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On the Existence of Moments for the Busy Period in a Single-Server Queue

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We obtain sufficient conditions for the finiteness of the moments of the single-server queue busy period in discrete and continuous time. In the cases of power and exponential moments, the conditions found coincide with the known necessary ones.

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1. Introduction and notation. Consider a sequence $\{(t_n, \sigma_n)\}$ of random vectors with nonnegative coordinates which are i.i.d. in n, but any dependence between the components is allowed. Put $a = \mathbf{E}t_1 - \mathbf{E}\sigma_1$. Assume that $a > 0$. Let $\xi_n = \sigma_n - t_n$, $S_n = \sum_{i=1}^n \xi_i$, and $\tau = \min\{n \geq 1: S_n \leq 0\} < \infty$ a.s. be the first passage time to the nonpositive half-line. Here $\mathbf{E}\xi_1 = -a < 0$. Let $B = \sum_{i=1}^{\tau} \sigma_i$.

In what follows, we investigate conditions for the finiteness of the moments $\mathbf{E}\exp(g(B))$ and $\mathbf{E}\exp(g(\tau))$, where the function g is taken from a rather wide class which includes the logarithmic ($g(x) = \alpha \log x$, $\alpha \geq 1$), the linear functions ($g(x) = cx$, $c > 0$), and functions between the logarithmic and linear (e.g., $g(x) = x^\beta$, $\beta \in (0, 1)$ and $g(x) = (\log x)^\gamma$, $\gamma > 1$).

The main motivation for our studies follows from queueing theory, where the random variables τ and B have simple interpretations and appear in various queueing problems. Indeed, consider a single-server queue with inter-arrival times $\{t_n\}$ and service times $\{\sigma_n\}$. Since $a > 0$, the queue is stable. Suppose that customer 1 arrives at the empty system at time instant 0. Then B is the (first) busy period of the system (in continuous time) and τ is the number of customers that are served within that busy period. Consider the discrete time imbedded Markov chain $\{W_n\}_{n\geq 1}$ where W_n represents the waiting time of customer n. Then $\tau = \min\{n \geq 2: W_n = 0\} - 1$ may be viewed as the "busy period in discrete time" (busy cycle). Not only are τ and B interesting on their own, but there are plenty of examples of the use of τ and B as estimates for other quantities in the queueing theory.

Note that our results are related to the recent studies of the tail asymptotics for the busy period distribution in the single-server queue with heavy tails (see, e.g., Borovkov 2000, Jelenkovic et al. 2002, Zwart 2001). Indeed, knowledge of the asymptotics of the tail distribution gives complete information on the existence of moments. However, the results in Borovkov (2000), Jelenkovic et al. (2002), and Zwart (2001) relate to particular subclasses of so-called subexponential distributions only (see also Remark 2). In addition, we are not aware of any tail asymptotic results for B when the distribution of σ has a "Weibull-type" tail $e^{-x^\alpha L(x)}$ with $\alpha \in (1/2, 1)$.

In the case $g(x) = \alpha \log x$, it is well known (see, e.g., Gut 1974a, Theorem 2.1) that $\mathbf{E}\tau^\alpha < \infty$ if and only if $\mathbf{E}(\xi_1^+)^\alpha < \infty$ (here $x^+ = \max(0, x)$) and if and only if $\mathbf{E}|\max(\xi_1, -L)|^\alpha < \infty$ for any number $L \geq 0$. In particular, when σ_1 and t_1 are independent, any of the conditions above is equivalent to the finiteness of $\mathbf{E}\sigma_1^\alpha$, and also to the finiteness of $\mathbf{E}B^\alpha$ (see, e.g., Ghahramani et al. 1989 and references therein).

For the exponential moments, it is known (see, for example, Borovkov 1962, Theorem 2) that if $\mathbf{E}\exp(c\xi_1) < \infty$ for some $c > 0$ then $\mathbf{E}\exp(u\tau)$ is finite for all $u < \theta$ and infinite for all $u > \theta$, where $e^{-\theta} = \inf_{\alpha>0} \mathbf{E}\exp(\alpha\xi_1) < 1$. It is shown in Doney (1989, Theorem 2) that $\mathbf{E}\exp(\theta\tau)$ is finite if some extra conditions hold. With regard to the existence of $\mathbf{E}\exp(uB)$, we are not aware of any general results for dependent σ_n and t_n. Note that, for the $D/GI/1$ queue with $t_n \equiv t = const$, if $\mathbf{E}\exp(c\sigma_1)$ is finite for some $c > 0$, then $\mathbf{E}\exp(uB)$ is finite for all $u < \theta/t$ and infinite for all $u > \theta/t$, where θ is as above. This is, in fact, an immediate consequence of the result for $\mathbf{E}\exp(u\tau)$.

