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Markov Processes and Related Fields

Sampling at a Random Time with a Heavy-Tailed Distribution*

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Abstract. Let $S_n = \xi_1 + \cdots + \xi_n$ be a sum of i.i.d. non-negative random variables, $S_0 = 0$. We study the asymptotic behaviour of the probability $\mathsf{P}\{X(T) > n\}$, $n \to \infty$, where $X(t) = \max\{n \geq 0 : S_n \leq t\}$, $t \geq 0$, is the corresponding renewal process. The stopping time T has a heavy-tailed distribution and is independent of $X(t)$. We treat two different approaches to the study: via the law of large numbers and by using the large deviation techniques. The first approach is applied to the case when T has a heavier tail than $\exp(-\sqrt{x})$. The second one is mostly applied to the case of the so-called "moderately heavy tails" when T has a lighter tail than $\exp(-\sqrt{x})$. As a corollary, the distributional Little's law allows us to obtain the tail asymptotics for a stationary queue length in a single server queue with subexponential service times. More generally, if a stable queueing system satisfies the distributional Little's law and if a stationary sojourn time distribution of a "typical" customer is heavy-tailed and its asymptotics is known, then the results of this paper provide a way for obtaining the tail asymptotics for a stationary queue length.

KEYWORDS: sums of i.i.d. random variables, renewal process, independent stopping time, heavy-tailed distribution, large deviations, Cramèr transform, rate function, single server queue, stationary queue length

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1. Introduction

Let ξ, ξ_n, n=1, 2, ..., be i.i.d. non-negative random variables with positive mean value $\mu \equiv \mathsf{E}\,\xi > 0$. Put $S_n = \xi_1 + \cdots + \xi_n$, $S_0 = 0$. Let $X(t) = \max\{n \geq 0 : S_n \leq t\}$ be the undelayed renewal process.

The main goal of the paper is to study the asymptotic behaviour of the probability $\mathsf{P}\{X(T) > n\}$, where T is a stopping time, independent of $\{X(t),\ t \geq 0\}$, with heavy-tailed distribution F.

Definition 1.1. A distribution F with an unbounded support is called *a heavy-tailed distribution* (or *a long-tailed distribution*) if $\overline{F}(x+t) \sim \overline{F}(x)$[1] as $x \to \infty$, for each fixed t.

Note that for any random variable T with a heavy-tailed distribution F it holds that $\mathsf{E}\exp(\varepsilon T) = \infty$ for each $\varepsilon > 0$ or, equivalently,

$$\overline{F}(x)e^{\varepsilon x} \to \infty \quad \text{as } x \to \infty. \tag{1.1}$$

We have

$$\begin{aligned} \mathsf{P}\{X(T) > n\} &= \mathsf{P}\{S_{n+1} \leq T\} \\ &= \mathsf{E}\{\mathsf{P}\{T \geq S_{n+1} \mid S_{n+1}\}\} = \mathsf{E}\,\overline{F}(S_{n+1} - 0), \end{aligned}$$

where $\overline{F}(x) = F((x,\infty))$ is the right tail of F. Hence, as $n \to \infty$, we concentrate on the quantity

$$\mathsf{E}\,\overline{F}(S_n) \equiv \mathsf{E}\exp(-g(S_n)),$$

where

$$g(x) = -\log \overline{F}(x). \tag{1.2}$$

By definition, $g(x) = 0$ for $x < 0$ and $g(x)$ is a non-decreasing function such that $g(x) \to \infty$ as $x \to \infty$.

Definition 1.2. A distribution F with the tail $\exp(-x^\beta)$, $x > 0$, $\beta > 0$, is called a *Weibull distribution with parameter* β.

In a recent paper [1], Asmussen, Klüppelberg and Sigman considered two distinct cases when T has either a heavier or a lighter tail than that of the Weibull distribution with parameter 1/2. In the first case, they showed that

$$\mathsf{P}\{X(T) > n\} \sim \overline{F}(n\mu), \quad n \to \infty,$$

under mild conditions on $X(t)$. In the second ("moderately heavy") case, the asymptotics is obtained under restriction that $X(t)$ is a homogeneous Poisson process with intensity $1/\mu$ and $\overline{F}(x) = \gamma(x)\exp(-x^\beta)$, where $\beta \in [1/2, 1)$ and γ

[1] Hereinafter, $a(x) \sim b(x)$ means that $a(x)/b(x) \to 1$ as $x \to \infty$.

is a continuous function satisfying certain conditions. In particular, it is shown that the tail of $X(T)$ is heavier than that of T/μ.

In this paper, we study both of the mentioned cases and find the asymptotics of $\mathsf{P}\{X(T) > n\}$ under more general conditions: for any renewal process $X(t)$ and for a wide class of functions F. Our technique differs from that of [1]: in Section 3, we make use of the law of large numbers (LLN), while the results of Sections 5–8 are based on tools from the large deviations theory.

Our investigations are motivated in particular by the following queueing problem: find the tail asymptotics of a stationary queue length Q in a stable "first-in-first-out" queueing system. Due to *distributional Little's law* (see [4]), Q is equal in distribution to $N(U)$ where $N(t)$ is a stationary (delayed) renewal input process with interarrival times ξ_n, U is independent of N and has the distribution of the stationary sojourn time of a "typical" customer. Assume that the asymptotics of $\mathsf{P}\{U > x\}$, $x \to \infty$, is known and is heavy-tailed. Then (see Section 2) $\mathsf{P}\{N(U) > n\} \sim \mathsf{P}\{X(U) > n\}$ as $n \to \infty$ and the asymptotics of $\mathsf{P}\{X(U) > n\}$ can be found by using the results of Sections 3 and 5–8.

For instance, let us consider a stable $GI/GI/1$ queue (among other examples, one can mention a FCFS multi-server queue, a tandem of queues, etc.). Let σ be a "typical" service time and $\rho = \mathsf{E}\,\sigma/\,\mathsf{E}\,\xi < 1$. Then the asymptotics of $\mathsf{P}\{U > x\}$, $x \to \infty$, is well known (see, e.g., [9]): it is either exponential if $\mathsf{E}\exp(\lambda\sigma)$ is finite for a certain $\lambda > 0$ or

$$\mathsf{P}\{U > x\} \sim \rho(1-\rho)^{-1}\,\mathsf{P}\{\sigma^{(s)} > x\} \qquad \text{as } x \to \infty$$

if the distribution of $\sigma^{(s)}$,

$$\mathsf{P}\{\sigma^{(s)} > x\} = (\mathsf{E}\,\sigma)^{-1}\int_x^\infty \mathsf{P}\{\sigma > y\}dy,$$

is subexponential (see definition in [9]). In [1], one can find a further discussion and references on the topic and related problems.

The paper is organized as follows. Section 2 contains some preliminary results. In particular, we show that the asymptotics of $N(U)$ and $X(U)$ do not differ in the heavy-tailed case. In Section 4 we give definitions and properties of the Cramèr transform and the rate function and obtain uniform estimates of the probability for the sum S_n to be in a compact set.

The main results of the paper concerning $\mathsf{E}\exp(-g(S_n))$ are obtained in Sections 3 and 5–8. We use two different approaches: (a) linear approximations of $g(S_n)$ around $n\mu$ and the LLN (see Section 3) and (b) large deviations techniques (see Sections 5–8).

Section 3 deals with the case $g(x) = o(x^{1/2})$. In [1], it is shown that the equivalence $\mathsf{P}\{X(T) > n\} \sim \overline{F}(n\mu)$ holds for a general process $\{X(t),\ t \ge 0\}$ satisfying the CLT, under certain conditions on the distribution of T. We obtain an analogue of this result for a general renewal process which may not satisfy the CLT; we do not assume the existence of $\mathsf{E}\,\xi^2$. Conditions on the distribution of T are also weakened.

In Section 5, we obtain a general result concerning the desired asymptotics of functions g such that $g(x) = o(x)$. Namely,

$$\mathsf{E}\exp(-g(S_n)) \sim \exp(-H_n(t_n)), \qquad n \to \infty,$$

where $H_n(x) = g(x) + n\Lambda(x/n)$, Λ is the rate function of ξ and t_n, $t_n \le n\mu$, is a root of a certain equation.

In Section 6, we investigate the properties of H_n, t_n and the asymptotic behaviour of $H_n(t_n)$ under further restrictions on g. Specific examples and applications of the results are given when $\overline{F}(x) = x^\gamma \exp(-x^\beta)$ with $\gamma > 0$ and $\beta \in (0, 2/3)$; see Corollaries 6.1, 6.2 and 6.4. A counterexample to the asymptotics in [1, Remark 4.4] is given, and the proper convergence rate is derived in (6.11).

In Section 7, we develop a generalization of the results of Section 6 by using an iterative procedure. In general, it is not easy to obtain t_n. The iterative procedure replaces t_n by some $t_n^{(k)}$ in a finite number of steps, where $t_n^{(k)}$ is easy to derive and where the replacement of $H_n(t_n)$ by $H_n(t_n^{(k)})$ causes only an error of $o(1)$ as $n \to \infty$. Special assumptions on g are required. This leads to a methodology for asymptotic estimation for a large class of heavy-tailed distributions F.

