0 + z_1]) \\ &\quad \times \mathbf{P}(cT^{(2,\widetilde{y}_0)} > \widetilde{x}_0 + z_1 - x_1)\, \mathbf{P}(Z_1 \in [z_1 . z_2]).\end{aligned}$$

Denote by R_0 the value of the rightmost side of the above inequality (note that it is positive). Then, for any $(x, y) \in V_1$,

$$\mathbf{P}_{x,y}((X_1, Y_1) \in V_1) \ge R_0 > 0.$$

Take some small positive $\varepsilon < (z_2 - z_1)/4$ (to be specified later). Choose $t_2 > 0$ so large that $x_2 := x_1 + ct_2 > \widetilde{x}_0 + z_2$ and $y_0 \exp\{-\alpha(x_2 - \widetilde{x}_0)\} \le \varepsilon$. Let $b = \varphi(x_2 - z_2)$ and note that $b > 0$. Then, for any $(x, y) \in V_1$,

$$\begin{aligned}&\mathbf{P}(x + cT_{x,y} \in [x_2, x_2 + \varepsilon], y \exp\{-\alpha T_{x,y}\} \le \varepsilon) \\ &\qquad \ge b\varepsilon\, \mathbf{P}(T^{(1,x_1)} > (x_2 - x_1)/c)\, \mathbf{P}(T^{(2,y_0)} > (x_2 - x_1)/c).\end{aligned}$$

Denote by R_1 the right-hand side of the inequality above (which is a positive number). Then, for any $(x, y) \in V_1$ and for $(X_1, Y_1) = (x + T_{x,y} - Z_1, y \exp\{-\alpha T_{x,y}\} + k)$,

$$\mathbf{P}_{x,y}((X_1, Y_1) \in [x_2 - z_1 - \varepsilon, x_2 - z_1] \times [k, k + \varepsilon]) \ge R_1 \frac{b\varepsilon}{z_2 - z_1} =: R_2 > 0.$$

Let

$$\widehat{V} = [x_2 - z_1 - \varepsilon, x_2 - z_1] \times [k, k + \varepsilon].$$

From the construction above, one may conclude that, for any $(x, y) \in V$,

$$\inf_{(x,y)\in V} \mathbf{P}_{x,y}((X_1, Y_1) \in \widehat{V}) \geq R_2 > 0 \tag{7.2}$$

(since $V \subset V_1$) and then that, by the Markov property,

$$\mathbf{P}_{x,y}((X_2, Y_2) \in \widehat{V}) \geq R_0 \cdot \inf_{(x,y)\in V_1} \mathbf{P}_{x,y}((X_1, Y_1)\widehat{V}) = R_0 R_2 > 0. \tag{7.3}$$

Now take $\varepsilon > 0$ so small that one can choose positive numbers t_3 and t_4 such that $t_4 > t_3 > z_2$, that

$$k_2 := k \exp\{-\alpha t_3\} > (k + \varepsilon) \exp\{-\alpha t_4\} =: k_1,$$

and that

$$\delta := \varepsilon + c(t_4 - t_3) < \frac{z_2 - z_1}{2}.$$

Then, for any $y \in [k, k + \varepsilon]$, we have the inclusion

$$[k_1, k_2] \subseteq [y \exp\{-\alpha t_4\}, y \exp\{-\alpha t_3\}].$$

For any $(x, y) \in \widehat{V}$ denote by $g_{x,y}(u)$ a density function of random variable $y \exp\{-\alpha T_{x,y}\}$ (which clearly has an absolutely continuous distribution).