In a number of papers (see, e.g., Kalashnikov 1978, Chapter 4, Theorem 2.1, and the list of references therein) conditions for the finiteness of the moments of τ are studied in a more general stochastic setting (for Markov chains), but the functions $G(x) \equiv \exp g(x)$ considered are assumed to be convex and their derivatives are assumed to be concave (in particular, the function $G(x) = x^\alpha$ satisfies these conditions iff $\alpha \in [1, 2]$).

Some other papers (see, e.g., Gut 1974b, Alsmeyer 1987, and the list of references therein) deal with the problem of the finiteness of the moments of the first hitting time on some set (under more general stochastic assumptions).

In order to get unified results, fix two nonnegative numbers k_0 and k_1 and put $\gamma_n = k_0 + k_1\sigma_n$, $\Gamma = \sum_{i=1}^{\tau} \gamma_i$. Note that $\Gamma = B$ if $k_0 = 0$, $k_1 = 1$ and $\Gamma = \tau$ if $k_0 = 1$, $k_1 = 0$. First, in Theorem 2.1, we obtain sufficient conditions for the existence of $\mathbf{E}\exp(g(\Gamma))$ for the class of eventually nondecreasing and concave functions g. In particular, for power moments, the conditions obtained are also necessary for both the moments of τ and of B. For exponential moments, our conditions are also necessary for the moments of τ. Second, we consider (see Theorem 2.2) a subclass of functions g that are lighter than any linear function ($g(x) = o(x)$, $x \to \infty$) and for which relatively simple conditions imply the conditions of Theorem 2.1.

The paper is organized as follows. In §2 we state the main results of the paper. We illustrate our results by examples in §3. Section 4 contains the proofs of the results.

2. The results. Recall that we consider a sequence $\{(t_n, \sigma_n)\}$ of random vectors with nonnegative coordinates which are i.i.d. in n, but any dependence between the components of the vectors is allowed. Let $\xi_n = \sigma_n - t_n$, with $\mathbf{E}\xi_1 = -a < 0$, $S_n = \sum_{i=1}^{n} \xi_i$, and $\tau = \min\{n \geq 1\colon S_n \leq 0\}$. Fix two nonnegative numbers k_0 and k_1 and consider $\Gamma = \sum_{i=1}^{\tau} \gamma_i$, where $\gamma_i = k_0 + k_1\sigma_i$. In particular, if $k_0 = 0$, $k_1 = 1$, then $\Gamma = B = \sum_{i=1}^{\tau} \sigma_i$ (the first busy period in the queueing context), and if $k_0 = 1$, $k_1 = 0$, then $\Gamma = \tau$ (the number of customers served within the first busy period or the first busy period in discrete time in the queueing context).

THEOREM 2.1. *Let g be a function bounded from above on compact sets and eventually nondecreasing and concave.*

Suppose that, for some $c > -k_1$,

$$\mathbf{E}e^{g(c\xi_1^+ + k_1\sigma_1)} < \infty, \tag{2.1}$$

and there exist $L > 0$ and $x_0 \geq 0$ such that if $\tilde{\xi}_1 = \max(\xi_1, -L)$, then $\mathbf{E}\tilde{\xi}_1 < 0$ and, for all $x \geq x_0$,

$$\mathbf{E}\exp\bigl\{g(x + c\tilde{\xi}_1 + \gamma_1) - g(x)\bigr\} \leq 1. \tag{2.2}$$

Then

$$\mathbf{E}e^{g(\Gamma)} < \infty. \tag{2.3}$$

In particular, for $c > 0$, condition (2.1) with $k_1 = 0$ and condition (2.2) with $\gamma_1 = 1$ imply the finiteness of $\mathbf{E}\exp(g(\tau))$. Similarly, for $c > -1$, (2.1) with $k_1 = 1$ and (2.2) with $\gamma_1 = \sigma_1$ imply the finiteness of $\mathbf{E}\exp(g(B))$.

Remark 1. The result of Theorem 2.1 also holds for functions g for which the concavity and nondecreasing conditions are replaced by the following one:

$$g(x+y) \le g(x) + g(y^+) + K \quad \text{for some } K \text{ and all } x \ge x_0. \tag{2.4}$$

Remark 2. Note that $\sigma_1 \ge \xi_1^+ = (\sigma_1 - t_1)^+$ a.s. Hence, (2.1) and (2.2) hold if for some $c > 0$,

$$\mathbf{E}e^{g((c+k_1)\sigma_1)} < \infty, \tag{2.5}$$

and $\mathbf{E}\exp\{g(x + c\hat{\xi}_1 + \gamma_1) - g(x)\} \le 1$ for all $x \ge x_0$, where $\hat{\xi}_1 = \sigma_1 - \min(t_1, L) \ge \tilde{\xi}_1$ for some $L > 0$ such that $\mathbf{E}\hat{\xi}_1 < 0$.