In Section 8 (Theorem 8.1), we apply the results of Section 6 to the case of the standard Poisson process $X(t)$ and $\overline{F}(x) = \exp(-u(x) - x^\beta)$, $\beta \in (0, 1)$, for a suitable $u(x)$.

2. Preliminaries

In the sequel we use the following general qualitative results.

Lemma 2.1. *Let F be any distribution satisfying condition* (1.1). *If $\delta > 0$, then $\mathsf{P}\{S_n \le n(\mu-\delta)\} = o(\mathsf{E}\,\overline{F}(S_n))$ as $n \to \infty$.*

Proof. By the Chebyshev inequality in the exponential form, for $\lambda > 0$ we have the following estimate:

$$\begin{aligned}\mathsf{P}\{S_n \le n(\mu-\delta)\} &= \mathsf{P}\{\exp(-\lambda(S_n - n(\mu-\delta)) \ge 1\} \\ &\le \mathsf{E}\exp(-\lambda(S_n - n(\mu-\delta))) \;=\; (\mathsf{E}\exp(-\lambda(\xi_1 - \mu + \delta)))^n.\end{aligned}$$

Since $\mathsf{E}(\xi_1-\mu+\delta) = \delta > 0$, $\mathsf{E}\exp(-\lambda(\xi_1-\mu+\delta)) < 1$ for sufficiently small $\lambda > 0$. Thus, there exists $\varepsilon > 0$ such that

$$\mathsf{P}\{S_n \le n(\mu-\delta)\} = o(e^{-n\varepsilon}) \quad \text{as } n \to \infty.$$

According to the law of large numbers, $\mathsf{P}\{S_n \le 2n\mu\} \to 1$ as $n \to \infty$. Hence,

$$\mathsf{E}\,\overline{F}(S_n) \ge \overline{F}(2n\mu)\,\mathsf{P}\{S_n \le 2n\mu\} \sim \overline{F}(2n\mu).$$

Now the relation of the lemma follows from condition (1.1). □

Lemma 2.2. *Let two distributions F and G be tail equivalent, i.e. $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$. If condition* (1.1) *holds, then $\mathsf{E}\,\overline{F}(S_n) \sim \mathsf{E}\,\overline{G}(S_n)$ as $n \to \infty$.*

Proof. Putting $\delta = \mu/2$, from Lemma 2.1 we obtain

$$\begin{aligned} \mathsf{E}\,\overline{F}(S_n) &\sim \mathsf{E}\{\overline{F}(S_n);\ S_n > n\mu/2\}, \\ \mathsf{E}\,\overline{G}(S_n) &\sim \mathsf{E}\{\overline{G}(S_n);\ S_n > n\mu/2\} \end{aligned}$$

as $n \to \infty$. Now the conclusion of Lemma 2.2 follows from the condition $\overline{F}(x) \sim \overline{G}(x)$. □

Lemma 2.3. *Let $\xi_0 \ge 0$ be a random variable independent of everything else. If F is a heavy-tailed distribution, then*

$$\mathsf{E}\,\overline{F}(\xi_0 + S_n) \sim \mathsf{E}\,\overline{F}(S_n) \qquad \text{as } n \to \infty.$$

Proof. A heavy-tailed distribution satisfies condition (1.1). By Lemma 2.2, for each fixed $t > 0$, $\mathsf{E}\,\overline{F}(t + S_n) \sim \mathsf{E}\,\overline{F}(S_n)$ as $n \to \infty$. We also have $\overline{F}(t + S_n) \le \overline{F}(S_n)$. Therefore, the representation

$$\frac{\mathsf{E}\,\overline{F}(\xi_0 + S_n)}{\mathsf{E}\,\overline{F}(S_n)} = \int_0^\infty \frac{\mathsf{E}\,\overline{F}(t + S_n)}{\mathsf{E}\,\overline{F}(S_n)}\,\mathsf{P}\{\xi_0 \in dt\}$$

and the Lebesgue bounded convergence theorem complete the proof. □

Corollary 2.1. *Consider a stable GI/GI/1 queue* (*see Introduction*)*. Let the distribution of U be heavy-tailed. Then*

$$\mathsf{P}\{Q > n\} = \mathsf{P}\{N(U) > n\} \sim \mathsf{P}\{X(U) > n\} \qquad \text{as } n \to \infty,$$

where N is the stationary (*delayed*) *input renewal process and X is the undelayed one.*

3. Asymptotic behaviour of $\mathsf{E}\exp(-g(S_n))$ in the case $g(x) = o(x^{1/2})$

3.1. Decomposition of a characteristic-type function at zero point

In [8], the series expansion for the characteristic function $\mathsf{E}\exp(i\lambda\eta)$ of a random variable η was established at the point $\lambda = 0$ under assumption that η has some moment of non-integer order. The following lemma extends some results of [8] to a wider class of functionals of η.

Lemma 3.1. *Let η be a random variable and let $f(\lambda) : \mathbf{R} \to \mathbf{C}$ be a continuous function such that $|f(\lambda)| \le f_0$ for some $f_0 < \infty$. Define the function $\widehat{f}(\lambda) \equiv \mathsf{E}\, f(\lambda\eta)$, $\lambda \in \mathbf{R}$. Fix a positive integer k. Let f have the kth order derivative,*

$$\sup_\lambda |f^{(k)}(\lambda)| = f_k < \infty,$$

and let $\mathsf{E}\,|\eta|^\alpha$ be finite for some $\alpha \in [k-1, k)$. Then the function $\widehat{f}(\lambda)$ admits the following decomposition, as $\lambda \to 0$:

$$\widehat{f}(\lambda) = \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} \mu_j \lambda^j + o(\lambda^\alpha),$$

where $\mu_j = \mathsf{E}\,\eta^j$.

Proof. Since $\mathsf{E}\,|\eta|^\alpha < \infty$, $\mathsf{E}\{\eta^j;\ |\eta| > c\} = o(1/c^{\alpha-j})$ as $c \to \infty$, for any $j \in [0, k-1]$. Therefore, there exists a function $d(c)$ such that $d(c)/c \to 0$ as $c \to \infty$ and

$$\mathsf{P}\{|\eta| > d(c)\} = o(1/c^\alpha), \tag{3.1}$$

$$\mathsf{E}\{|\eta|^j;\ |\eta| > d(c)\} = o(1/c^{\alpha-j}), \quad j \le k-1. \tag{3.2}$$

We have

$$\begin{aligned} &\Big|\widehat{f}(\lambda) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} \mu_j \lambda^j\Big| \\ &\quad\le \Big|\mathsf{E}\Big\{f(\lambda\eta) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} \eta^j \lambda^j;\ |\eta| \le d(c)\Big\}\Big| \\ &\qquad + f_0\,\mathsf{P}\{|\eta| > d(c)\} + \sum_{j=0}^{k-1} \frac{|f^{(j)}(0)|}{j!} |\lambda|^j\, \mathsf{E}\{|\eta|^j;\ |\eta| > d(c)\}. \end{aligned} \tag{3.3}$$

Taking into account that

$$\Big|f(z) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j\Big| \le \frac{f_k}{k!} |z|^k$$

for any $z \in \mathbf{R}$, $k - \alpha > 0$ and $d(c) = o(c)$ as $c \to \infty$, we get

$$\begin{aligned} &\Big|\mathsf{E}\Big\{f(\lambda\eta) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} \eta^j \lambda^j;\ |\eta| \le d(c)\Big\}\Big| \\ &\quad\le f_k |\lambda|^k\, \mathsf{E}\{|\eta|^k;\ |\eta| \le d(c)\} \\ &\quad\le f_k |\lambda|^k\, \mathsf{E}\{|\eta|^\alpha / |\eta|^{\alpha-k};\ |\eta| \le d(c)\} \\ &\quad\le f_k |\lambda|^k d^{k-\alpha}(c)\, \mathsf{E}\,|\eta|^\alpha = |\lambda|^k o(c^{k-\alpha}). \end{aligned} \tag{3.4}$$

Putting $c = 1/\lambda$ and substituting (3.1)–(3.4) into (3.3), we reach the conclusion of the lemma. □

3.2. LLN in the case $\mathsf{E}\,|\eta|^\alpha < \infty$ for some $\alpha \in [1,2)$

Let η_n, $n \ge 1$, be i.i.d. random variables. For $T_n = \eta_1 + \cdots + \eta_n$, the following Kolmogorov – Marcinkiewicz strong law of large numbers is valid (see, e.g., [6, p. 243]).

Lemma 3.2. *Let $\mathsf{E}\,|\eta_1|^\alpha$ be finite for some $\alpha \in [1,2)$ and let $\mathsf{E}\,\eta_1 = 0$. Then $T_n/n^{1/\alpha} \to 0$ as $n \to \infty$ almost surely.*

Note that the weak convergence $T_n/n^{1/\alpha} \Rightarrow 0$ follows from Lemma 3.1. Indeed, for any fixed $\lambda \in \mathbf{R}$,

$$\mathsf{E}\exp(i\lambda T_n/n^{1/\alpha}) = (\mathsf{E}\exp(i\lambda\eta_1/n^{1/\alpha}))^n = (1 + o((n^{-1/\alpha})^\alpha))^n \to 1 \quad \text{as } n \to \infty.$$

3.3. Asymptotic behaviour of $\mathsf{E}\exp(-g(S_n))$

The following theorem deals with a function $d(x) = o(x^{1/2})$, and, in particular, with any Weibull distribution with parameter $\beta \in (0, 1/2)$.

