ON OPTIMALITY OF THE FCFS DISCIPLINE IN MULTISERVER QUEUEING SYSTEMS AND NETWORKS

S. G. Foss and N. I. Chernova

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§ 1. Introduction

In multiserver queueing systems, various service disciplines are used, e.g., "first-come-first-served" (FCFS), "last-come-first-served" (LCFS), cyclic, time sharing disciplines, etc. Therefore, the natural problems of comparing service disciplines were studied by many authors.

We restrict our consideration to disciplines which do not allow time sharing and service interruption. In other words, we assume that, at any time instant, each server may serve at most one customer. Once started, a service continues until completion; afterwards the customer leaves the system.

Usually, the state of the system is described by the sequence of finite-dimensional random vectors $W_n = (W_{n,1}, \dots, W_{n,m})$, $n \geq 0$ (with m standing for the number of servers), where $W_{n,j}$ denotes the total workload at the jth server just after the nth customer arrival, i.e., the amount of time needed by server j to complete the services of all customers (including the nth) who have arrived up to this time instant. We are interested in minimizing the distribution of a certain functional $\phi(W_n)$ for every fixed n. The minimization problem for functionals of joint distributions $\phi(W_n, W_{n+1}, \dots, W_{n+k})$ is interesting as well. The following questions arise: (a) for which disciplines can we solve the optimization problem; (b) how may we describe a class of valid functionals $\{\phi\}$?

Alongside W_n we also consider the sequence d_n of the actual waiting times (or the sojourn times) of customers in the system and analyze the problem of minimizing the distributions of certain functionals of this sequence.

We further assume that the sequence $\{s_n\}$ of service times consists of independent identically distributed (i.i.d.) random variables (r.v.'s) and is independent of the other input characteristics.

We use the following conventions. We say that an r.v. ξ is stochastically greater than an r.v. ξ^0 and write $\xi \geq_{st} \xi^0$ if $\mathbf{P}(\xi \geq x) \geq \mathbf{P}(\xi^0 \geq x)$ for all real x. A random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ is stochastically greater than a random vector $\boldsymbol{\xi}^0 = (\xi_1^0, \dots, \xi_n^0)$,

$$\boldsymbol{\xi} \geq_{st} \boldsymbol{\xi}^0, \tag{1}$$

if there exists a coupling $(\boldsymbol{\eta}, \boldsymbol{\eta}^0)$ of both vectors on a common probability space, $\boldsymbol{\eta} =_D \boldsymbol{\xi}$ and $\boldsymbol{\eta}^0 =_D \boldsymbol{\xi}^0$, such that $\boldsymbol{\eta} \geq \boldsymbol{\eta}^0$ a.s. coordinate-wise. Note that (1) implies the following inequality: for every vector $x = (x_1, \dots, x_n)$,

$$\mathbf{P}(\boldsymbol{\xi} \geq x) \geq \mathbf{P}(\boldsymbol{\xi}^0 \geq x). \tag{2}$$

The converse is false in general: for $n \geq 2$, inequality (2) does not imply (1).

Denote by T^0 the FCFS service discipline (see Example 1) and consider another discipline T. We endow all quantities of the system governed by the FCFS discipline with the superscript 0 .

Seemingly, the first substantial and quite general results on optimality of the FCFS discipline were proved in [1]. Firstly, it was shown therein that, for every $n = 1, 2, \dots$, every *Schur convex* function $\phi : \mathbb{R}^m \to \mathbb{R}$, and all *admissible* service disciplines T, the following inequality holds:

$$\phi(W_n) \geq_{st} \phi(W_n^0). \tag{3}$$

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In particular, (3) implies that

$$\max_j W_{n,j} \geq_{st} \max_j W^0_{n,j} \tag{4}$$

(see Section 2 for the definitions of *Schur convexity* and *admissibility*). Further, denote by v_n the time instant of the nth customer's service completion. Then the following is true (see [1]): for every natural number n,

$$R(v_1, \ldots, v_n) \geq_{st} R\big(v^0_1, \ldots, v^0_n\big), \tag{5}$$

where, for every vector $x = (x_1, \ldots, x_n)$, the vector $R(x)$ is obtained from x by permuting coordinates in nondecreasing order. In particular, (5) implies the relation

$$\sum_1^n d_j \geq_{st} \sum_1^n d^0_j. \tag{6}$$

If the systems with the FCFS and the T disciplines are both stable then

$$\frac{1}{n}\sum_1^n d_j \to \mathbf{E}d, \quad \frac{1}{n}\sum_1^n d^0_j \to \mathbf{E}d^0 \text{ a.s.},$$

where d and d^0 stand for the corresponding stationary waiting times. Then (6) implies the inequality

$$\mathbf{E}\, d \geq \mathbf{E}\, d^0. \tag{7}$$

Validity of (4) was discussed earlier by several authors. In [2, p. 220] it was asserted that this inequality holds almost surely. But Stoyan [3] gave a counterexample and conjectured that (4) is valid in the stochastic sense. A proof of (4) was proposed in [4], but it appears to have been based on incorrect arguments.

Wolff [5] considered stable $GI/GI/m$ systems with the FCFS discipline T^0 and the *cyclic* discipline T (the cyclic discipline is defined in Example 2). In this case, he proved the following generalization of (7): for every increasing convex function f,

$$\mathbf{E}f(d) \geq \mathbf{E}f(d^0). \tag{8}$$

Whitt [6] (see also [7]) considered the cyclic T discipline, too. He showed (by example) that, in general, d and d^0 are noncomparable; i.e., the inequality $d \geq_{st} d^0$ may fail.

A slightly different class of service disciplines, where the order of service may differ from the order of arrival, was studied in [8] (in particular, that class contains the LCFS discipline). Results similar to (3) and (5) were established.

In [9], the following assertion was stated (we formulate it in our notation): for every $k = 1, 2, \ldots$,

$$\min_{0\leq l\leq k} \max_j (W_{n+l,j} - \tau_{n+l+1})^+ \geq_{st} \min_{0\leq l\leq k} \max_j \big(W^0_{n+l,j} - \tau_{n+l+1}\big)^+, \tag{9}$$

but a correct proof was provided only for $k = 1$.

Another variant of the proof of inequality (3) was proposed in [10]. The paper [11] contains a short description of results in [1, 8, 9] together with proofs of the main results in particular cases. In the survey [12], the proof of the main result of [1, 11] was reproduced and some other problems were considered.

Recent publications (see, e.g., [13] and others) show that, using comparison theorems, enables us to obtain new results for systems with the FCFS discipline such as: statements on the existence of moments for the stationary waiting time, construction of upper bounds for tail distributions of certain characteristics, etc.

We prove the following new results. First, we establish a natural generalization of (6):

$$h(d_1, \ldots, d_n) \geq_{st} h\big(d^0_1, \ldots, d^0_n\big) \tag{10}$$

for every Schur-convex function h. In particular, (10) implies that (8) is valid for every future-independent service discipline (not only for the cyclic one). A more general assertion for the joint distributions of several Schur-convex functions is also valid (see Remark 2).

An interesting application of (10) was given in [14] (with a reference to the present paper unpublished at that time). We discuss it in more detail in § 4.