Moreover, for $c > 0$, (2.1) and (2.5) are equivalent if σ_1 and t_1 are independent.

Remark 3. In the exponential case ($e^{g(x)} = e^{ux}$), Theorem 2.1 gives the well-known sufficient condition for the finiteness of $\mathbf{E}e^{g(\tau)}$: if $\mathbf{E}e^{k\xi_1} < \infty$ for some $k > 0$, then $\mathbf{E}e^{u\tau} < \infty$ for all $u < \theta$, where $e^{-\theta} = \inf_{\alpha>0} \mathbf{E}e^{\alpha\xi_1}$.

To see this, let α_0 be such that $\mathbf{E}e^{\alpha_0\xi_1} = e^{-\theta}$ (note that α_0 always exists). Take an arbitrary $u < \theta$. The conditions of Theorem 2.1 are valid for $c = \alpha_0/u$ and $g(x) = ux$, since (2.1) follows from the finiteness of $\mathbf{E}e^{\alpha_0\xi_1}$ and (2.2) from the inequalities:

$$\mathbf{E}\exp\big(g(x + c\xi_1 + 1) - g(x)\big) = \mathbf{E}\exp(\alpha_0\xi_1 + u) = e^{-\theta+u} < 1.$$

Hence $\mathbf{E}e^{g(\tau)} < \infty$ as required.

We now provide some conditions which yield (2.2) for functions g which increase slower than linear functions. We write $\ln^+ x = \max(0, \ln x^+)$. Let h' denote the left derivative of any concave function h. Recall that $a = -\mathbf{E}\xi_1$. Let $b = \mathbf{E}\sigma_1$.

Theorem 2.2. *Let the function g be such that the function $h(x) = g(x) - \ln^+ x$ is bounded from above on compact sets, eventually nondecreasing and concave, and further $h(x) = o(x)$ as $x \to \infty$. If*

$$\mathbf{E}e^{g(c\xi_1^+ + k_1\sigma_1)} < \infty \qquad \textit{for some } c > (k_0 + k_1 b)/a, \tag{2.6}$$

then (2.1) *and* (2.2) *hold, and hence* $\mathbf{E}e^{g(\Gamma)} < \infty$.

Corollary 2.1. *Under the conditions of Theorem* 2.2 *the following statements hold via specific choices of k_0 and k_1. If*

$$\mathbf{E}e^{g(c\xi_1^+)} < \infty \qquad \textit{for some } c > 1/a, \tag{2.7}$$

then

$$\mathbf{E}e^{g(\tau)} < \infty, \tag{2.8}$$

and if

$$\mathbf{E}e^{g(c\xi_1^+ + \sigma_1)} < \infty \qquad \textit{for some } c > b/a, \tag{2.9}$$

then

$$\mathbf{E}e^{g(B)} < \infty. \tag{2.10}$$

In particular, if σ_n and t_n are independent, then conditions (2.7) *and* (2.9) *are equivalent to* (2.5) *with $k_1 = 0$ and $k_1 = 1$, respectively.*

We further give an analogue of the result above for functions which are only asymptotically equivalent to nondecreasing and concave ones.

Corollary 2.2. *Let the function g be bounded from above on compact sets and satisfy the following conditions*:

(a) $\limsup_{x\to\infty} g(x)/g(rx) \equiv b(r) < 1$ *for all* $r > 1$;

(b) $g(x) = o(x)$, *as* $x \to \infty$; *and*

(c) *there exists a nondecreasing and concave function* g_1 *such that*

$$g(x) = g_1(x)(1 + o(1)).$$

If

$$\mathbf{E}e^{g(c\xi_1^+)} < \infty \qquad \text{for some } c > 1/a, \tag{2.11}$$

then

$$\mathbf{E}e^{g(\tau)} < \infty. \tag{2.12}$$

Similarly, if

$$\mathbf{E}e^{g(c\sigma_1)} < \infty \qquad \text{for some } c > 1 + b/a, \tag{2.13}$$

then

$$\mathbf{E}e^{g(B)} < \infty. \tag{2.14}$$

Proof of Corollary 2.2. The proofs of (2.12) and (2.14) are similar to each other, and we provide only the first.

Note that for any random variable η and for any two functions v and v_1 which are bounded from above on compact sets, if $v(x) = (1 + o(1))v_1(x) \to \infty$ as $x \to \infty$ and if $\mathbf{E}e^{v(\eta)}$ is finite, then $\mathbf{E}e^{v_1(c_1\eta)}$ is finite for all $c_1 < 1$.