Theorem 3.1. *Let $\gamma \in (1/2, 1]$. For any function $d(x)$ such that $d(x) \to \infty$ as $x \to \infty$, let*

$$g(x + x^\gamma/d(x)) = g(x) + o(1) \qquad \text{as } x \to \infty. \tag{3.5}$$

If $\mathsf{E}\,\xi^{1/\gamma}$ is finite, then

$$\mathsf{E}\exp(-g(S_n)) \sim \exp(-g(n\mu)) = \overline{F}(n\mu) \qquad \text{as } n \to \infty.$$

Remark 3.1. Given $\gamma \le 1$, the function $g(x) = x^{1-\gamma}$ satisfies condition (3.5).

Corollary 3.1. *Let $\overline{F}(x) \sim x^{\gamma_1}\exp(-cx^\beta)$, $\gamma_1 \in \mathbf{R}$, $c > 0$, $\beta \in (0, 1/2)$. If $\mathsf{E}\,\xi^{1/(1-\beta)}$ is finite, then $\mathsf{E}\,\overline{F}(S_n) \sim \overline{F}(n\mu)$ as $n \to \infty$.*

Proof of Theorem 3.1. Let a twice differentiable function f be such that $f(x) = \exp(x)$ for $x \le 1$ and let the second derivative of f be bounded. Due to the conditions $1/\gamma \in [1,2)$ and $\mathsf{E}\,\xi^{1/\gamma} < \infty$, Lemma 3.1 implies that $\mathsf{E}\,f(\lambda(\xi - \mu)) = 1 + o(\lambda^{1/\gamma})$ as $\lambda \to 0$. Taking into account that ξ is non-negative we obtain for $\lambda \in (-1/\mu, 0)$

$$\mathsf{E}\,e^{\lambda(\xi-\mu)} = \mathsf{E}\,f(\lambda(\xi - \mu)) = 1 + o(\lambda^{1/\gamma}) \quad \text{as } \lambda \uparrow 0.$$

Therefore, there exists a sequence $d_1(x) \to \infty$ such that

$$\mathsf{E}\exp(-d_1(n\mu)(\xi - \mu)/n^\gamma) = 1 + o(1/n) \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned}\mathsf{E}\exp(-d_1(n\mu)(S_n-n\mu)/n^\gamma) &= \big(\mathsf{E}\exp(-d_1(n\mu)(\xi-\mu)/n^\gamma)\big)^n \\ &= (1+o(1/n))^n \to 1.\end{aligned}$$

On the other hand, according to Lemma 3.2, there exists a function $d_2(x) \to \infty$ such that $d_2(n\mu)(S_n - n\mu)/n^\gamma$ converges weakly to 0. We set $d(x) = \min(d_1(x), d_2(x))$. Then $d(x) \to \infty$ and

$$\mathsf{E}\exp(-d(n\mu)(S_n-n\mu)/n^\gamma) \to 1 \qquad \text{as } n\to\infty \tag{3.6}$$

and $d(n\mu)(S_n - n\mu)/n^\gamma$ converges weakly to 0. Taking into account that it implies the convergence

$$\mathsf{E}\big\{\exp(-d(n\mu)(S_n-n\mu)/n^\gamma);\ d(n\mu)|S_n-n\mu|/n^\gamma \le 1\big\} \to 1,$$

from (3.6) we deduce that

$$\mathsf{E}\big\{\exp(-d(n\mu)(S_n-n\mu)/n^\gamma);\ d(n\mu)(S_n-n\mu)/n^\gamma \le -1\big\} \to 0 \qquad \text{as } n\to\infty. \tag{3.7}$$

Now turn to condition (3.5). First it implies that

$$g(S_n)-g(n\mu) \equiv g\Big(n\mu + \frac{n^\gamma}{d(n\mu)} d(n\mu)(S_n - n\mu)/n^\gamma\Big) - g(n\mu)$$

converges weakly to 0 as $n \to \infty$. Therefore, by (3.6),

$$\mathsf{E}\big\{\exp(-(g(S_n)-g(n\mu)));\ d(n\mu)(S_n-n\mu)/n^\gamma > -1\big\} \to 1. \tag{3.8}$$

Second, (3.5) provides the existence of a constant c such that

$$g(y)-g(x) \ge -\mu^\gamma(x-y)d(x)/x^\gamma - c$$

for all $x > 0$ and $y \in [x/2, x]$. Therefore, the inequality

$$\exp(-(g(S_n)-g(n\mu))) \le \exp(-d(n\mu)(S_n-n\mu)/n^\gamma + c)$$

holds if $S_n \in [n\mu/2, n\mu]$. ¿From this inequality and from (3.7), we obtain the convergence

$$\mathsf{E}\big\{\exp(-(g(S_n)-g(n\mu)));\ d(n\mu)(S_n-n\mu)/n^\gamma \le -1,\ S_n \ge n\mu/2\big\} \to 0.$$

Together with Lemma 2.1, this convergence implies that

$$\mathsf{E}\big\{e^{-g(S_n)};\ d(n\mu)(S_n-n\mu)/n^\gamma \le -1\big\} = o(e^{-g(n\mu)}) \qquad \text{as } n\to\infty.$$

Combining this with (3.8) we complete the proof. □

4. Some estimates via the rate function

In this section, we remove the assumption that ξ is positive and that $\mu = \mathsf{E}\,\xi$ is finite. Define $\varphi(\lambda) = \mathsf{E}\exp(\lambda\xi)$. In the sequel we assume that

$$\lambda_- \equiv \inf\{\lambda \le 0: \ \varphi(\lambda) < \infty\} < 0.$$

Then, in particular, $\mu \in (-\infty, \infty]$.

4.1. The rate function and the Cramèr transform

We recall the notions and some properties of the rate function and the Cramèr transform. We set

$$\alpha_- = \lim_{\lambda\to\lambda_-} \varphi'(\lambda)/\varphi(\lambda) = \lim_{\lambda\to\lambda_-} d\ln\varphi(\lambda)/d\lambda.$$

It is well known (see, e.g., [3, Chapter XVI, § 7]) that the function $\ln\varphi(\lambda)$ is strictly convex. Hence, $\alpha_- < \varphi'(0)/\varphi(0) = \mu$, where $\varphi'(0)$ is the left derivative at zero.

If the random variable ξ is bounded from below, then $\lambda_- = -\infty$ and α_- coincides with essential infimum of ξ, i.e. $\alpha_- = \inf\{x\colon\ \mathsf{P}\{\xi < x\} > 0\}$. If the random variable ξ is unbounded from below and $\lambda_- = -\infty$, then $\alpha_- = -\infty$. If the random variable ξ is from below and λ_- is finite, then α_- may take any value from the interval $[-\infty, \mu)$.

Since the function $\ln\varphi(\lambda)$ is strictly convex, for $\alpha \in (\alpha_-, \mu]$ the function $\alpha\lambda - \ln\varphi(\lambda)$ has a unique maximum, say, at point $\lambda(\alpha)$. Then

$$d\ln\varphi(\lambda)/d\lambda\big|_{\lambda=\lambda(\alpha)} = \alpha.$$

The function $\lambda(\alpha)$ is increasing and $\lambda(\mu) = 0$. The function

$$\Lambda(\alpha) \equiv \sup_{\lambda}\,\{\alpha\lambda - \ln\varphi(\lambda)\} = \alpha\lambda(\alpha) - \ln\varphi(\lambda(\alpha)), \quad \alpha \in (\alpha_-, \mu], \tag{4.1}$$

is called *the rate function.* By differentiating (4.1) we have $\Lambda'(\alpha) = \lambda(\alpha)$; therefore, the function Λ is strictly convex. We have $\Lambda(\mu)=\Lambda'(\mu)=0$. For convenience, we set $\Lambda(\alpha) = \lambda(\alpha) = 0$ for all $\alpha \ge \mu$.

Definition 4.1. A random variable $\xi^{(\alpha)}$ with the distribution

$$\mathsf{P}\{\xi^{(\alpha)} \in du\} = e^{\lambda(\alpha)u}\,\mathsf{P}\{\xi \in du\}/\varphi(\lambda(\alpha))$$

is called *the Cramèr transform* of the random variable ξ.

Note that $\xi^{(\alpha)} = \xi$ for $\alpha \ge \mu$. By definition,

$$\begin{aligned} \mathsf{E}\,\xi^{(\alpha)} &= \varphi'(\lambda)/\varphi(\lambda)\big|_{\lambda=\lambda(\alpha)} = \begin{cases} \alpha, & \text{if} \quad \alpha < \mu, \\ \mu, & \text{if} \quad \alpha \ge \mu, \end{cases} \\ \sigma^{(\alpha)2} \equiv \operatorname{Var}\xi^{(\alpha)} &= \varphi''(\lambda)/\varphi(\lambda)\big|_{\lambda=\lambda(\alpha)} - \alpha^2 = (\ln\varphi(\lambda))''\big|_{\lambda=\lambda(\alpha)}. \end{aligned}$$

By differentiating the relation $\varphi'(\lambda)/\varphi(\lambda)\big|_{\lambda=\lambda(\alpha)} = \alpha$ with respect to $\alpha \le \mu$ and taking into account that $\Lambda'(\alpha) = \lambda(\alpha)$ we get

$$\sigma^{(\alpha)2} = 1/\Lambda''(\alpha). \tag{4.2}$$

In particular, if $\operatorname{Var}\xi < \infty$, then $\Lambda''(\mu) = 1/\operatorname{Var}\xi > 0$.