Second, we give an example, showing that (9) may fail in general for $k \geq 2$. For $k = 1$, we prove the following generalization of (9): for arbitrary Schur-convex functions ϕ_1, ϕ_2 and for all $n = 1, 2, \dots$, $x_1, x_2 \in \mathbb{R}$

$$\mathbf{P}(\phi_1(W_n) \geq x_1, \phi_2(W_{n+1}) \geq x_2) \geq \mathbf{P}\big(\phi_1\big(W_n^0\big) \geq x_1, \phi_2\big(W_{n+1}^0\big) \geq x_2\big). \tag{11}$$

The paper contains four sections. In § 2, we introduce the main definitions and notations and state the results and corollaries. The proofs are given in § 3. In § 4, further possible generalizations are considered together with examples of applications.

§ 2. Main Definitions and Statements

2.1. The model description. Consider a queueing system with m servers, governed by the independent sequences of r.v.'s $\{\tau_i\}_{i\geq 1}$ and $\{s_i\}_{i\geq 1}$, by the initial vector $W_0 = (W_{0,1}, W_{0,2}, \dots, W_{0,m})$, and by a service discipline of a certain class. Here τ_1 is the arrival time of customer 1 and, for $i \geq 2$, τ_i is the interarrival time between the $(i-1)$st and ith customers. The r.v. s_i is the service time of the ith customer. For $j = 1, \dots, m$, the coordinate $W_{0,j} \geq 0$ of W_0 is the first time instant when server j can begin customer service. Define a service discipline as a random sequence $T = \{T_n\}_{n\geq 1}$, where T_n stands for the number of the station where the nth customer is served.

Let all r.v.'s be defined on a probability space $\langle \Omega, \mathscr{F}, \mathbf{P} \rangle$. Denote by $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ the set of natural numbers. Let $t_n = \sum_{i=1}^{n} \tau_i$ be the nth arrival instant.

The sequence of service times $\{s_i\}_{i\geq 1}$ is assumed to consist of i.i.d. random variables which do not depend on W_0 and $\{\tau_i\}_{i\geq 1}$.

The sequence $T = \{T_n\}_{n\geq 1}$ determines the service procedure as follows.

Set $v_{0,j} = W_{0,j}$ for $1 \leq j \leq m$. The service of the first customer proceeds at station T_1. If $T_1 = k$ then the service starts at the time instant $u_1 = \max\{v_{0,k}, t_1\}$ and continues for s_1 units of time. Before the time instant u_1, server k is out of service. Given the event $\{T_1 = k\}$, we put $v_{1,j} = v_{0,j}$ for $j \neq k$, and $v_1 = v_{1,k} = u_1 + s_1$.

The following r.v.'s are defined inductively for each $n \in \mathbb{N}$: u_n is the time instant when the service of customer n starts and $v_{n,j}$ is the last departure epoch of customers with numbers $1, \dots, n$ from station j.

Given the event $\{T_{n+1} = k\}$, server k cannot serve any of the customers numbered $n{+}1, n{+}2, \dots$, until the time instant $u_{n+1} = \max\{v_{n,k}, t_{n+1}\}$. The service of customer $n{+}1$ starts at station $T_{n+1} = k$ at the time instant u_{n+1}. Given the event $\{T_{n+1} = k\}$, we put

$$v_{n+1,j} = v_{n,j} \ \text{ if } j \neq k, \ \text{ and } \ v_{n+1} = v_{n+1,k} = u_{n+1} + s_{n+1}.$$

Note that, for $i < j$, the inequality $u_i + s_i \leq u_j$ holds a.s. on the event $\{T_i = T_j\}$.

A service discipline T is called *admissible* if it exhibits the "future independence" property.

Definition 1. A service discipline $T = \{T_n\}_{n\geq 1}$ is *admissible* in the system $\Sigma(W_0, \{\tau_i\}, \{s_i\})$ if, for $n \geq 1$, whatever the set of natural numbers $\{k_1, \dots, k_n\}$, the following equality holds:

$$\begin{aligned} &\mathbf{P}\big\{T_1 = k_1, \dots, T_n = k_n \mid W_0; \{\tau_i\}_{i=1}^{\infty}; \{s_i\}_{i=1}^{\infty}\big\} \\ &= \mathbf{P}\big\{T_1 = k_1, \dots, T_n = k_n \mid W_0; \{\tau_i\}_{i=1}^{\infty}; \{s_i\}_{i=1}^{n-1}\big\}. \end{aligned} \tag{12}$$

The workload at station j is measured as the length of the time interval between t_n and the last departure epoch of customers with numbers $1, \dots, n$ from server j. Thus, it is equal to $W_{n,j} = (v_{n,j} - t_n)^+$. The vectors W_n obey the evident recurrence relation $W_n = (W_{n-1} - i\tau_n)^+ + s_n e_{T_n}$, where e_k stands for the unit vector with all but kth coordinates equal to zero and the kth coordinate equal to one, and $i = (1, \dots, 1)$ is the vector of ones.

We consider some examples of admissible service disciplines.

EXAMPLE 1 (FCFS discipline). For $n \in \mathbb{N}$ put

$$T_n = \min_{1\le j\le m}\{j : v_{n-1,j} = \min_{1\le k\le m} v_{n-1,k}\} = \arg\min_{1\le j\le m}\{v_{n-1,j}\}.$$

Here the $\arg\min_{1\le j\le m}\{v_{n-1,j}\}$ means the smallest j such that $v_{n-1,j} \le v_{n-1,k}$ for all $1 \le k \le m$.

EXAMPLE 2 (cyclic discipline). Put $T_{nk+j} = j$ for $n, k \ge 0$ and $j = 1, \dots, m$.

EXAMPLE 3 (random discipline). Here the r.v.'s $\{T_n\}$ are mutually independent, do not depend on $\{W_0, \{\tau_i\}, \{s_i\}\}$, and are distributed uniformly on the set $\{1, 2, \dots, m\}$ of station numbers: $\mathbf{P}(T_n = j) = 1/m$ for $j = 1, \dots, m$.

EXAMPLE 4 (the discipline introduced in [14]). Each of the first $m-2$ customers chooses the station with the minimal waiting time among the stations that are not chosen by the previous customers. The customers numbered $m-1$, $m, \dots$ may not choose any of the stations already chosen by the previous $m-2$ customers. From the two stations left, each such customer chooses the server with the smallest waiting time.

Define

$$A_n = \begin{cases} \{1, \dots, m\} \setminus \{T_1, \dots, T_{n-1}\} & \text{for } 1 \le n \le m-2, \\ \{1, \dots, m\} \setminus \{T_{n-m+2}, \dots, T_{n-1}\} & \text{for } n > m-2, \end{cases}$$

the set of stations "available" for customer n. Then $T_n = \arg\min_{j\in A_n}\{v_{n-1,j}\}$.

2.2. Schur convexity. For an arbitrary vector $x = (x_1, \dots, x_n)$, denote by $x_{(1)} \le x_{(2)} \le \dots \le x_{(n)}$ the vector obtained from x by permuting coordinates in nondecreasing order. We need the following partial ordering.