Hence, without loss of generality, we can assume that the function g_1 satisfies (2.11) with the same constant c.

By the condition (a), the function $h_1(x) = g_1(x) - \ln^+(x)$ is eventually monotone and concave. Thus, the statement of Corollary 2.1 holds for the function g_1.

Consider $c_1 \in (1/a, c)$. Then the statement of Corollary 2.1 is also valid for the function $\tilde{g}(x) \equiv g_1(cx/c_1)$ if one replaces c by c_1 in (2.7). Hence, $\mathbf{E}e^{g_1(c\tau/c_1)} < \infty$ and $\mathbf{E}e^{g(\tau)} < \infty$. □

Remark 4. The sharpness of the conditions obtained may be illustrated as follows:

(a) It follows from the Strong Law of Large Numbers (see also, e.g., the proof of Heyde 1964, Lemma 3) that, for any $0 < \rho < 1$ and $\varepsilon > 0$, for all sufficiently large n,

$$\mathbf{P}(S_n > 0) \geq n\rho\mathbf{P}\big(\xi_1 + a > (a + \varepsilon)n\big).$$

(b) By applying the circular permutation method and Feller (1971, Chapter 12, §6, Lemma 2) it follows that

$$\mathbf{P}(\tau > n) \geq \mathbf{P}(S_n > 0)/n.$$

Combining (a) and (b) we get that, for any $0 < \rho < 1$ and $0 < c < 1/a$,

$$\mathbf{P}(\tau > n) \geq \rho\mathbf{P}(c\xi_1 > n)$$

for all sufficiently large n.

Thus if $\mathbf{E}e^{g(\tau)} < \infty$, then $\mathbf{E}e^{g(c\xi_1)} < \infty$ for all $c < 1/a$.

3. Examples. In this section, several examples of functions g which satisfy the conditions of Theorem 2.2 are considered. We also discuss the paper of Borovkov (2000), in which conditions for the finiteness of $\mathbf{E}\exp(g(\tau))$ are obtained for a specific class of functions g. We show that the class of functions considered is strictly included in the one from Corollary 2.2.

Example 1. The following are examples of functions g which satisfy the conditions of Theorem 2.2.

- Let $g(x) = \alpha \ln x$ for $x \ge x_1 \ge 1$, where $\alpha \ge 1$. Then (2.1) is equivalent to the finiteness of $\mathbf{E}(\xi_1^+)^\alpha$ in case of $\Gamma = \tau$ and to the finiteness of $\mathbf{E}(\sigma_1)^\alpha$ in case of $\Gamma = B$. As remarked in the Introduction, it is well known that these conditions are necessary and sufficient for the finiteness of $\mathbf{E}\tau^\alpha$ and $\mathbf{E}B^\alpha$, respectively.
- Let $g(x) = \alpha(\ln x)^\beta$ for $x \ge x_1 \ge 1$, where $\alpha > 0$, $\beta > 1$. Then (2.8) follows from the condition $\mathbf{E}e^{g(c\xi_1^+)} < \infty$ for some $c > 1/a$ and (2.10) follows from the condition $\mathbf{E}e^{g(c\xi_1^+ + \sigma_1)} < \infty$ for some $c > b/a$.
- Let $g(x) = x^\alpha$ for $x \ge x_1 \ge 0$, where $\alpha \in (0, 1)$. Then condition (2.7) reduces to $\mathbf{E}e^{c(\xi_1^+)^\alpha} < \infty$ for some $c > (1/a)^\alpha$ and condition (2.9) reduces to $\mathbf{E}e^{(c\xi_1^+ + \sigma_1)^\alpha} < \infty$ for some $c > b/a$.
- Let $g(x) = x/\ln x$ for $x \ge x_1 > 1$. If $\mathbf{E}e^{c\xi_1/\ln \xi_1}\mathbf{I}(\xi_1 > x_1) < \infty$ for some $c > 1/a$, then (2.8) holds. The finiteness of $\mathbf{E}e^{(c\xi_1^+ + \sigma_1)/\ln(c\xi_1^+ + \sigma_1)}\mathbf{I}(c\xi_1^+ + \sigma > x_1)$ for some $c > b/a$ implies (2.10).
- As observed above in Corollary 2.2, the result of Theorem 2.2 is also valid for functions of the form $g(x)(1 + o(1))$. Consider, for example, the function $g(x) = x^\alpha(1 + o(1))$. The finiteness of $\mathbf{E}\exp\{c(\xi_1^+)^\alpha\}$ for some $c > (1/a)^\alpha$ implies (2.12), and the finiteness of $\mathbf{E}\exp\{c{\sigma_1}^\alpha\}$ for some $c > (1 + b/a)^\alpha$ results in (2.14).