4.2. Estimate of the probability that S_n is in a compact set

Let r be a parameter from a certain parameter set. For any r, let $\{\xi_{rk}\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with an arbitrary distribution, $S_{rn} = \xi_{r1} + \cdots + \xi_{rn}$, $n \ge 1$.

For any $c > 0$, define a random variable $\xi_{r1}^{[c]}$ with the distribution

$$\mathsf{P}\{\xi_{r1}^{[c]} \in B\} \equiv \mathsf{P}\{\xi_{r1} \in B \mid \xi_{r1} \in [-c, c]\}.$$

We need the following extension of Theorem 9 in [7, Chapter III] to the parametric case.

Lemma 4.1. *If there exists a constant $c > 0$ such that*

$$\inf_r \mathsf{P}\{\xi_{r1} \in [-c, c]\} > 0, \tag{4.3}$$

$$\inf_r \operatorname{Var}\xi_{r1}^{[c]} > 0, \tag{4.4}$$

then there exists a constant c_1 such that

$$\sup_{x \in \mathbb{R}} \mathsf{P}\{S_{rn} \in [x, x+1]\} \le c_1/\sqrt{n}$$

for each $n \ge 1$.

Remark 4.1. Lemma 4.1 can be derived from uniform estimates of the concentration function (see, e.g., [5, § 2.2]). We prefer however to give a direct proof, which follows [7, Chapter III, § 2].

Proof of Lemma 4.1. Let $\psi_r^{[c]}(\lambda) = \mathsf{E}\exp(i\lambda\xi_{r1}^{[c]})$ be the characteristic function of $\xi_{r1}^{[c]}$. Since the random variables $|\xi_{r1}^{[c]}|$ are bounded by c, the following representation holds:

$$\begin{aligned} \psi_r^{[c]}(\lambda) &= \exp(i\lambda\mathsf{E}\,\xi_{r1}^{[c]})\,\mathsf{E}\exp(i\lambda(\xi_{r1}^{[c]} - \mathsf{E}\,\xi_{r1}^{[c]})) \\ &= \exp(i\lambda\mathsf{E}\,\xi_{r1}^{[c]})\Big(1 - \lambda^2\Big(\frac{\operatorname{Var}\xi_{r1}^{[c]}}{2} + \theta_r(\lambda)\Big)\Big), \end{aligned}$$

where $\theta_r(\lambda) \to 0$ as $\lambda \to 0$ uniformly in r. Taking into account (4.4), we conclude that there exists $\delta > 0$ such that for $|\lambda| \le \delta$ and for any r,

$$|\psi_r^{[c]}(\lambda)| \le 1 - \delta\lambda^2. \tag{4.5}$$

Let $\psi_r(\lambda) = \mathsf{E}\exp(i\lambda\xi_{r1})$ be the characteristic function of ξ_{r1}. Since

$$\psi_r(\lambda) = \psi_r^{[c]}(\lambda)\,\mathsf{P}\{\xi_{r1} \in [-c, c]\} + \mathsf{E}\{\exp(i\lambda\xi_{r1});\ |\xi_{r1}| > c\},$$

from (4.5) we obtain that

$$|\psi_r(\lambda)| \le (1-\delta\lambda^2)\,\mathsf{P}\{\xi_{r1} \in [-c, c]\} + \mathsf{P}\{|\xi_{r1}| > c\} \le 1 - \varepsilon\lambda^2 \tag{4.6}$$

for $|\lambda| \le \varepsilon$, where $\varepsilon = \delta \min_r \mathsf{P}\{\xi_{r1} \in [-c, c]\}$ is positive, due to condition (4.3).

Consider a random variable η with the characteristic function (see [3, Chapter XVI, § 3])

$$\psi(\lambda) = \mathsf{E}\, e^{i\lambda\eta} = \begin{cases} 1 - |\lambda|/\varepsilon, & \text{if} \quad |\lambda| \le \varepsilon; \\ 0, & \text{if} \quad |\lambda| > \varepsilon. \end{cases}$$

We assume η to be independent of S_{rn}. The characteristic function of the sum $S_{rn} + \eta$ is equal to $\psi_r^n(\lambda)\psi(\lambda)$. Since the function $|\psi(\lambda)|$ is integrable, the following inverse formula (see [3, Chapter XV, § 3]) holds for every $y < z$:

$$\mathsf{P}\{S_{rn} + \eta \in [y, z]\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda y} - e^{-i\lambda z}}{i\lambda} \psi_r^n(\lambda)\psi(\lambda) d\lambda.$$

Therefore, for $c_2 = (z - y)/2\pi$,

$$\mathsf{P}\{S_{rn} + \eta \in [y, z]\} \le c_2 \int_{-\infty}^{\infty} |\psi_r(\lambda)|^n \psi(\lambda) d\lambda \le c_2 \int_{-\varepsilon}^{\varepsilon} (1-\varepsilon\lambda^2)^n d\lambda,$$

by the definition of $\psi(\lambda)$ and (4.6). Since $1 - \varepsilon\lambda^2 \le \exp(-\varepsilon\lambda^2)$,

$$\mathsf{P}\{S_{rn} + \eta \in [y, z]\} \le c_2 \int_{-\infty}^{\infty} \exp(-n\varepsilon\lambda^2)\, d\lambda = \frac{c_2\sqrt{\pi}}{\sqrt{n\varepsilon}} \equiv \frac{c_3}{\sqrt{n}}. \tag{4.7}$$

Let u be such that $\mathsf{P}\{|\eta| \le u\} \ge 1/2$. Put $y = x-u$ and $z = x+1+u$. We get the inequalities

$$\mathsf{P}\{S_{rn}+\eta \in [y, z]\} \ge \mathsf{P}\{S_{rn} \in [x, x+1], |\eta| \le u\} \ge \mathsf{P}\{S_{rn} \in [x, x+1]\}/2.$$

Combining this with (4.7) we obtain the assertion of the lemma. □

4.3. Estimate for the probability that S_n is in a compact set via the rate function

Lemma 4.2. *Let $\alpha_1 \in (\alpha_-, \mu]$. Then there exists a constant c_1 such that the following inequality holds for each $n \ge 1$ and $\alpha \in [\alpha_1, \infty)$:*

$$\mathsf{P}\{S_n \in [n\alpha-1, n\alpha)\} \le c_1 e^{-n\Lambda(\alpha)}/\sqrt{n}. \tag{4.8}$$

Remark 4.2. Under the assumption $\mathsf{E}|\xi|^3 < \infty$, Lemma 4.2 follows from the uniform large deviation principle (see, for example, [2, Lemmas 3 and 4]).

Proof of Lemma 4.2. Consider independent copies $\xi_n^{(\alpha)}$, $n \ge 1$, of the random variable $\xi^{(\alpha)}$. Let $S_n^{(\alpha)} = \xi_1^{(\alpha)} + \cdots + \xi_n^{(\alpha)}$. The following inverse formula holds:

$$\mathsf{P}\{S_n \in du\} = \varphi^n(\lambda(\alpha)) e^{-\lambda(\alpha)u}\, \mathsf{P}\{S_n^{(\alpha)} \in du\}.$$

Hence,

$$\begin{aligned}
\mathsf{P}\{S_n \in [n\alpha-1, n\alpha)\} &= \varphi^n(\lambda(\alpha)) \int_{n\alpha-1}^{n\alpha} e^{-\lambda(\alpha)u}\, \mathsf{P}\{S_n^{(\alpha)} \in du\} \\
&\le \varphi^n(\lambda(\alpha)) e^{-\lambda(\alpha)\alpha n}\, \mathsf{P}\{S_n^{(\alpha)} \in [n\alpha-1, n\alpha)\} \\
&= e^{-n\Lambda(\alpha)}\, \mathsf{P}\{S_n^{(\alpha)} \in [n\alpha-1, n\alpha)\},
\end{aligned} \tag{4.9}$$

by definition (4.1) of the rate function $\Lambda(\alpha)$. The family of random variables $\{\xi_1^{(\alpha)},\ \alpha \ge \alpha_1\}$ satisfies the conditions of Lemma 4.1 with respect to $r = \alpha$. Hence

$$\mathsf{P}\{S_n^{(\alpha)} \in [n\alpha-1, n\alpha)\} \le c_1/\sqrt{n}, \tag{4.10}$$

for some c_1, uniformly in $n \ge 1$ and $\alpha \in [\alpha_1, \infty)$. Substituting (4.10) into (4.9), we obtain (4.8). □

5. Asymptotic behaviour of $\mathsf{E}\exp(-g(S_n))$ in the general case, via the linear approximation at another point

From our point of view, the tools of Section 3 (linear approximations at the point $n\mu$ plus LLN or even CLT) cannot be used to treat the asymptotic behaviour of $\mathsf{E}\exp(-g(S_n)) = \mathsf{E}\,\overline{F}(S_n)$ in the general case, $g(x) = o(x)$. For example, if $g(x) = x^\beta$ with $\beta \in (1/2, 1)$, then the advantage of the linear approximation of $g(S_n)$ at the point $n\mu$ is doubtful. It seems reasonable to approximate the value of $g(S_n)$ at a point which is located to the left of $n\mu$.