Then direct computations show that

$$c_0 := \inf_{(x,y)\in\widehat{V}} \inf_{u\in[k_1,k_2]} g_{x,y}(u). \tag{7.4}$$

Indeed, let

$$c_1 = \inf_{0\leq t\leq k_2/k_1} \frac{\ln(1 + t)}{t}$$

and

$$c_2 = \inf_{(x,y)\in\widehat{V}} \inf_{\ln(k/k_2)\leq a<b\leq\ln((k+\varepsilon)/k_1)} \frac{\mathbf{P}(a \leq T_{x,y} \leq b)}{b - a}.$$

Then both c_1 and c_2 are positive and, for $[a, b] \subseteq [k_1, k_2]$,

$$\mathbf{P}\big(y \exp\{-\alpha T_{x,y}\} \in [a, b]\big) = \mathbf{P}\Big(\frac{\ln(y/b)}{\alpha} \leq T_{x,y} \leq \frac{\ln(y/a)}{\alpha}\Big) \geq \frac{c_1 c_2}{\alpha}(b - a),$$

so (7.4) holds with $c_0 = c_1 c_2/\alpha$.

Furthermore, let $x_3 = x_2 - z_1 - \varepsilon + t_3$. Note that if $(x, y) \in \widehat{V}$ and $T_{x,y} \in [t_3, t_4]$, then $x + cT_{x,y} \in [x_3, x_3 + \delta]$. Then, by condition (CZ), given $x + cT_{x,y} = v \in [x_3, x_3+\delta]$, the random variable $v - Z_1$ has an absolutely continuous component with a uniform distribution on the interval $[x_3 - (z_1 + z_2)/2, x_3 - z_1]$.

We may therefore conclude that, for any $(x, y) \in \widehat{V}$,

$$\mathbf{P}_{x,y}\big((X_1, Y_1) \in \cdot\big) \geq 2(z_2 - z_1)^{-1} h c_0^{-1} \mu(\cdot) \tag{7.5}$$

where μ is a two-dimensional uniform distribution on the rectangle

$$V_2 := [x_3 - (z_1 + z_2)/2, x_3 - z_1] \times [k_1, k_2],$$

coefficient h is from condition (CZ), and c_0 is from (7.4).

It follows from inequalities (7.2), (7.3), and (7.5), that condition (SC2) is satisfied with $k = 2$ and with $m_1 = 2$ and $m_2 = 3$, and this completes the proof. □

Corollary 7.1. *Assume again that $k < \alpha$ and that condition* (SZ) *holds. Consider the Markov chain $(\widetilde{X}_n, \widetilde{Y}_n)$ and let $0 \leq S_1 < S_2 < \ldots S_k < \ldots$ be consecutive times of the ends of discrete-time "cycles" where random vectors $(\widetilde{X}_{S_k}, \widetilde{Y}_k)$ have uniform distribution in the rectangle V_2. Then random variables $l_k = S_k - S_{k-1}$, $k \geq 2$, are i.i.d. with a finite mean.*

A proof of this result can be found, for instance, in [1].

8. Stability in continuous time

Theorem 8.1. *Under condition (CZ),*

(1) *there exists a unique stationary version of the continuous-time Markov process $(X(t), Y(t))$ (which is also ergodic);*

(2) *for any initial value $X(0) = X_0 = x$, $Y(0) = Y_0 = y$, the process $(X(t), Y(t))$ converges to the stationary one in the total variation norm.*

Proof. Consider again the embedded Markov chain $(\widetilde{X}_n, \widetilde{Y}_n)$ and its cycles of length l_k. Then the corresponding cycles in continuous time are defined as

$$L_k = \sum_{i=S_{k-1}+1}^{S_k} \widehat{T}_{\widehat{X}_i, \widehat{Y}_i}, \quad k = 1, 2, \ldots$$

which are again i.i.d. for $k \geq 2$.

Then a proof of the theorem follows from the two results below, Statement 8.1 and Statement 8.2 (see, for instance, [1], Proposition 3.8, p.203 or [2], Section 7, or [3], Chapter 3). □

Statement 8.1. *The distribution of random variables L_k, $k \geq 2$ has an absolutely continuous component with a density function which is separated from* 0 *on a finite time interval of positive length.*

Statement 8.2. $\mathbf{E}L_2$ *is finite.*

Statement 8.1 may be verified directly using arguments similar to those in the previous section. Furthermore, since $C := \sup_{x,y} \mathbf{E}\widetilde{T}_{x,y} < \infty$,

$$\mathbf{E}L_2 \leq C\mathbf{E}l_2 < \infty,$$

and the result follows.

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