Example 2. The paper of Borovkov (2000) deals with functions g which satisfy the condition

$$g(x+y) - g(x) = \frac{\alpha g(x)}{x} y(1 + o(1)), \qquad y = o(x),\ 0 < \alpha < 1. \tag{3.1}$$

This condition is related to slow variation with remainder as treated, for example, in de Haan and Stadtmuller (1996), Goldie and Smith (1987), Bingham et al. (1987), and de Haan and Resnick (1996). Note that (see, e.g., Theorem 2.2.2 in Goldie and Smith 1987, Theorem 3.12.2 in Bingham et al. 1987) this is a subclass of regularly varying functions

$$g(x) = x^\alpha L(x), \qquad 0 < \alpha < 1, \tag{3.2}$$

where $L(x)$ is a slowly varying function such that $\ln L(x) = C + o(1/x) + \int_1^x (\varepsilon(t)/t)\, dt$ for some function $\varepsilon(t) \to 0$ as $t \to \infty$ and some constant C.

For such functions g, the finiteness of $\mathbf{E}\exp(g(\tau))$ follows from Borovkov (2000, Corollary 5.1). On the other hand, this result also follows from Corollary 2.2. To show this, consider, for $x \ge 1$, the nondecreasing function

$$\tilde{g}(x) = \int_1^x \frac{\alpha g(z)}{z}\, dz.$$

Without loss of generality we can assume that $g(x) = x^\alpha L(x)$, where $\ln L(x) = \int_1^x (\varepsilon(t)/t)\, dt$ for a function $\varepsilon(t) \to 0$ as $t \to \infty$.

Since $g'(x) = (\alpha g(x)/x)(1 + o(1))$, as $x \to \infty$,

$$g(x) = \tilde{g}(x)(1 + o(1)), \qquad \text{as } x \to \infty.$$

Also, $\tilde{g}$ is eventually concave, since its derivative is an eventually decreasing function. This follows since, for $y > 0$, by (3.1),

$$\alpha\frac{g(x+y)}{x+y} - \alpha\frac{g(x)}{x} \le \alpha(\alpha - 1)\frac{g(x)}{x(x+y)} y(1 + o(1)) < 0, \qquad \text{as } x \to \infty.$$

Below we show by example that the class of (eventually) nondecreasing and concave functions is wider than the class of functions (3.2).

EXAMPLE 3. Let $0 < \alpha < 1 < \beta$. Let the function h be defined by $h(x_n) = x_n^\alpha$ for

$$x_0 = 2, \qquad x_{n+1} = x_n^\beta$$

with piecewise linear interpolation elsewhere. Then h is a monotone concave function.

Put $\gamma = \alpha + \beta - \alpha\beta$ and $\tilde{x}_n = x_n^\gamma$. Then $\gamma \in (1, \beta)$ and $\tilde{x}_n \in (x_n, x_{n+1})$ for all n. Further, as $n \to \infty$,

$$\frac{h(\tilde{x}_n)}{2\tilde{x}_n^{\alpha/\gamma}} \to 1.$$

Thus h cannot be represented in the form $x^\delta L(x)$ for any $\delta \in (0, 1)$ and slowly varying function L.

4. The proofs.

PROOF OF THEOREM 2.1. Without loss of generality we can assume that $g(x) = 0$ for $x \leq 0$. In order to avoid technicalities, assume that g is nondecreasing and concave on $[0, \infty)$. Let $\tilde{\xi}_n = \max(\xi_n, -L)$ and $\widetilde{S}_n = \sum_1^n \tilde{\xi}_i$ for $n = 1, 2, \ldots$, and put $\tilde{\tau} = \min\{n \geq 1 : \widetilde{S}_n \leq 0\}$. Note that $\tilde{\tau} \geq \tau$ since $\tilde{\xi}_i \geq \xi_i$ a.s.

Let $\psi_i = c\tilde{\xi}_i + \gamma_i$, $\Psi_n = \sum_{i=1}^n \psi_i$. Then

$$\Gamma = \sum_{i=1}^{\tau} \gamma_i \leq \sum_{i=1}^{\tilde{\tau}} \gamma_i = \sum_{i=1}^{\tilde{\tau}} (\gamma_i + c\tilde{\xi}_i) - c\widetilde{S}_{\tilde{\tau}} \equiv \Psi_{\tilde{\tau}} - c\widetilde{S}_{\tilde{\tau}},$$

where $-c\widetilde{S}_{\tilde{\tau}} \leq c^+ L$ a.s.

Since

$$g(\Gamma) \leq g\big(\Psi_{\tilde{\tau}} - c\widetilde{S}_{\tilde{\tau}}\big) \leq g(\Psi_{\tilde{\tau}}) + g(c^+ L),$$

$\mathbf{E}e^{g(\Gamma)} < \infty$ if $\mathbf{E}e^{g(\Psi_{\tilde{\tau}})} < \infty$.