We assume that there exist functions $g^*(x)$ and $d(x)$ such that $g^*(x) \ge 0$ is continuous, $g^*(x) \to 0$, $d(x) \to \infty$ as $x \to \infty$ and

$$|g(y) - g(x) - g^*(x)(y-x)| \le \varepsilon_1(x), \tag{5.1}$$

uniformly in $\{y : |y-x| \le d(x)\sqrt{x}\}$, where $\varepsilon_1(x) \downarrow 0$. In particular, the distribution F is heavy-tailed. We set

$$\begin{aligned} H(x) &= H_n(x) \equiv g(x) + n\Lambda(x/n), \\ h(x) &= h_n(x) \equiv g^*(x) + \lambda(x/n); \end{aligned}$$

here the function $h(x)$ plays the role of the "derivative" for $H(x)$.

Fix $\alpha_0 \in (\alpha_-, \mu)$. Note that $\alpha_- \ge 0$, because $\mathsf{P}\{\xi \ge 0\} = 1$. Since $g^*(x) \ge 0$ and $\lambda(\mu) = 0$, $h(n\mu) \ge 0$. Since $g^*(x) \to 0$, the value of $h(n\alpha_0)$ tends to the negative limit $\lambda(\alpha_0) < 0$ as $n \to \infty$. In view of these relations and because of the continuity of g^* and λ, for n large enough, there exists at least one point $t = t_n \in (n\alpha_0, n\mu]$ such that $h(t) = 0$, i.e.

$$g^*(t) = -\lambda(t/n). \tag{5.2}$$

If there are many solutions to equation (5.2), we can take any of them. Note that there is no solution to equation (5.2) in the domain $[0, n\alpha_0]$ for all sufficiently large n. Therefore, $n\mu - t_n \le n(\mu - \alpha_0)$ for all sufficiently large n. Since we may take $\alpha_0 \in (\alpha_-, \mu)$ as close to μ as we like, $n\mu - t_n = o(n)$ as $n \to \infty$. In the next section, we prove that in "regular" cases, $n\mu - t \sim ng^*(n\mu)\sigma^2$ if $\sigma^2 = \operatorname{Var}\xi$ is finite.

Theorem 5.1. *Let* (5.1) *hold. Moreover, let for any $x > 0$ and $y \ge (\alpha_0/\mu)x$,*

$$|g(y) - g(x) - g^*(x)(y-x)| \le \varepsilon_2(x)(1 + |y-x|^2/x), \tag{5.3}$$

where $\varepsilon_2(x) \downarrow 0$ as $x \to \infty$. If $\sigma^2 = \operatorname{Var}\xi$ is finite, then

$$\mathsf{E}\,\overline{F}(S_n) \equiv \mathsf{E}\exp(-g(S_n)) = \exp(-H(t))(1 + o(1)) \quad \text{as } n \to \infty.$$

Remark 5.1. Note that (5.1) implies (5.3) in the domain $|y-x| \le d(x)\sqrt{x}$ for $\varepsilon_2(x) = \varepsilon_1(x)$.

Remark 5.2. Conditions (5.1) and (5.3) hold if the function g is twice differentiable and $xg''(x) \to 0$ as $x \to \infty$. In this case, $g^*(x) = g'(x)$. For instance, if $g(x) = x^\beta$, $\beta \in (0,1)$, then $g^*(x) = g'(x) = \beta x^{\beta-1}$,

$$0 \ge y^\beta - x^\beta - \beta x^{\beta-1}(y-x) \ge -(1-\beta)x^{\beta-1}(y-x)^2/x,$$

and one can take $d(x) = o(x^{(1-\beta)/2})$ and $\varepsilon_2(x) = (1-\beta)x^{\beta-1}$.

Proof of Theorem 5.1. We use the estimate

$$\begin{aligned} &\big|\mathsf{E}\exp(-g(S_n)) - \mathsf{E}\exp(-g(t) - g^*(t)(S_n - t))\big| \\ &\quad\le \mathsf{E}\big\{\big|\exp(-g(S_n)) - \exp(-g(t) - g^*(t)(S_n - t))\big|;\ |S_n-t| \le d(n)\sqrt{n}\big\} \\ &\qquad + \mathsf{E}\big\{\exp(-g(S_n)) + \exp(-g(t) - g^*(t)(S_n - t));\ S_n-t < -d(n)\sqrt{n}\big\} \\ &\qquad + \mathsf{E}\big\{\exp(-g(S_n)) + \exp(-g(t) - g^*(t)(S_n - t));\ S_n-t > d(n)\sqrt{n}\big\} \\ &\quad\equiv E_1 + E_2 + E_3. \end{aligned}$$

Condition (5.1) yields

$$E_1 \le (\exp(\varepsilon_1(t)) - 1)\,\mathsf{E}\{\exp(-g(S_n));\ |S_n - t| \le d(n)\sqrt{n}\} = o(\mathsf{E}\exp(-g(S_n))).$$

The rest part of the proof is based on Lemmas 5.3–5.7, which are stated and proved below. Lemmas 5.3 and 5.7 imply that

$$E_2 = o(\mathsf{E}\exp(-g(S_n)) + \exp(-H(t))).$$

It follows from Lemmas 5.4–5.6 that $E_3 = o(\mathsf{E}\exp(-g(S_n)) + \exp(-H(t)))$ as well. Therefore,

$$\mathsf{E}\exp(-g(S_n)) = (1+o(1))\,\mathsf{E}\exp(-g(t) - g^*(t)(S_n - t)) + o(\exp(-H(t))). \quad (5.4)$$

We have

$$\mathsf{E}\exp(-g^*(t)(S_n - t)) = \varphi^n(-g^*(t))\exp(g^*(t)t) = \exp(g^*(t)t + n\ln\varphi(-g^*(t)))$$

(as before, $\varphi(\lambda) = \mathsf{E}\exp(\lambda\xi)$). By the choice of the point t, $g^*(t) = -\lambda(t/n)$. Hence,

$$\mathsf{E}\exp(-g^*(t)(S_n - t)) = \exp(-\lambda(t/n)t + n\ln\varphi(\lambda(t/n))).$$

By the definition of the point $\lambda(t/n)$, the power of the exponent is equal to $-n\Lambda(t/n)$. Therefore,

$$\mathsf{E}\exp(-g(t) - g^*(t)(S_n - t)) = \exp(-g(t) - n\Lambda(t/n)) = \exp(-H(t)).$$

Substituting this into (5.4) completes the proof of Theorem 5.1. □

Lemma 5.1. *Under the conditions of Theorem* 5.1, *there exist a constant* $\delta > 0$ *and a sequence* $\varepsilon_3(n) \downarrow 0$ *such that*

$$H(x) \ge H(t) + \delta(x-t)^2/n - \varepsilon_3(n) \quad (5.5)$$

for any $x \in [n\alpha_0, n\mu + (n\mu - t)]$. *In addition, if* $|x - t| = o(\sqrt{n})$, *then*

$$H(x) = H(t) + o(1). \quad (5.6)$$

Proof. Since $t \ge n\alpha_0$, (5.3) and (5.2) imply

$$\begin{aligned} g(x) &\ge g(t) + g^*(t)(x-t) + o(1) + o((x-t)^2/t) \\ &= g(t) - \lambda(t/n)(x-t) + o(1) + o((x-t)^2/n), \end{aligned} \quad (5.7)$$

as $n \to \infty$, for any x. If $x \in [n\alpha_0, n\mu]$, then from the Taylor expansion, for some $\theta = \theta(x)$ between t/n and x/n,

$$\Lambda(x/n) = \Lambda(t/n) + \lambda(t/n)\frac{x-t}{n} + \lambda'(\theta)\frac{(x-t)^2}{2n^2}. \quad (5.8)$$

It follows from (5.7) and (5.8) that for $x \in [n\alpha_0, n\mu]$,

$$H(x) \geq H(t) + \lambda'(\theta)\frac{(x-t)^2}{2n} + o(1) + o((x-t)^2/n) \quad \text{as } n \to \infty.$$

Due to (4.2), $\lambda'(\alpha) = 1/\sigma^{(\alpha)2}$ for $\alpha \leq \mu$. Hence

$$\inf_{u \in [\alpha_0, \mu]} \lambda'(u) \equiv 9\delta > 0.$$

Therefore, $\lambda'(\theta) \geq 9\delta$ and, for $x \in [n\alpha_0, n\mu]$,

$$H(x) \geq H(t) + 4\delta(x-t)^2/n - \varepsilon_3(n), \quad \varepsilon_3(n) \downarrow 0 \text{ as } n \to \infty.$$

Thus (5.5) is proved in the domain $x \in [n\alpha_0, n\mu]$. It remains to consider the domain $x \in [n\mu, n\mu+(n\mu-t)]$. If $x \geq n\mu$, then $\Lambda(x/n) = 0$ and $H(x) = g(x) = H(n\mu) + g(x) - g(n\mu)$. Hence for $x \in [n\mu, n\mu+(n\mu-t)]$,

$$\begin{aligned} H(x) \;\geq\; H(n\mu) \;&\geq\; H(t) + 4\delta(n\mu - t)^2/n - \varepsilon_3(n) \\ &\geq\; H(t) + \delta(x-t)^2/n - \varepsilon_3(n), \end{aligned}$$

and (5.5) is proved.