DEFINITION 2. Given two vectors $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l)$, we write $x \lhd y$ if

$$\sum_{i=k}^{l} x_{(i)} \le \sum_{i=k}^{l} y_{(i)}$$

for all $1 \le k \le l$.

DEFINITION 3. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *Schur-convex*, if the inequality $f(x) \le f(y)$ holds for arbitrary vectors $x \lhd y$.

It is easy to show that the class of Schur-convex functions coincides with the following class H_n.

DEFINITION 4. We say that a function $h : \mathbb{R}^n_+ \to \mathbb{R}$ *belongs to the class* H_n, if the following inequality holds for every $1 \le j < n$ and all $(a_j, a_{j+1}) \lhd (b_j, b_{j+1})$:

$$h(a_1, \dots, a_j, a_{j+1}, \dots, a_n) \le h(a_1, \dots, b_j, b_{j+1}, \dots, a_n).$$

2.3. Main results. Consider two admissible disciplines $T^{(1)}$ and $T^{(2)}$. For $n \in \mathbb{N}$ and for each discipline $T^{(i)}$, denote by $d_n^{(i)} = u_n^{(i)} - t_n$ the waiting time of customer n.

The coordinate $W_{n,j}^{(i)} = (v_{n,j}^{(i)} - t_n)^+$ of the vector $W_n^{(i)} = (W_{n,1}^{(i)}, \dots, W_{n,m}^{(i)})$ equals the workload at station j at the time instant t_n.

We define two ways of comparison between service disciplines.

DEFINITION 5. We write $T^{(1)} \preccurlyeq_n T^{(2)}$ if, for every $h \in H_n$,

$$h\big(d_1^{(1)}, \dots, d_n^{(1)}\big) \le_{st} h\big(d_1^{(2)}, \dots, d_n^{(2)}\big). \tag{13}$$

DEFINITION 6. We write $T^{(1)} \preccurlyeq_n T^{(2)}$ if the following relation is valid for all $\phi_1, \phi_2 \in H_m$ and $x_1, x_2 \in \mathbb{R}$:

$$\mathbf{P}\big(\phi_1\big(W_{n-1}^{(1)}\big) \ge x_1,\ \phi_2\big(W_n^{(1)}\big) \ge x_2\big) \le \mathbf{P}\big(\phi_1\big(W_{n-1}^{(2)}\big) \ge x_1,\ \phi_2\big(W_n^{(2)}\big) \ge x_2\big). \tag{14}$$

Theorem 1. *For every $n \in \mathbb{N}$, the service discipline* FCFS *is no worse than any other admissible discipline T; i.e.,*

$$T^0 \preccurlyeq_n T, \quad T^0 \prec\!\!\!\sim_n T.$$

REMARK 1. We exhibit an example below, showing that the inequality $T^0 \prec\!\!\!\sim_n T$ may fail if one considers the joint distribution of more than two random vectors in (14). In particular, the inequality

$$\begin{aligned}&\mathbf{P}\big(\phi_1\big(W^0_{n-2}\big) \geq x_1, \phi_2\big(W^0_{n-1}\big) \geq x_2, \phi_3\big(W^0_n\big) \geq x_3\big)\\ &\leq \mathbf{P}(\phi_1(W_{n-2}) \geq x_1, \phi_2(W_{n-1}) \geq x_2, \phi_3(W_n) \geq x_3),\end{aligned}$$

for the FCFS discipline T^0 and an arbitrary admissible discipline T fails in general.

EXAMPLE 5. Consider a two-server queueing system with $\tau_1 = \tau_2 = \tau_3 = 0$, $W_{0,1} = c > 0 = W_{0,2}$. Assume that the r.v.'s s_1, s_2, s_3 are i.i.d. and take only two values $\alpha < \beta$ with respective probabilities $1-p$ and p. We compare the FCFS discipline T^0 with the discipline T given by $T_1 = 1$, $T_n = \arg\min_{j=1,2}\{W_{n-1,j}\}$ for $n = 2, 3$. Here we put $W_n = W_{n-1} + s_n e_{T_n}$.

We consider the functions $\phi_i(x_1, x_2) = \max(x_1, x_2)$, $i = 1, 2, 3$, in the class H_2 and show the existence of values of $c, \alpha, \beta, x_1, x_2, x_3$ such that

$$\begin{aligned}P \equiv\ &\mathbf{P}(\max_j W_{1,j} \geq x_1, \max_j W_{2,j} \geq x_2, \max_j W_{3,j} \geq x_3)\\ &\leq \mathbf{P}\big(\max_j W^0_{1,j} \geq x_1, \max_j W^0_{2,j} \geq x_2, \max_j W^0_{3,j} \geq x_3\big) \equiv P^0.\end{aligned} \tag{15}$$

Note that

$$W_1 = \begin{pmatrix} c+s_1 \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} c+s_1 \\ s_2 \end{pmatrix}, \quad W^0_1 = \begin{pmatrix} c \\ s_1 \end{pmatrix}.$$

Assume that x_1, x_2 satisfy the inequalities: $\beta \geq x_1 = x_2 > c+\alpha$, $2\beta > c+\alpha+\beta \geq x_3 > \alpha+\beta$, $x_3 > c+\beta$. Then $\max_j W_{1,j} \geq x_1$ if and only if $s_1 = \beta$. On the event $\{s_1 = \beta\}$, we have $T_2 = 1$, $T_3 = 2$. The inequality $\max_j W_{2,j} \geq x_2$ in (15) is always true, and the inequality $\max_j W_{3,j} \geq x_3$ becomes

$$\{\max_j W_{3,j} = \max(c+\beta, s_2+s_3) \geq x_3\} = \{s_2 = s_3 = \beta\}.$$

Hence, $P = p^3$.

We now calculate P^0. The event $\big\{\max_j W^0_{1,j} = \max(c, s_1) \geq x_1\big\}$ in (15) takes place only if $s_1 = \beta > c$. The inequality $\big\{\max_j W^0_{2,j} = \max(c+s_2, s_1) \geq x_2\big\}$ is valid always.

On the event $\{s_1 = \beta, s_2 = \beta\}$, we have

$$\big\{\max_j W^0_{3,j} = \max(c+\beta, \beta+s_3) \geq x_3\big\} = \{s_3 = \beta\}.$$

On the event $\{s_1 = \beta, s_2 = \alpha\}$, we have

$$\big\{\max_j W^0_{3,j} = \max(c+\alpha+s_3, \beta) \geq x_3\big\} = \{s_3 = \beta\}.$$

Therefore,

$$P^0 = \mathbf{P}(s_1 = \beta, s_2 = \beta, s_3 = \beta) + \mathbf{P}(s_1 = \beta, s_2 = \alpha, s_3 = \beta) = p^2 > P.$$

Furthermore, we show that (9) may fail either. Put $x = x_1 = x_2 = \beta$, $x_3 = c+\alpha+\beta$, $\tau_1 = \tau_2 = \tau_3 = 0$ and $\tau_4 = c+\alpha$. Then

$$\{\max_j (W_{3,j} - \tau_4)^+ \geq x\} = \{\max_j W_{3,j} \geq x_3\}$$

and

$$\mathbf{P}\big(\min_{1\leq i\leq 3} \max_j (W_{i,j} - \tau_{i+1})^+ \geq x\big) = p^3 < p^2 = \mathbf{P}\big(\min_{1\leq i\leq 3} \max_j \big(W^0_{i,j} - \tau_{i+1}\big)^+ \geq x\big).$$

§ 3. Proof of Theorem 1

First, we prove the following analog of Lemma 3.1 in [11].