Since $\xi_n^+ - L \leq \tilde{\xi}_n \leq \xi_n^+$, it follows from (2.1) that $\mathbf{E}\exp\{g(\psi_1^+)\} < \infty$. Therefore, for all $n = 2, 3, \ldots$,

$$\mathbf{E}\exp(g(\Psi_n)) \leq \mathbf{E}\exp\left(\sum_1^n g(\psi_i^+)\right) = \big(\mathbf{E}\exp\{g(\psi_1^+)\}\big)^n < \infty. \tag{4.1}$$

Consider random variables $Z_n = e^{g(\Psi_{\tilde{\tau}})}\mathbf{I}(\tilde{\tau} \leq n)$. Observe that $Z_n \leq Z_{n+1}$ a.s. for all n and $Z_n \to e^{g(\Psi_{\tilde{\tau}})}$ a.s. as $n \to \infty$. Hence, $\mathbf{E}Z_n \to \mathbf{E}e^{g(\Psi_{\tilde{\tau}})} \leq \infty$. Since

$$0 \leq Z_n \leq \sum_{i=1}^n \exp\{g(\Psi_i)\},$$

$\mathbf{E}Z_n$ is finite for all n.

Now we show that $\lim_{n\to\infty} \mathbf{E}Z_n$ is also finite. Put $\tilde{\tau}_n = \min(\tilde{\tau}, n)$ and introduce random variables $U_n = \exp(g(\Psi_{\tilde{\tau}_n}))$, that is,

$$U_n = \exp\left(g\left(\sum_{i=1}^{\tilde{\tau}_n} \psi_i\right)\right) = \exp\left(g\left(\sum_{i=1}^n \psi_i \mathbf{I}(\tilde{\tau} \geq i)\right)\right) \leq \exp\left(\sum_1^n g(\psi_i^+)\right).$$

It follows from (4.1) that $\mathbf{E}U_n < \infty$. Clearly, $Z_n \leq U_n$ a.s. Therefore, it is sufficient to prove that

$$\sup_n \mathbf{E}U_n < \infty. \tag{4.2}$$

Consider

$$\mathbf{E}U_{n+1} - \mathbf{E}U_n = \mathbf{E}(U_{n+1} - U_n) = \mathbf{E}((U_{n+1} - U_n)\mathbf{I}(\tilde{\tau} \geq n+1)) = \mathbf{E}V_n,$$

where

$$V_n = \{\mathbf{E}(U_{n+1}/\mathscr{F}_n) - U_n\}\mathbf{I}(\tilde{\tau} \geq n+1),$$

and $\mathscr{F}_n$ is a σ-algebra generated by the random variables $\sigma_1, \ldots, \sigma_n, t_1, \ldots, t_n$.

Since $U_n = \exp(g(\Psi_n))$ and $U_{n+1} = \exp(g(\Psi_{n+1}))$ when $\tilde{\tau} \geq n+1$, one can represent V_n as

$$V_n = e^{g(\Psi_n)}\mathbf{E}\{e^{g(\Psi_n+\psi_{n+1})-g(\Psi_n)} - 1/\mathscr{F}_n\}\mathbf{I}(\tilde{\tau} \geq n+1). \tag{4.3}$$

Let $G_n = \sum_{i=1}^n \sigma_i$.

Suppose first that $k_0 > 0$. If $\tilde{\tau} \geq n+1$, then $\Psi_n = k_0 n + k_1 G_n + c\widetilde{S}_n \geq k_0 n + (k_1 + c)\widetilde{S}_n > k_0 n$ since $\widetilde{S}_n > 0$ a.s. Then, from (2.2) and (4.3), $V_n \leq 0$ a.s. for all $n \geq x_0/k_0$. Hence, $\mathbf{E}U_{n+1} \leq \mathbf{E}U_n$, and (4.2) follows.

Suppose now that $k_0 = 0$. Then, from (4.3), for all $\delta > 0$,

$$V_n \leq E_1 + E_2,$$

where

$$E_1 = e^{g(\Psi_n)}\mathbf{E}\{e^{g(\Psi_n+\psi_{n+1})-g(\Psi_n)} - 1/\mathscr{F}_n\}\mathbf{I}(\tilde{\tau} \geq n+1, G_n > \delta n)$$

and

$$E_2 = \mathbf{E}\{e^{g(\Psi_n+\psi_{n+1})}/\mathscr{F}_n\}\mathbf{I}(G_n \leq \delta n).$$

We make a particular choice of δ below.