Now let $|x - t| = o(\sqrt{n})$. Then it follows from condition (5.1) that

$$g(x) = g(t) + g^*(t)(x-t) + o(1) = g(t) - \lambda(t/n)(x-t) + o(1). \tag{5.9}$$

Further, $\sup_{u \in [\alpha_0, \mu]} \lambda'(u) < \infty$. From (5.8) we obtain

$$\Lambda(x/n) = \Lambda(t/n) + \lambda(t/n)(x-t)/n + o(1/n) \tag{5.10}$$

and (5.9) together with (5.10) prove (5.6). □

Following the lines of the proof of Lemma 5.1, we obtain the following lemma.

Lemma 5.2. *Under the conditions of Theorem* 5.1, *there exist a constant* $\delta > 0$ *and a sequence* $\varepsilon_3(n) \downarrow 0$ *such that*

$$\widetilde{H}(x) \geq \widetilde{H}(t) + \delta(x-t)^2/n - \varepsilon_3(n) \tag{5.11}$$

for any $x \in [n\alpha_0, n\mu+(n\mu-t)]$, *where*

$$\widetilde{H}(x) = \widetilde{H}_n(x) \equiv g(t) + g^*(t)(x-t) + n\Lambda(x/n).$$

Lemma 5.3. *Let condition* (5.1) *hold. Then*

$$\begin{aligned} \mathsf{E}\left\{\exp(-g(S_n));\, S_n < n\alpha_0\right\} &= o(\mathsf{E}\exp(-g(S_n))), \\ \mathsf{E}\left\{\exp(-g(t) - g^*(t)(S_n - t));\, S_n < n\alpha_0\right\} &= o(\mathsf{E}\exp(-g(S_n))) \end{aligned}$$

as $n \to \infty$.

Proof. The first relation is implied by Lemma 2.1, since F is heavy-tailed and $\alpha_0 < \mu$. Since $g^*(x) \to 0$, the second relation follows from the estimate

$$\begin{aligned} \mathsf{E}\left\{ \exp(-g(t) - g^*(t)(S_n - t));\ S_n < n\alpha_0 \right\} &\le \exp(g^*(t)t)\, \mathsf{P}\{S_n < n\alpha_0\} \\ &= \exp(o(n))\, \mathsf{P}\{S_n < n\alpha_0\}, \end{aligned}$$

and from the proof of Lemma 2.1. □

Lemma 5.4. *Let* $\sigma^2 \equiv \mathrm{Var}\, \xi$ *be finite. Then*

$$\mathsf{E}\left\{ \exp(-g(S_n));\ S_n > n\mu + d(n)\sqrt{n}/2 \right\} = o(\mathsf{E}\exp(-g(S_n))) \qquad \text{as } n \to \infty.$$

Proof. Due to the CLT

$$\mathsf{E}\exp(-g(S_n)) \ge \exp(-g(n\mu))\, \mathsf{P}\{S_n \le n\mu\} \sim \exp(-g(n\mu))/2 \qquad \text{as } n \to \infty.$$

Using this estimate and the Chebyshev inequality, we complete the proof:

$$\begin{aligned} &\mathsf{E}\{\exp(-g(S_n)); S_n \ge n\mu + d(n)\sqrt{n}/2\} \\ &\quad \le \exp(-g(n\mu))\, \mathsf{P}\{S_n \ge n\mu + d(n)\sqrt{n}/2\} \\ &\quad \le 4\sigma^2 \exp(-g(n\mu))/d^2(n) = o(\mathsf{E}\exp(-g(S_n))). \end{aligned}$$

□

Lemma 5.5. *Under the conditions of Theorem* 5.1

$$\mathsf{E}\left\{ \exp(-g(t) - g^*(t)(S_n - t));\ S_n > n\mu + d(n)\sqrt{n}/2 \right\} = o(\exp(-H(t)))$$

as $n \to \infty$.

Proof. By the Chebyshev inequality, we have

$$\begin{aligned} &\mathsf{E}\left\{ \exp(-g(t) - g^*(t)(S_n - t));\ S_n \ge n\mu + d(n)\sqrt{n}/2 \right\} \\ &\quad \le \exp(-g(t) - g^*(t)(n\mu - t))\, \mathsf{P}\{S_n \ge n\mu + d(n)\sqrt{n}/2\} \\ &\quad \le 4\sigma^2 \exp(-g(t) - g^*(t)(n\mu - t))/d^2(n) \\ &\quad = o(\exp(-g(t) - g^*(t)(n\mu - t))). \end{aligned}$$

In view of Lemma 5.2,

$$g(t) + g^*(t)(n\mu - t) = \widetilde{H}(n\mu) \ge \widetilde{H}(t) + o(1) = H(t) + o(1).$$

This completes the proof of the lemma. □

Lemma 5.6. *Under the conditions of Theorem* 5.1*, as* $n \to \infty$*, we have*

$$\begin{aligned} E_+ &\equiv \mathsf{E}\big\{ \exp(-g(S_n));\, t{+}d(n)\sqrt{n} \le S_n < n\mu{+}d(n)\sqrt{n}/2 \big\} \\ &= o(\exp(-H(t))); \qquad (5.12) \end{aligned}$$

$$\begin{aligned} &\mathsf{E}\big\{ \exp(-g(t) - g^*(t)(S_n - t));\, t{+}d(n)\sqrt{n} \le S_n < n\mu{+}d(n)\sqrt{n}/2 \big\} \\ &\quad = o(\exp(-H(t))). \qquad (5.13) \end{aligned}$$

Proof. First,

$$E_+ \le E'_+ \equiv \mathsf{E}\big\{ \exp(-g(S_n)); t + d(n)\sqrt{n} \le S_n < n\mu + (n\mu - t) \big\}.$$

Indeed, if $d(n)\sqrt{n}/2 < n\mu - t$, then the domain of integration in E'_+ is larger than in E_+. If $d(n)\sqrt{n}/2 \ge n\mu - t$, then

$$t + d(n)\sqrt{n} = t + d(n)\sqrt{n}/2 + d(n)\sqrt{n}/2 \ge n\mu + d(n)\sqrt{n}/2$$

and the domain of integration in E_+ is empty, as well as in E'_+. In this case $E_+ = E'_+ = 0$.

Put $t_1 = t + d(n)\sqrt{n}$ and $t_2 = n\mu + (n\mu - t)$. We have

$$\begin{aligned} E'_+ &= \sum_{k=t_1}^{t_2} \mathsf{E}\{\exp(-g(S_n)); k{-}1 \le S_n < k\} \\ &\le \sum_{k=t_1}^{t_2} \exp(-g(k-1))\, \mathsf{P}\{S_n \in [k{-}1, k)\} \\ &\le c_4 \sum_{k=t_1}^{t_2} \exp(-g(k))\, \mathsf{P}\{S_n \in [k{-}1, k)\}, \end{aligned}$$

since $\exp(-g(k-1)) \sim \exp(-g(k))$. Applying Lemma 4.2 we obtain

$$E'_+ \le \frac{c_5}{\sqrt{n}} \sum_{k=t_1}^{t_2} e^{-g(k)} e^{-n\Lambda(k/n)} = \frac{c_5}{\sqrt{n}} \sum_{k=t_1}^{t_2} e^{-H(k)}. \qquad (5.14)$$

According to the first assertion of Lemma 5.1 we have

$$\begin{aligned} \sum_{k=t_1}^{t_2} e^{-H(k)} &= e^{-H(t)} \sum_{k=t_1}^{t_2} e^{-(H(k)-H(t))} \\ &\le e^{-H(t)} \sum_{k=t_1}^{t_2} \exp(-\delta(k-t)^2/n + \varepsilon_3(n)) \\ &\sim e^{-H(t)} \sum_{k=d(n)\sqrt{n}}^{t_2-t} \exp(-\delta k^2/n) \end{aligned}$$

$$\leq \; e^{-H(t)} \int_{d(n)\sqrt{n}}^{\infty} \exp(-\delta x^2/n)\, dx$$

$$= \; e^{-H(t)}\sqrt{n} \int_{d(n)}^{\infty} \exp(-\delta y^2)\, dy \; = \; o(e^{-H(t)}\sqrt{n}),$$

since $d(n) \to \infty$. Substituting this into (5.14) we get (5.12); (5.13) can be obtained in the same way, by using Lemma 5.2. □

Lemma 5.7. *Under the conditions of Theorem* 5.1, *as* $n \to \infty$, *we have*

$$E_- \;\equiv\; \mathsf{E}\big\{\exp(-g(S_n));\, n\alpha_0 \leq S_n < t - d(n)\sqrt{n}\big\} \;=\; o(e^{-H(t)});$$

$$\mathsf{E}\big\{\exp(-g(t) - g^*(t)(S_n - t));\, n\alpha_0 \leq S_n < t - d(n)\sqrt{n}\big\} \;=\; o(e^{-H(t)}).$$