Lemma 1. *Let $W_0 = (W_{0,1}, \dots, W_{0,m})$ be a nonrandom vector and $W_{0,r} \geq W_{0,l}$ for some fixed $r \neq l$. Let T be an admissible discipline in the system $\Sigma(W_0, \{\tau_i\}, \{s_i\})$ such that $T_1 = r$ and $T_2 = l$. Then*

(a) *there exist r.v.'s s'_1 and s'_2 with the following properties: they are i.i.d., coincide in distribution with s_1, and are independent of $\{\tau_i\}$, $\{s_i\}_{i>2}$, and $s'_1 + s'_2 = s_1 + s_2$ a.s.;*

(b) *in the system $\Sigma(W_0, \{\tau_i\}, \{s'_1, s'_2, s_3, s_4, \dots\})$, there exists an admissible discipline T' such that $T'_1 = l, T'_2 = r$ and, for every $i \geq 3$,*

$$W'_{i-1} \lhd W_{i-1} \quad \text{a.s.}, \tag{16}$$

$$d'_i = \big(W'_{i-1,T'_i} - \tau_i\big)^+ \leq d_i = \big(W_{i-1,T_i} - \tau_i\big)^+ \quad \text{a.s.}, \tag{17}$$

$$d'_1 \leq d_1, \quad (d'_1, d'_2) \lhd (d_1, d_2) \quad \text{a.s.}, \tag{18}$$

and

$$\mathbf{P}\big(\phi_1\big(W'_{i-2}\big) \geq x_1,\ \phi_2\big(W'_{i-1}\big) \geq x_2\big) \leq \mathbf{P}\big(\phi_1\big(W_{i-2}\big) \geq x_1,\ \phi_2\big(W_{i-1}\big) \geq x_2\big) \tag{19}$$

for all functions $\phi_1, \phi_2 \in H_m$ and all $x_1, x_2 \in \mathbb{R}$.

Proof. Put $A = \{W_{0,r} - \tau_1 - \tau_2 < 0\}$. It is easy to show that the r.v.'s

$$s'_1 = s_1 I(A) + s_2 I(\overline{A}); \quad s'_2 = s_2 I(A) + s_1 I(\overline{A})$$

satisfy item (a) of Lemma 1. For instance, the i.i.d. property for s'_j, $j = 1, 2$, (as well as the independence of τ_1, τ_2) follows from the corresponding properties of $\{s_i\}$: by the Total Probability Law,

$$\mathbf{P}(s'_1 \in B) = \mathbf{P}(s_1 \in B, A) + \mathbf{P}(s_2 \in B, \overline{A}) = \mathbf{P}(s_1 \in B)\mathbf{P}(A) + \mathbf{P}(s_1 \in B)\mathbf{P}(\overline{A}) = \mathbf{P}(s_1 \in B).$$

As was proved in [1] (see also [11, Lemma 3.1]), (16) is valid for the discipline T' in the system with service times $\{s'_1, s'_2, s_3, s_4, \dots\}$, where T' is defined as follows. Put $T'_1 = l$, $T'_2 = r$ a.s. and, for $i > 2$,

(a) on the event $\overline{A}$, put $T'_i = T_i$ a.s.;

(b) on the event A, put

$$T'_i = \begin{cases} r, & \text{if } T_i = l, \\ l, & \text{if } T_i = r, \\ T_i & \text{otherwise.} \end{cases}$$

For convenience, we reproduce the proof of (16) in [11], proving (17) simultaneously.

Proof of (16) and (17). First, note that, on the event $\overline{A}$,

$$\begin{gathered} W_{2,l} = (W_{0,l} - \tau_1 - \tau_2)^+ + s_2 \geq W'_{2,l} = ((W_{0,l} - \tau_1)^+ + s_2 - \tau_2)^+, \\ W_{2,r} = W_{0,r} - \tau_1 - \tau_2 + s_1 = W'_{2,r} \quad \text{a.s.} \end{gathered} \tag{20}$$

and $W_{2,j} = W'_{2,j}$ a.s., for any other j. On the event A,

$$W_{2,l} = s_2 = W'_{2,r}, \quad W_{2,r} = ((W_{0,r} - \tau_1)^+ + s_1 - \tau_2)^+ = W'_{2,l} \quad \text{a.s.} \tag{21}$$

and $W_{2,j} = W'_{2,j}$ a.s., for any other j. It is easy to show that (20) and (21) hold true for the corresponding coordinates of W_i and W'_i with any $i \geq 2$. Therefore, (16) holds. Moreover, $W_{i-1,T_i} \geq W'_{i-1,T'_i}$ a.s., for all $i > 2$; therefore, (17) holds.

Proof of (18). Note that

$$d_1 = \max\{d_1, d_2\} = (W_{0,r} - \tau_1)^+ \geq \max\{d'_1, d'_2\} = \max\{(W_{0,l} - \tau_1)^+, (W_{0,r} - \tau_1 - \tau_2)^+\} \quad \text{a.s.}$$

On the event $\overline{A}$,

$$d_1 + d_2 = W_{0,r} - \tau_1 + (W_{0,l} - \tau_1 - \tau_2)^+ \geq W_{0,r} - \tau_1 + (W_{0,l} - \tau_1)^+ - \tau_2 = d_1' + d_2' \quad \text{a.s.}$$

On the event A, we have $d_2 = 0 = d_2'$; whence $d_1 + d_2 = d_1 \geq d_1' = d_1' + d_2'$ a.s. Then $(d_1', d_2') \lhd (d_1, d_2)$ a.s. by Definition 2, and (18) follows.

Proof of (19). Note that, for $i > 3$, (19) follows directly from (16). We prove (19) for $i = 3$. By the Total Probability Law, we can prove (19) separately on the events A and $\overline{A}$.

On the event A,

$$\begin{pmatrix} W_{1,l} \\ W_{1,r} \end{pmatrix} = \begin{pmatrix} (W_{0,l} - \tau_1)^+ \\ (W_{0,r} - \tau_1)^+ + s_1 \end{pmatrix} \lhd \begin{pmatrix} (W_{0,l} - \tau_1)^+ + s_1 \\ (W_{0,r} - \tau_1)^+ \end{pmatrix} = \begin{pmatrix} W_{1,l}' \\ W_{1,r}' \end{pmatrix} \quad \text{a.s.} \tag{22}$$

and $W_2' \lhd W_2$ a.s. due to (16). The latter proves (19) on the event A.

Once s_1, s_2, s_1', and s_2' are independent of τ_1 and τ_2, it suffices to prove (19) on $\overline{A}$ only for fixed τ_1 and τ_2.

For brevity, put $b = W_{0,r} - \tau_1 \geq a = (W_{0,l} - \tau_1)^+ \geq 0$, $\tau = \tau_2$. On $\overline{A}$, we have $b > \tau$. Let $a < b$. The vectors W_i and W_i', $i = 1, 2$, differ from one another only in two coordinates (rth and lth); so we can prove (19) with functions $\phi_1, \phi_2 \in H_2$.