If $\tilde{\tau} \geq n+1$ and $G_n > \delta n$, then $\widetilde{S}_n > 0$ and $\Psi_n = k_1 G_n + c\widetilde{S}_n > (k_1 - c^-)\delta n$, where $c^- = \max(-c, 0)$. Thus, from (2.2), $E_1 \leq 0$ a.s. for all $n \geq x_0/\delta(k_1 - c^-)$.

If $G_n \leq \delta n$, then

$$\Psi_n = k_1 G_n + c\widetilde{S}_n \leq (k_1 + c^+)G_n \leq (k_1 + c^+)\delta n.$$

Since g is concave, there exist $u > 0$ and $\tilde{x} > 0$ such that $g(x) \leq ux/(k_1 + c^+)$ for all $x \geq \tilde{x}$.

Hence $g(\Psi_n) \leq g((k_1 + c^+)\delta n) \leq u\delta n$ for all $n \geq \tilde{x}/\delta(k_1 + c^+)$.

Note that

$$\mathbf{P}(G_n \leq \delta n) \leq e^{\alpha\delta n}\mathbf{E}e^{-\alpha G_n} = (e^{\alpha\delta}\mathbf{E}e^{-\alpha\sigma_1})^n$$

for $\alpha > 0$. Therefore, for all $n \geq \max(x_0/\delta(k_1 - c^-), \tilde{x}/\delta(k_1 + c^+))$,

$$\mathbf{E}U_{n+1} - \mathbf{E}U_n \leq \mathbf{E}e^{g(\psi_1^+)}e^{u\delta n}\mathbf{P}(G_n \leq \delta n) \leq \mathbf{E}e^{g(\psi_1^+)}(e^{\delta(\alpha+u)}\mathbf{E}e^{-\alpha\sigma_1})^n.$$

Let $\beta(\delta) = e^{\delta(\alpha+u)}\mathbf{E}e^{-\alpha\sigma_1}$ and put $\alpha = 2u\delta/b$, where $b = \mathbf{E}\sigma_1$.

We have $\beta'(0) = -u < 0$. Then there exists $\delta_0 > 0$ such that $\beta_0 \equiv \beta(\delta_0) < 1$ and

$$\mathbf{E}U_{n+1} - \mathbf{E}U_n \leq \mathbf{E}e^{g(\psi_1^+)}\beta_0^n \equiv k(n).$$

Since $\sum_{n=1}^\infty k(n) < \infty$, the proof of Theorem 2.1 is completed. □

The proof of Theorem 2.2 is based on the following lemma.

LEMMA 4.1. *Let ψ be a random variable such that $\mathbf{E}\psi < 0$ and $\mathbf{P}(\psi \geq -M) = 1$ for some $M \in (0, \infty)$. For any function H, if $H(0) \geq 0$ and*

$$\mathbf{E}e^{H(\psi)+\ln^+\psi} < \infty, \tag{4.4}$$

then there exists $\varepsilon_0 > 0$ such that

$$\mathbf{E}\exp\min\{\varepsilon\psi, H(\psi^+)\} \leq 1 \tag{4.5}$$

and

$$\mathbf{E}\psi\exp\min\{\varepsilon\psi, H(\psi^+)\} < 0 \tag{4.6}$$

for all $\varepsilon \in [0, \varepsilon_0]$.

Proof. For any fixed $q > 0$ and for any $\varepsilon > 0$,

$$E \equiv \mathbf{E}e^{\min\{\varepsilon\psi, H(\psi^+)\}} \le E_1 + E_2,$$

where

$$E_1 = \mathbf{E}e^{\varepsilon\psi}\mathbf{I}\left(\psi < \frac{q}{\varepsilon}\right) \quad \text{and} \quad E_2 = \mathbf{E}e^{H(\psi)}\mathbf{I}\left(\psi \ge \frac{q}{\varepsilon}\right).$$

Consider first E_1. The following estimates are valid:

$$e^{\varepsilon\psi} \le 1 + \frac{e^q - 1}{q}\varepsilon\psi, \qquad 0 \le \psi < \frac{q}{\varepsilon},$$

$$e^{\varepsilon\psi} \le 1 + \varepsilon\psi + \frac{M^2}{2}\varepsilon^2, \qquad \psi \in [-M, 0].$$

Therefore,

$$E_1 \le 1 + \left(\frac{e^q - 1}{q}\mathbf{E}\psi^+ - \mathbf{E}\psi^-\right)\varepsilon + \frac{M^2}{2}\varepsilon^2. \tag{4.7}$$

Now choose $q > 0$ such that $(e^q - 1)/q < \mathbf{E}\psi^-/\mathbf{E}\psi^+$.