Proof. It follows the lines of the proof of Lemma 5.6. For example, let us show that $E_- = o(e^{-H(t)})$. Put $t_3 = n\alpha_0$ and $t_4 = t - d(n)\sqrt{n}$. Then

$$E_- \leq c_4 \sum_{k=t_3}^{t_4} e^{-g(k)}\, \mathsf{P}\{S_n \in [k-1, k)\}.$$

Applying Lemma 4.2 and the first assertion of Lemma 5.1, we obtain

$$E_- \;\leq\; \frac{c_5}{\sqrt{n}} \sum_{k=t_3}^{t_4} e^{-H(k)} \;\leq\; \frac{c_6 e^{-H(t)}}{\sqrt{n}} \sum_{k=t_3}^{t_4} \exp(-\delta(k-t)^2/n)$$

$$\leq\; \frac{c_6 e^{-H(t)}}{\sqrt{n}} \int_{-\infty}^{-d(n)\sqrt{n}} \exp(-\delta x^2/n)\, dx \;=\; o(e^{-H(t)}),$$

since $d(n) \to \infty$. The lemma is proved. □

6. Asymptotic behaviour of the point $t = t_n$ and the value of $e^{-H(t)}$, in particular cases

Let $\widehat{t} = \widehat{t}_n \in (n\alpha_0, n\mu]$ be any point satisfying the relation

$$g^*(\widehat{t}\,) = -\lambda(\widehat{t}/n) + o(1/\sqrt{n}) \quad \text{as } n \to \infty. \tag{6.1}$$

In particular, one can take $\widehat{t}_n = t_n$, where t_n is from (5.2).

Lemma 6.1. *Under the conditions of Theorem* 5.1, $\widehat{t}_n = t_n + o(\sqrt{n})$ *and* $\mathsf{E}\exp(-g(S_n)) \sim \exp(-H(\widehat{t}\,))$ *as* $n \to \infty$.

Proof. It follows from (6.1) and (5.2) that

$$\begin{aligned} g^*(\widehat{t}\,) - g^*(t) &= \lambda(t/n) - \lambda(\widehat{t}/n) + o(1/\sqrt{n}) \;=\; \lambda'(\theta)(t-\widehat{t}\,)/n + o(1/\sqrt{n}) \\ &= (t-\widehat{t}\,)/n\sigma^2 + o(1/\sqrt{n} + |t-\widehat{t}|/n) \qquad (6.2) \end{aligned}$$

as $n \to \infty$, since $\lambda'(\theta) \to 1/\sigma^2$. Condition (5.3) implies

$$\begin{aligned} g(t) &= g(\widehat{t}\,) + g^*(\widehat{t}\,)(t-\widehat{t}\,) + o(1 + (t-\widehat{t}\,)^2/n); \\ g(\widehat{t}\,) &= g(t) + g^*(t)(\widehat{t}-t) + o(1 + (t-\widehat{t}\,)^2/n). \end{aligned}$$

Thus,

$$(g^*(\widehat{t}\,) - g^*(t))(t-\widehat{t}\,) = o(1 + (t-\widehat{t}\,)^2/n). \qquad (6.3)$$

Combining this with (6.2) we obtain

$$(t-\widehat{t}\,)^2/n = o(|t-\widehat{t}|/\sqrt{n} + 1) = o\big((|t-\widehat{t}|/\sqrt{n} + 1)^2\big) \qquad \text{as } n \to \infty.$$

Hence, $(t-\widehat{t}\,)/\sqrt{n} = o(|t-\widehat{t}|/\sqrt{n} + 1)$, which implies the first assertion of the lemma. Now the second assertion follows from (5.6) and Theorem 5.1. □

Lemma 6.2. *Let $u(x)$ be a function satisfying the condition*:

$$u(x+y) = u(x) + o(1 + |y|/\sqrt{x}) \qquad (6.4)$$

as $x \to \infty$ uniformly in $y \ge -x$. Then, under the conditions of Theorem 5.1,

$$\mathsf{E}\exp(-u(S_n) - g(S_n)) \sim \exp(-u(t) - H(t)) \quad \text{as } n \to \infty.$$

Remark 6.1. Typical examples of functions u satisfying (6.4) are x^β for $\beta < 1/2$ and $(\log x)^\beta$ for any $\beta \in \mathbf{R}$.

Proof of Lemma 6.2. Taking into account that $|y/\sqrt{x}| \le 1 + y^2/x$, we obtain the relation

$$u(x+y) = u(x) + o(1 + y^2/x).$$

Note that the function $g_1(x) \equiv u(x) + g(x)$ satisfies conditions (5.1) and (5.3) with $g_1^* = g^*$. In this case, $H_1 = u + H$ and $h_1 = h$. Hence t satisfies the equation $g_1^*(t) = -\lambda(t/n)$ and, due to Theorem 5.1, $\mathsf{E}\exp(-g_1(S_n)) \sim \exp(-H_1(t))$. The proof is complete. □

Lemma 6.3. *Let $\sigma^2 \equiv \operatorname{Var}\xi$ be finite. Then any solution t_n to equation* (5.2) *satisfies*

$$n\mu - t_n = (1+o(1))ng^*(t_n)\sigma^2 \quad \text{as } n \to \infty. \qquad (6.5)$$

If

$$g^*(x - o(x)) = g^*(x) + o\left(g^*(x) + 1/\sqrt{x}\right) \quad \text{as } x \to \infty, \qquad (6.6)$$

then

$$n\mu - t_n = (1+o(1))ng^*(n\mu)\sigma^2 + o(\sqrt{n}) \quad \text{as } n \to \infty. \tag{6.7}$$

Moreover,

$$n\mu - t_n = (1+o(1))ng^*(n\mu)\sigma^2 \quad \text{as } n \to \infty, \tag{6.8}$$

provided that

$$g^*(x-o(x)) = (1+o(1))g^*(x) \quad \text{as } x \to \infty. \tag{6.9}$$

Remark 6.2. The function x^β, $\beta \in \mathbf{R}$, satisfies condition (6.9).

Proof of Lemma 6.3. Recall that $n\mu - t_n = o(n)$ as $n \to \infty$. By the Taylor expansion, $\lambda(t/n) = \lambda(\mu) + \lambda'(\theta)(t/n - \mu)$, where $\theta \in [t/n, \mu]$. Since $\lambda(\mu) = 0$ and $\lambda'(\mu) = 1/\sigma^2$, from (5.2) we obtain that

$$g^*(t) = (1/\sigma^2 + o(1))(\mu - t/n),$$

which is equivalent to (6.5). Further, if condition (6.6) holds then $g^*(t) = (1 + o(1))g^*(n\mu) + o(1/\sqrt{n})$, which together with (6.5) implies relation (6.7). Finally, (6.5) and (6.9) imply (6.8). □

Corollary 6.1. *Assume that the conditions of Theorem* 5.1 *hold. If* $g^*(x) = o(1/\sqrt{x})$ *as* $x \to \infty$, *then* $\mathsf{E}\exp(-g(S_n)) \sim \exp(-g(n\mu))$ *as* $n \to \infty$, *which coincides with the asymptotics in Theorem* 3.1.

Proof. Since $t_n \ge n\alpha_0$, $g^*(t_n) = o(1/\sqrt{n})$. By this relation, it follows from (6.5) that $n\mu - t = o(\sqrt{n})$. Applying (5.6), we deduce $H(t) = H(n\mu) + o(1)$. Now the corollary follows from Theorem 5.1. □

Theorem 6.1. *Assume that the conditions of Theorem* 5.1 *and* (6.6) *hold. Then*

$$H(t_n) = g(n\mu) - (1+o(1))n(g^*(n\mu)\sigma)^2/2 + o(1) \quad \text{as } n \to \infty.$$

Proof. By Lemma 6.3 we have

$$n\mu - t_n = (1+o(1))ng^*(n\mu)\sigma^2 + o(\sqrt{n}).$$

Put $\Delta \equiv (1+o(1))ng^*(n\mu)\sigma^2$. By virtue of (5.6), $H(t_n) = H(n\mu - \Delta) + o(1)$ as $n \to \infty$. From (5.3) we have

$$\begin{aligned} g(n\mu - \Delta) &= g(n\mu) - g^*(n\mu)\Delta + o(1 + \Delta^2/n) \\ &= g(n\mu) - (1+o(1))n(g^*(n\mu)\sigma)^2 + o(1). \end{aligned}$$

Since $\Lambda(\mu) = 0$, $\Lambda'(\mu) = 0$ and $\Lambda''(\mu) = 1/\sigma^2$, we get

$$n\Lambda((n\mu - \Delta)/n) = n(1/\sigma^2 + o(1))\Delta^2/2n^2 \sim n(g^*(n\mu)\sigma)^2/2.$$

Now the assertion of the theorem follows from the last two relations. □

Corollary 6.2. *Assume that the conditions of Theorem* 5.1 *and* (6.6) *hold. If* $g^*(x) = O(1/\sqrt{x})$ *as* $x \to \infty$, *then*

$$\mathsf{E}\exp(-g(S_n)) \sim \exp(-g(n\mu) + n(g^*(n\mu)\sigma)^2/2)$$

as $n \to \infty$.