We have

$$\begin{pmatrix} W_{1,r} \\ W_{1,l} \end{pmatrix} = \begin{pmatrix} b + s_1 \\ a \end{pmatrix}, \quad \begin{pmatrix} W_{2,r} \\ W_{2,l} \end{pmatrix} = \begin{pmatrix} b + s_1 - \tau \\ (a - \tau)^+ + s_2 \end{pmatrix},$$

$$\begin{pmatrix} W_{1,r}' \\ W_{1,l}' \end{pmatrix} = \begin{pmatrix} b \\ a + s_2 \end{pmatrix}, \quad \begin{pmatrix} W_{2,r}' \\ W_{2,l}' \end{pmatrix} = \begin{pmatrix} b + s_1 - \tau \\ (a + s_2 - \tau)^+ \end{pmatrix}.$$

For $x_1, x_2 \in \mathbb{R}$, define the events

$$A_0 \equiv A_0(x_1, x_2, \phi_1, \phi_2) = \left\{ \phi_1 \begin{pmatrix} b + s_1 \\ a \end{pmatrix} \geq x_1,\ \phi_2 \begin{pmatrix} b + s_1 - \tau \\ (a - \tau)^+ + s_2 \end{pmatrix} \geq x_2 \right\},$$

$$A_1 \equiv A_1(x_1, x_2, \phi_1, \phi_2) = \left\{ \phi_1 \begin{pmatrix} b + s_1 \\ a \end{pmatrix} \geq x_1, \quad \phi_2 \begin{pmatrix} b + s_1 - \tau \\ (a + s_2 - \tau)^+ \end{pmatrix} \geq x_2 \right\},$$

$$A_2 \equiv A_2(x_1, x_2, \phi_1, \phi_2) = \left\{ \phi_1 \begin{pmatrix} b \\ a + s_2 \end{pmatrix} \geq x_1,\ \phi_2 \begin{pmatrix} b + s_1 - \tau \\ (a + s_2 - \tau)^+ \end{pmatrix} \geq x_2 \right\}.$$

In order to prove (19), it suffices to show that

$$\mathbf{P}(A_0) \geq \mathbf{P}(A_1) \geq \mathbf{P}(A_2). \tag{23}$$

The first inequality in (23) follows from the monotonicity of $\phi_2 \in H_2$ in the second argument. We prove the second inequality. Set $u_1 = \min\{s_1, s_2\}$, $u_2 = \max\{s_1, s_2\}$ and consider the events C_1, C_2, C_3:

$$C_1 = \left\{ x_1 \leq \phi_1 \begin{pmatrix} b + u_1 \\ a \end{pmatrix} \right\}, \quad C_2 = \left\{ x_1 > \phi_1 \begin{pmatrix} b + u_2 \\ a \end{pmatrix} \right\},$$

$$C_3 = \left\{ \phi_1 \begin{pmatrix} b + u_1 \\ a \end{pmatrix} < x_1 \leq \phi_1 \begin{pmatrix} b + u_2 \\ a \end{pmatrix} \right\}.$$

By the Total Probability Law, for $j = 1, 2$,

$$\mathbf{P}(A_j) = \mathbf{P}(A_j C_1) + \mathbf{P}(A_j C_2) + \mathbf{P}(A_j C_3) \equiv P_{j1} + P_{j2} + P_{j3}. \tag{24}$$

We prove that $P_{1i} \geq P_{2i}$ for $i = 1, 2, 3$.

1. Note that $C_1 \subseteq \left\{x_1 \le \phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix}\right\}$. Hence,

$$\begin{aligned} P_{11} = \mathbf{P}\left\{\phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix} \ge x_1,\ \phi_2 \begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix} \ge x_2,\ C_1\right\} \\ = \mathbf{P}\left\{\phi_2 \begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix} \ge x_2,\ C_1\right\} \\ \ge \mathbf{P}\left\{\phi_1 \begin{pmatrix} b \\ a+s_2 \end{pmatrix} \ge x_1,\ \phi_2 \begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix} \ge x_2,\ C_1\right\} = P_{21}. \end{aligned} \tag{25}$$

2. Observe that

$$C_2 \subseteq \left\{x_1 > \phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix}\right\}.$$

Therefore,

$$C_2 \cap \left\{\phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix} \ge x_1\right\} = \varnothing \text{ and } P_{12} = 0.$$

Moreover,

$$C_2 \subseteq \left\{x_1 > \phi_1 \begin{pmatrix} b+s_2 \\ a \end{pmatrix}\right\} \subseteq \left\{x_1 > \phi_1 \begin{pmatrix} b \\ a+s_2 \end{pmatrix}\right\},$$

whence

$$C_2 \cap \left\{\phi_1 \begin{pmatrix} b \\ a+s_2 \end{pmatrix} \ge x_1\right\} = \varnothing,$$

and, therefore, $P_{22} = 0 = P_{12}$.

3. Consider the following events

$$D_0 = \{s_1 = s_2\}, \quad D_1 = \{s_1 < s_2\}, \quad D_2 = \{s_1 > s_2\}.$$

Clearly, $D_0 \cap C_3 = \varnothing$. Thus, for $j = 1, 2$,

$$P_{j3} = \mathbf{P}(A_j C_3 D_1) + \mathbf{P}(A_j C_3 D_2). \tag{26}$$

Consider the term P_{13}. The equality

$$C_3 \cap D_1 = \left\{s_1 < s_2,\ \phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix} < x_1 \le \phi_1 \begin{pmatrix} b+s_2 \\ a \end{pmatrix}\right\}$$

implies $\mathbf{P}(A_1 C_3 D_1) = 0$. Note that

$$C_3 \cap D_2 = \left\{s_2 < s_1,\ \phi_1 \begin{pmatrix} b+s_2 \\ a \end{pmatrix} < x_1 \le \phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix}\right\} \subseteq \left\{\phi_1 \begin{pmatrix} b+s_1 \\ a \end{pmatrix} \ge x_1\right\},$$

and, therefore,

$$P_{13} = \mathbf{P}\left\{\phi_2 \begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix} \ge x_2,\ C_3,\ D_2\right\}. \tag{27}$$

Consider the term P_{23}. The equality

$$C_3 \cap D_2 \subseteq \left\{\phi_1 \begin{pmatrix} b \\ a+s_2 \end{pmatrix} < x_1\right\},$$

implies both $\mathbf{P}(A_2C_3D_2) = 0$ and

$$P_{23} = \mathbf{P}(A_2C_3D_1) \le \mathbf{P}\left\{\phi_2\begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix} \ge x_2,\ C_3,\ D_1\right\}$$
$$= \mathbf{P}\left\{\phi_2\begin{pmatrix} b+s_2-\tau \\ (a+s_1-\tau)^+ \end{pmatrix} \ge x_2,\ C_3,\ D_2\right\}. \qquad (28)$$

The latter equation follows since s_1 and s_2 are i.i.d. r.v.'s.