For E_2, the following estimate is valid:

$$E_2 \le \frac{\varepsilon}{q}\mathbf{E}\psi e^{H(\psi)}\mathbf{I}\left(\psi \ge \frac{q}{\varepsilon}\right). \tag{4.8}$$

From condition (4.4), $\mathbf{E}\psi e^{H(\psi)}\mathbf{I}(\psi > q/\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon_1 > 0$ such that

$$K(q, \varepsilon_1) \equiv -\left(\frac{e^q - 1}{q}\mathbf{E}\psi^+ - \mathbf{E}\psi^- + \frac{1}{q}\mathbf{E}\psi e^{H(\psi)}\mathbf{I}\left(\psi \ge \frac{q}{\varepsilon_1}\right)\right) > 0. \tag{4.9}$$

From (4.7)–(4.9), it follows that

$$E \le 1 - K(q, \varepsilon_1)\varepsilon + \frac{M^2}{2}\varepsilon^2 < 1$$

for all $\varepsilon \le \varepsilon_2 \equiv 2K(q, \varepsilon_1)/M^2$. Putting $\varepsilon_0 \equiv \min(\varepsilon_1, \varepsilon_2)$, we obtain (4.5).

We now prove (4.6). For $\varepsilon > 0$, since $H(0) \ge 0$,

$$J \equiv \mathbf{E}\psi e^{\min\{\varepsilon\psi, H(\psi^+)\}} \le \mathbf{E}\eta_{1,\varepsilon} + \mathbf{E}\eta_{2,\varepsilon},$$

where

$$\eta_{1,\varepsilon} = \psi e^{\varepsilon\psi}\mathbf{I}\left(\psi \le \frac{q}{\varepsilon}\right) \quad \text{and} \quad \eta_{2,\varepsilon} = \psi e^{H(\psi)}\mathbf{I}\left(\psi > \frac{q}{\varepsilon}\right).$$

Observe that $\eta_{1,\varepsilon} \to \psi$ a.s. as $\varepsilon \to 0$. Moreover, for any $\varepsilon > 0$,

$$\eta^+_{1,\varepsilon} = \psi e^{\varepsilon\psi}\mathbf{I}\left(0 \le \psi \le \frac{q}{\varepsilon}\right) \le e^q\psi^+ \quad \text{and} \quad \eta^-_{1,\varepsilon} = -\psi e^{\varepsilon\psi}\mathbf{I}(\psi < 0) \le \psi^-.$$

Hence the dominated convergence theorem implies

$$\mathbf{E}\eta_{1,\varepsilon} \to \mathbf{E}\psi < 0 \qquad \text{as } \varepsilon \to 0.$$

It follows from (4.4) that $\mathbf{E}\eta_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$. Hence $J \to \mathbf{E}\psi < 0$ as $\varepsilon \to 0$ and the result follows. □

Proof of Theorem 2.2. Put $\psi = c\tilde{\xi}_1 + \gamma_1$, where $\tilde{\xi}_1 = \max(\xi_1, -L)$ and $L > 0$ is chosen such that $\tilde{a} \equiv -\mathbf{E}\tilde{\xi}_1 > (k_0 + k_1 b)/c$. It is sufficient to verify the condition (2.2) of Theorem 2.1 (the other conditions of that theorem are trivially satisfied), i.e., to show that there exists x_0 such that

$$\mathbf{E}e^{g(x+\psi)-g(x)} \le 1 \tag{4.10}$$

for all $x \ge x_0$.

It follows from the conditions of the theorem that there exists $x_1 \ge 0$ such that $h(x)$ is monotone and concave for all $x \ge x_1$.

Put $M = cL > 0$ and $x_2 = x_1 + M$. For all $x \ge x_2$ and $y \ge -M$, the following inequalities hold:

$$h(x+y) - h(x) \le h(x_2 + y^+) - h(x_2) \quad \text{and} \quad h(x+y) - h(x) \le h'(x)y.$$

Therefore

$$\begin{aligned} g(x+y) - g(x) &= h(x+y) - h(x) + \ln\left(1 + \frac{y}{x}\right) \\ &\le \min(h'(x)y, h(x_2 + y^+) - h(x_2)) + \ln\left(1 + \frac{y}{x}\right). \end{aligned}$$

Observe that $\mathbf{E}\psi < 0$, since $c > (k_0 + k_1 b)/\tilde{a}$ and $\mathbf{P}(\psi \ge -M) = 1$. Put $H(x) = h(x_2 + x) - h(x_2)$. Choose ε_0 the same as in Lemma 4.1 and then $x_0 \ge x_2$ so large that $h'(x) \le \varepsilon_0$ for all $x \ge x_0$.

Then (4.10) follows from

$$\mathbf{E}e^{g(x+\psi)-g(x)} \le \mathbf{E}e^{\min(h'(x)\psi,\, H(\psi^+))}\left(1 + \frac{\psi}{x}\right)$$

and Lemma 4.1. □

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