If $s_2 < s_1$ then, by Definition 2,

$$\begin{pmatrix} b+s_2-\tau \\ (a+s_1-\tau)^+ \end{pmatrix} \lhd \begin{pmatrix} b+s_1-\tau \\ (a+s_2-\tau)^+ \end{pmatrix}.$$

Therefore, (27) and (28) imply the inequality $P_{13} \ge P_{23}$.

We have thus proved (23) and (19), completing the proof of Lemma 1. $\square$

Now, we fix an arbitrary $n \in \mathbb{N}$. For $0 \le k \le n$, we write $T^{(1)}_{(1,n)} = T_{(1,k)} \cup T^0_{(k+1,n)}$ if $T^{(1)}_i = T_i$ for $1 \le i \le k$ and

$$T^{(1)}_i = T^0_i = \arg\min_{1\le j\le m}\{v_{i-1,j}\}$$

for $k+1 \le i \le n$. This notation means that the first k customers are served according to the discipline T, while the customers numbered $k+1 \le i \le n$ are served according to the FCFS discipline. We write T^0_n instead of $T^0_{(n,n)}$. For brevity, we put $T_{(1,0)} \cup T^0_{(1,n)} = T^0_{(1,n)}$.

The proof of the theorem is based on the "backward induction argument" of [1, Lemma 2]. To provide the induction step, we need the following

Lemma 2. *For arbitrary integers k, n, $1 \le k \le n$, and for an arbitrary discipline $T^{(1)}$ of the form $T^{(1)}_{(1,n)} = T_{(1,k)} \cup T^0_{(k+1,n)}$, there exists a service discipline $T^{(2)}$ such that*

$$T^{(2)}_{(1,k)} = T_{(1,k-1)} \cup T^0_k \quad \text{and} \quad T^{(2)} \preccurlyeq_n T^{(1)}, \quad T^{(2)} \preccurlyeq_n T^{(1)}.$$

PROOF. The statement is clear for $k = n$. Indeed, in this case $T^{(1)}_{(1,n)} = T_{(1,n)}$ and $T^{(2)}_{(1,n)} = T_{(1,n-1)} \cup T^0_n$. By the definition of T^0, we have $d^{(2)}_n \le d_n$ a.s., and $T^{(2)} \preccurlyeq_n T$ because of the monotonicity of $h \in H_n$ in its last argument.

The vectors $W^{(2)}_n$ and W_n may only differ in the rth and lth coordinates, and only if $T_n = r \ne l = T^{(0)}_n$. The vectors coincide if $T_n = T^{(0)}_n$. In all cases

$$W^{(2)}_n \lhd W_n \quad \text{a.s.};$$

therefore, $\phi_2\big(W^{(2)}_n\big) \le \phi_2(W_n)$ a.s. whatever $\phi_2 \in H_m$, and $T^{(2)} \preccurlyeq_n T$.

For every fixed k, $1 \le k \le n-1$, if $T_k = T^0_k$, then the discipline $T^{(2)} \equiv T^{(1)}$ already satisfies the equality $T^{(2)}_{(1,n)} = T_{(1,k-1)} \cup T^0_{(k,n)}$.

Assume $T_k = r$ and $l = \arg\min_j\{v_{k-1,j}\} = \arg\min_j\{W_{k-1,j}\}$. By the Total Probability Law, we only have to prove the lemma on the event $\{r \ne l\}$.

In what follows, we assume $T_k = r \ne l = T^0_k$.

By the induction assumption, $T^{(1)}$ coincides with the FCFS discipline in the $(k+1)$st step; i.e., $T^{(1)}_{k+1} = T^0_{k+1} = \arg\min_j\{W_{k,j}\}$. Since $W_{k-1,l} \le W_{k-1,j}$ for every j and $W_{k,j} = (W_{k-1,j} - \tau_k)^+ + s_k I(j = r)$, the event $T^{(1)}_{k+1} = j \ne l$ may occur if and only if $W_{k,l} = W_{k,j} = 0$.

Fix j and assume the event $\{T_k = r \neq l = T_k^0\} \cap \{T_{k+1}^0 = j \neq l\}$ to occur. Then, instead of $T^{(1)}$, we can consider a new discipline $T^{(3)}$ such that $T_{k+i}^{(3)} = l$ if $T_{k+i}^{(1)} = j$ and $T_{k+i}^{(3)} = j$ if $T_{k+i}^{(1)} = l$ for every $i \geq 1$. Clearly, $d_{k+i}^{(1)} = d_{k+i}^{(3)}$, $W_{k+i}^{(1)} = R(W_{k+i}^{(3)})$ a.s.

In other words, we may assume throughout the proof that the equality $T_{k+1}^{(1)} = l$ holds a.s. on the event $\{T_k = r \neq l = T_k^0\}$.

Now, fix $1 \leq k \leq n-1$ and apply Lemma 1 to the "initial" vector W_{k-1} instead of W_0 and to the driving sequences $\{\tau_i\}_{i \geq k}$, $\{s_i\}_{i \geq k}$ instead of $\{\tau_i\}_{i \geq 1}$ and $\{s_i\}_{i \geq 1}$. More precisely, use Lemma 1 for a given $\{W_{k-1} = \text{const}\}$.

So we construct the discipline T' that coincides with T for first $k-1$ customers and coincides with T^0 for customer k. By (17) and (18), the following holds: for every function $h \in H_n$

$$h\big(d_1^{(1)}, \ldots, d_{k-1}^{(1)}, d_k', d_{k+1}', \ldots, d_n'\big) \leq h\big(d_1^{(1)}, \ldots, d_{k-1}^{(1)}, d_k^{(1)}, d_{k+1}^{(1)}, \ldots, d_n^{(1)}\big) \quad \text{a.s.} \tag{29}$$

Inequality (19) implies that, for $1 \leq k \leq n-1$,

$$\mathbf{P}\big(\phi_1\big(W_{n-1}'\big) \geq x_1,\ \phi_2\big(W_n'\big) \geq x_2\big) \leq \mathbf{P}\big(\phi_1\big(W_{n-1}^{(1)}\big) \geq x_1,\ \phi_2\big(W_n^{(1)}\big) \geq x_2\big). \tag{30}$$

In the system with the discipline T', replace the sequence of service times $s_1, \ldots, s_{k-1}, s_k', s_{k+1}', s_{k+2}, \ldots$ by $\{s_i\}$. Denote the resultant discipline by $T^{(2)}$. Inequality (29) remains valid, but now in the stochastic sense, and inequality (30) remains valid.

We have thus constructed the discipline $T^{(2)}$ such that $T^{(2)} \preccurlyeq_n T^{(1)}$, $T^{(2)} \preccurlyeq_n T^{(1)}$. Moreover, $T_{(1,k)}^{(2)} = T_{(1,k)}' = T_{(1,k-1)}^{(1)} \cup T_k^0$ by construction. $\square$

We take $n \in \mathbb{N}$ and prove, by use of backward induction, the following statement.

(I) For every $0 \leq k \leq n-1$ and every discipline T, the service discipline $T_{(1,n)}^{(1)} = T_{(1,k)} \cup T_{(k+1,n)}^0$ is no worse than T, i.e., the following inequalities hold:

$$T^{(1)} \preccurlyeq_n T, \quad T^{(1)} \preccurlyeq_n T. \tag{31}$$

Indeed, for $k = n-1$, (I) follows directly from Lemma 2. Assume that (I) holds for all $k \geq k_0 + 1$ less than n. Prove it for $k = k_0$.

By the induction assumption, for an arbitrary discipline T, the discipline $T_{(1,n)}^{(1)} = T_{(1,k_0+1)} \cup T_{(k_0+2,n)}^0$ is no worse than T in the sense of (31). By Lemma 2, there exists $T^{(2)}$ which is no worse than $T^{(1)}$ and such that $T_{(1,k_0+1)}^{(2)} = T_{(1,k_0)} \cup T_{k_0+1}^0$. By the induction assumption,

$$T_{(1,n)}^{(3)} = T_{(1,k_0+1)}^{(2)} \cup T_{(k_0+2,n)}^0 = T_{(1,k_0)} \cup T_{k_0+1}^0 \cup T_{(k_0+2,n)}^0 \equiv T_{(1,k_0)} \cup T_{(k_0+1,n)}^0$$

is no worse than $T^{(2)}$. Therefore, (I) holds for $k = k_0$.

In particular, Theorem 1 follows from (I) on taking $k = 0$. $\square$

Remark 2. Using (17) and (18), we can easily prove the following generalization of the inequality $T^0 \preccurlyeq_n T$: for every $k = 1, 2, \ldots$, arbitrary natural $n_1, \ldots, n_k$, and arbitrary functions $h_i \in H_{n_i}$, $i = 1, \ldots, k$, the following inequality holds

$$(h_1(d_1, \ldots, d_{n_1}), \ldots, h_k(d_1, \ldots, d_{n_k})) \geq_{st} \big(h_1\big(d_1^0, \ldots, d_{n_1}^0\big), \ldots, h_k\big(d_1^0, \ldots, d_{n_k}^0\big)\big). \tag{32}$$

§ 4. Generalizations and Applications of the Results

4.1. Possible generalizations. The statements of Theorem 1 may be generalized to networks of multiserver queues with state-independent routings (say, to Jackson-type networks). It suffices to give a proof for networks with deterministic routings and thereafter to make use of the Total Probability Law.

One of the possible generalizations is formulated below as Statement 1 for which we only give here some heuristic arguments. As a preliminary, we introduce a more general class of service disciplines which allows possible "delays" in commencement of service.

In what follows, we assume that a service discipline T is defined as a two-dimensional sequence

$$T = \{T_n, \Delta_n\}. \tag{33}$$

Here $\Delta_n \geq 0$ are random delays with the following meaning: if $T_1 = k$ then the service of the first customer starts not at the time instant $\max\{v_{0,k}, t_1\}$, but later at $u_1 = \max\{v_{0,k}, t_1\} + \Delta_1$, i.e., with delay Δ_1. We put $v_{1,j} = v_{0,j}$ for $j \neq k$, and $v_1 = v_{1,k} = u_1 + s_1$. For $n \geq 2$, $v_{n,j}$ and u_n are defined inductively. Assume $v_{n,j}$ and u_n to be defined. For a fixed k, if the event $T_{n+1} = k$ takes place, then the service of the $n+1$st customer starts at the time instant $u_{n+1} = \max\{v_{n,k}, t_{n+1}\} + \Delta_{n+1}$ (with delay Δ_{n+1}). Then we put $v_{n+1,j} = v_{n,j}$ for $j \neq k$, and $v_{n+1} = v_{n+1,k} = u_{n+1} + s_{n+1}$.

We say that a service discipline in the class (33) is *admissible*, if, for arbitrary n, $k_1, \ldots, k_n$ and arbitrary measurable sets $B_1, \ldots, B_n$, the following holds

$$\begin{aligned} &\mathbf{P}\big\{T_1 = k_1, \Delta_1 \in B_1, \ldots, T_n = k_n, \Delta_n \in B_n \mid W_0; \{\tau_i\}_{i=1}^{\infty}; \{s_i\}_{i=1}^{\infty}\big\} \\ &= \mathbf{P}\big\{T_1 = k_1, \Delta_1 \in B_1, \ldots, T_n = k_n, \Delta_n \in B_n \mid W_0; \{\tau_i\}_{i=1}^{\infty}; \{s_i\}_{i=1}^{n-1}\big\}. \end{aligned} \tag{34}$$

By coordinate-wise monotonicity of Schur-convex functions, a service discipline with delays is "worse" than the corresponding discipline without delays which is admissible in the sense of Definition 1. Applying Theorem 1 to the latter discipline, we can show that the statement of Theorem 1 holds true for the disciplines in the class (33) which are admissible in the sense of (34).

Remark 3. It is possible to consider further generalizations of the class of service disciplines (for example, we can assume that the order of service may differ from the order of arrival (see [8])) and prove optimality of the FCFS discipline in this case.

Now, consider a queueing network with K stations numbered $1, \ldots, K$, where each station is a multiserver queue. Exogenous customers arrive at station 1. At each station, upon service completion a customer either goes to some other station or leaves the system according to the following rule. Assume that a sequence of integers $l_{k,1}, l_{k,2}, \ldots$ is given for every $k = 1, \ldots, K$. Here the $l_{k,j}$'s take values in $1, 2, \ldots, K+1$, and have the following meaning: upon the jth service completion at station k, the customer is directed to station $l_{k,j}$ if $l_{k,j} \leq K$ or leaves the system if $l_{k,j} = K+1$. At an arbitrary station k, the service times $\{s_j^{(k)}\}$ form an i.i.d. sequence which is independent of everything else. Assume that the $l_{k,j}$ are selected in such a way that any customer may have only a finite number of services before leaving the network. For each station k, the service discipline $T^{(k)} = \big\{T_j^{(k)}, \Delta_j^{(k)}\big\}$ is used which is assumed to be *admissible* in the following sense: for all k, j, the random variables $T_{j+1}^{(k)}, \Delta_{j+1}^{(k)}$ may depend only on the local station history, i.e., on previous interarrival and service times at station k.

Also, assume that only a finite number of exogenous customers (say, n) can enter the system. Denote by z_j, $j = 1, \ldots, n$, the sojourn time of customer j. We give the following result without proof.

Statement 1. *For every Schur-convex function h,*

$$h\big(z_1^0, \ldots, z_n^0\big) \leq_{st} h(z_1, \ldots, z_n). \tag{35}$$

Here $(z_1^0, \ldots, z_n^0)$ stands for the vector of sojourn times in the network with FCFS discipline at all stations.

The proof of (35) is based on the same ideas as that of Theorem 1, but it is much more complicated. A statement analogous to (3) is also valid. We sketch a possible proof in the particular case of a tandem of two multiserver queues.

More precisely, consider a network of two stations (queues). Upon service completion at the first station, each customer goes to the second station and after service there it leaves the network.

In this particular case, the reader can prove (35) using the following idea. At the first station, replace the discipline $T^{(1)}$ by FCFS. Then all customers leave this station earlier (see (5)). Simultaneously, put delays at the entrance to the second station in such a way that all service commencements/completions at the second station remain the same as under $T^{(1)}$. Now, replace $T^{(2)}$ by the FCFS and, finally, remove the delays.

Remark 4. We should explain why (35) may be valid only for the joint distribution of the sojourn times of *all* n customers. The reason is that, in the scheme of proof, the following argument is crucial: at each station, the total number of services of the first n customers has to be the same for all service disciplines. If this is not the case, then (35) may fail.

Consider, for example, the tandem of two stations, where the first station is a two-server queue and the second is a single-server queue. Assume that only two customers arrive at the first station. The customer whose service is completed first is directed to the second station, and the other customer leaves the network immediately after service completion at the first station.

Define the initial state $W_0 = (W_{0,1}, W_{0,2})$ at the first station (assume $0 \le W_{0,1} < W_{0,2}$). Put $t_1 \le t_2$ for the arrival instants; put s_1, s_2 for the service times at the first station and s for the service time at the second station. Assume $t_1 = t_2 = W_{0,1}$, $s_1 \le W_{0,2} - W_{0,1}$ and $s > W_{0,2} - W_{0,1}$ a.s. Under the FCFS discipline, $z_1^0 = s_1 + s$ a.s. However, if we direct the first customer to the second server at the first station, then $z_1 = s_1 + W_{0,2} - W_{0,1}$. Therefore, for $n = 1$ (here n is less than the number of arrivals), inequality (35) fails.

Now, consider a network with infinitely many arrivals. Let $\tau_1 + \cdots + \tau_j$ be the time when customer j arrives. Assume that the network is stable under the FCFS discipline at all stations as well as under the disciplines $T^{(k)}$, $k = 1 \ldots, K$. Assume, moreover, that $\mathbf{E}f(z^0)$ and $\mathbf{E}f(z)$ exist for some convex and increasing function f, where z^0 and z stand for the stationary total sojourn times in the corresponding network. Then

$$\frac{1}{n}\sum_{j=1}^{n} f(z_j^0) \to \mathbf{E}f(z^0) \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} f(z_j) \to \mathbf{E}f(z) \quad \text{a.s.}$$

For every fixed n, denote by $z_{n,j}^0$ (or $z_{n,j}$), $j = 1, \ldots, n$, the total sojourn time of customer j in the modified network in which only the first n arrivals are allowed. It is easy to show that the normalized sum

$$\frac{1}{n}\sum_{j=1}^{n} f(z_{n,j}^0)$$

converges a.s. to $\mathbf{E}f(z^0)$. Similarly, $\frac{1}{n}\sum_{j=1}^{n} f(z_{n,j})$ converges a.s. to $\mathbf{E}f(z)$. Therefore, we can use (35) with the functions

$$h(z_1, \ldots, z_n) = \frac{1}{n}\sum_{1}^{n} f(z_j)$$

and obtain the following result.

Corollary 1. *Let z be a stationary sojourn time and assume the function f convex and increasing. Then*

$$\mathbf{E}f(z^0) \le \mathbf{E}f(z).$$

4.2. Applications. We suggest some applications of the results of § 2. Consider a multiserver queue $GI/GI/m$ with m servers and i.i.d. interarrival times τ_n which are independent of $\{s_n\}$. Put $a = \mathbf{E}\tau_1 > 0$, $b = \mathbf{E}s_1$ and assume the traffic intensity condition $\rho \equiv \frac{b}{ma} < 1$ holds.

1. In [14] the following statement was proved for the above queue with the FCFS discipline.

Theorem 2. *Assume that $(m-1)a > b$. If $\mathbf{E}s_1^c < \infty$ for $3/2 \le c \le 2$, then $\mathbf{E}(d^0)^{2c-2} < \infty$. If $\mathbf{E}s_1^c < \infty$ for $c > 2$, then $\mathbf{E}(d^0)^c < \infty$.*

The scheme of the proof in [14] was as follows: first, the existence of expectations was proved for the discipline of Example 4. Then the inequality $\mathbf{E}d^c \geq \mathbf{E}(d^0)^c$ was used. The latter inequality follows from (10) on taking the function $h = \frac{1}{n}\sum_{i=1}^{n}(d_i)^c$, $c \geq 1$ and letting n tend to infinity.

2. We also give two examples of obtaining upper bounds for tail asymptotics $\mathbf{P}(d^0 > x)$ in the $GI/GI/m$ queue defined above.

2.1. We make a comparison with the random discipline (see Example 3).

Put $x = y + z$, where $0 \leq y = y(x) \to \infty$ and $0 \leq z = z(x) \to \infty$ as $x \to \infty$. For all $\alpha > 1$,

$$\mathbf{P}(d^0 \geq x) = \mathbf{P}((d^0 - z)^+ \geq y) \leq \frac{\mathbf{E}((d^0 - z)^+)^\alpha}{y^\alpha} \leq \frac{\mathbf{E}((d - z)^+)^\alpha}{y^\alpha}.$$

Here d stands for the stationary waiting time in the multiserver queue with the random service discipline. The stationary workloads at all stations have the same distribution. Choosing a server at random and independently of the system state, we obtain a distribution of stationary waiting time which is the same as in the single-server queue $GI/GI/1$ with interarrival times τ_n and service times distributed as $\sigma_n = s_n I(T_n = 1)$; i.e., $\mathbf{P}(\sigma_n > x) = \frac{1}{m}\mathbf{P}(s_1 > x)$ for $x \geq 0$. For example, assume that the distribution of s_n is subexponential. Then the following property is well-known:

$$\mathbf{P}(d > x) \sim \frac{\rho}{1-\rho}\frac{1}{m}\int_x^\infty \mathbf{P}(s_1 > t)\, dt$$

as $x \to \infty$. We thus obtain

$$\mathbf{P}(d^0 \geq x) \leq (1 + o(1))\frac{\rho}{1-\rho}\frac{1}{my^\alpha}\int_z^\infty (t - z)^\alpha \mathbf{P}(s_1 > t)\, dt. \tag{36}$$

For specific distributions of the r.v. s_1, we can find the minimum over y of the RHS of (36).

2.2. We make a comparison with the cyclic discipline (see Example 2). The following inequalities hold:

$$\mathbf{P}(d^0 \geq x) \leq \mathbf{P}\big(W_1^0 + \dots + W_m^0 \geq mx\big) \leq \mathbf{P}(W_1 + \dots + W_m \geq mx),$$

where $(W_1, \dots, W_m)$ is the stationary vector of virtual waiting times. Assume the distribution of service times to be subexponential. We can show that

$$\mathbf{P}(W_1 + \dots + W_m \geq mx) \sim m\mathbf{P}(W > mx),$$

where W coincides in distribution with the stationary waiting time in the $GI/GI/1$ queue with the service times $\{s_n\}$ and interarrival times distributed as $\tau_1 + \dots + \tau_m$. Hence,

$$\mathbf{P}(d^0 \geq x) \leq (1 + o(1))\frac{\rho m}{1-\rho}\int_{mx}^\infty \mathbf{P}(s_1 > t)\, dt. \tag{37}$$

Examples show that, according to the distribution of s_1, either (36) or (37) may give a sharper estimate of $\mathbf{P}(d^0 \geq x)$.

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