In particular, the asymptotics $\mathsf{E}\exp(-g(S_n)) \sim \exp(-g(n\mu) + (c\sigma)^2/8\mu)$ holds if $\overline{F}(x) = x^\gamma \exp(-cx^\beta)$ and $\beta = 1/2$. If $\beta > 1/2$, then it is necessary to refine the asymptotics given in Lemma 6.3 and Theorem 6.1.

Lemma 6.4. *Let the conditions of Theorem* 5.1 *and* (6.6) *hold. Let*

$$g^*(x + O(xg^*(x))) = g^*(x) + O((g^*(x))^2) + o(1/\sqrt{x}) \quad \text{as } x \to \infty. \tag{6.10}$$

If $\mathsf{E}\,\xi^3$ *is finite, then*

$$n\mu - t_n = g^*(n\mu)n\sigma^2 + O(n(g^*(n\mu))^2) + o(\sqrt{n}) \quad \text{as } n \to \infty.$$

Remark 6.3. The function x^β, $\beta \in \mathbf{R}$, satisfies condition (6.10).

Remark 6.4. Since $\mathsf{E}\,\xi^3 < \infty$, $\lambda''(\mu) = -[\varphi'''(0) - 3\varphi''(0)\varphi'(0) + 2(\varphi'(0))^3]/\sigma^6 = -\mathsf{E}(\xi - \mu)^3/\sigma^6$ is finite.

Proof of Lemma 6.4. It follows from condition (6.10) that

$$g^*(t_n) = g^*(n\mu) + O((t_n - n\mu)^2/n^2) + o(1/\sqrt{n}).$$

Since $\lambda(\mu) = 0$, $\lambda'(\mu) = 1/\sigma^2$ and $\lambda''(\mu)$ is finite,

$$\lambda(t_n/n) = (t_n - n\mu)/n\sigma^2 + O((t_n - n\mu)^2/n^2).$$

Therefore, from (5.2) we obtain that

$$g^*(n\mu) + O((n\mu - t)^2/n^2) + o(1/\sqrt{n}) = (n\mu - t)/n\sigma^2.$$

Since $n\mu - t \sim ng^*(n\mu)\sigma^2$, the proof is complete. □

Corollary 6.3. *Under the conditions of Lemma* 6.4, *assume that* $g^*(x) = o(1/x^{1/4})$. *Then* $t_n = n\mu - g^*(n\mu)n\sigma^2 + o(\sqrt{n})$.

Corollary 6.4. *Under the conditions of Lemma* 6.4, *assume that* $g^*(x) = o(1/x^{1/3})$ *and*

$$g(x + y) = g(x) + g^*(x)y + o(1) + O(y^2 g^*(x)/x) \quad \text{as } x \to \infty.$$

Then

$$\mathsf{E}\exp(-g(S_n)) \sim \exp(-g(n\mu) + n(g^*(n\mu)\sigma)^2/2) \qquad \text{as } n \to \infty.$$

Proof of Corollary 6.4. Since $g^*(x) = o(1/x^{1/3})$, Corollary 6.3 and (5.6) imply $H(t_n) = H(n\mu - g^*(n\mu)n\sigma^2) + o(1)$ as $n \to \infty$. By Theorem 5.1, we obtain

$$\begin{aligned}
&\mathsf{E}\exp(-g(S_n)) \\
&\quad\sim \exp(-H(n\mu - g^*(n\mu)n\sigma^2)) \\
&\quad= \exp(-g(n\mu - ng^*(n\mu)\sigma^2) - n\Lambda(\mu - g^*(n\mu)\sigma^2)) \\
&\quad= \exp\big\{-g(n\mu) + n(g^*(n\mu)\sigma)^2 + o(1) + O(n(g^*(n\mu))^3) \\
&\qquad - n\lambda'(\mu)(g^*(n\mu)\sigma^2)^2/2 + O(n(g^*(n\mu))^3)\big\} \\
&\quad= \exp\big\{-g(n\mu) + n(g^*(n\mu)\sigma)^2 - n\lambda'(\mu)(g^*(n\mu)\sigma^2)^2/2 + o(1)\big\}.
\end{aligned}$$

Using the relation $\lambda'(\mu) = 1/\sigma^2$ we arrive at the conclusion. □

In particular, if $\overline{F}(x) = \exp(-x^\beta)$ and $\beta < 2/3$, then

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp -g(n\mu) + \beta^2\sigma^2(n\mu)^{2\beta-1}/2\mu.$$

If F is the exponential distribution with parameter $1/\mu$, then $\sigma^2 = \mu^2$; in this case we obtain the equivalence

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-(n\mu)^\beta + \beta^2(n\mu)^{2\beta-1}\mu/2). \tag{6.11}$$

For any $\beta \in (1/2, 2/3)$, the last inequality contradicts Remark 4.4 from [1], which states that

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-(n\mu)^\beta + (1-\beta)\beta^2(n\mu)^{2\beta-1}\mu). \tag{6.12}$$

Indeed, $(1-\beta)\beta^2 < \beta^2/2$ and (6.12) cannot be true.

7. Sequential approximations for t and $H(t)$

In the previous section, we found a number of first terms in the expansion of $H(t)$. Here we propose a sequential procedure which allows us to approximate t and, therefore, $H(t)$. We use the notation

$$L(\lambda) = (\ln\varphi(\lambda))' = \varphi'(\lambda)/\varphi(\lambda) = \mathsf{E}\,\xi e^{\lambda\xi}/\,\mathsf{E}\,e^{\lambda\xi}.$$

By the definition of $\lambda(\alpha)$, $\alpha = \varphi'(\lambda)/\varphi(\lambda)$ for $\lambda = \lambda(\alpha)$. Therefore, $L(\lambda(\alpha)) = \alpha$ and L is the inverse function to λ. Therefore, t is a root of equation (5.2) if and only if $t = nL(-g^*(t))$. If $0 < \lambda'(\mu) = 1/\sigma^2 < \infty$, then $\widehat{t} \in (n\alpha_0, n\mu]$ satisfies (6.1) if and only if

$$\widehat{t} = nL(-g^*(\widehat{t}\,)) + o(\sqrt{n}) \quad \text{as } n \to \infty. \tag{7.1}$$

Recall that $H(x) = g(x) + x\lambda(x/n) - n\ln\varphi(\lambda(x/n))$. Set

$$D(x) = g(x) - xg^*(x) - n\ln\varphi(-g^*(x)).$$

By (5.2), $H(t) = D(t)$. Since $\lambda(L(-g^*(x))) = -g^*(x)$,

$$\begin{aligned} D(x) - H\big(nL(-g^*(x))\big) &= g(x) - g\big(nL(-g^*(x))\big) + \big(nL(-g^*(x)) - x\big)g^*(x) \\ &= o\big(1 + (nL(-g^*(x)) - x)^2/x\big) \end{aligned}$$

owing to condition (5.1). Setting $x = \widehat{t}$, by Lemma 6.1 and (5.6) we get

$$\begin{aligned} D(\widehat{t}\,) &= H(nL(-g^*(\widehat{t}\,))) + o(1) = H(\widehat{t} + o(\sqrt{n})) + o(1) \\ &= H(t + o(\sqrt{n})) + o(1) = H(t) + o(1). \end{aligned}$$

Thus we obtained the following theorem.

Theorem 7.1. *Let $\widehat{t}$ satisfy* (7.1). *Then, under the conditions of Theorem* 5.1, $H(t) = D(\widehat{t}\,) + o(1)$ *as* $n \to \infty$.

Put $t^{(1)} = n\mu$ and $t^{(k+1)} = nL(-g^*(t^{(k)}))$ for any integer k. Let $L^* = \sup_{\lambda \le 0} L'(\lambda)$. We have

$$|t^{(2)} - t^{(1)}| = n|L(-g^*(n\mu)) - L(0)| \le L^* n g^*(n\mu), \tag{7.2}$$

$$\begin{aligned} |t^{(k+1)} - t^{(k)}| &= n|L(-g^*(t^{(k)})) - L(-g^*(t^{(k-1)}))| \\ &\le L^* n|g^*(t^{(k)}) - g^*(t^{(k-1)})|. \end{aligned} \tag{7.3}$$

Lemma 7.1. *For any fixed k, $t^{(k)} \sim n\mu$ as $n \to \infty$. In addition, if there exists $\delta > 0$ such that*

$$g^*(x) = o(x^{-\delta}), \tag{7.4}$$

$$g^*(x+v) - g^*(x) = o(vx^{-1-\delta}) \tag{7.5}$$

as $x \to \infty$ uniformly in $v > 0$, then, for any fixed $k \ge 1/2\delta$,

$$H(t) = H(t^{(k)}) + o(1) = D(t^{(k)}) + o(1) \qquad \text{as } n \to \infty.$$

Proof. Since $g^*(n\mu) \to 0$, it follows from (7.2) that $t^{(2)} \sim n\mu$ as $n \to \infty$. Then, by (7.3), $t^{(k)} \sim n\mu$ as $n \to \infty$, for any fixed k.

Due to (7.3) and (7.5), for all sufficiently large n,

$$|t^{(k+1)} - t^{(k)}| = o(|t^{(k)} - t^{(k-1)}|n^{-\delta}).$$

By induction, (7.2) and (7.4) we get

$$t^{(k+1)} - t^{(k)} = o(|t^{(2)} - t^{(1)}|n^{-(k-1)\delta}) = o(nn^{-k\delta}) = o(\sqrt{n}).$$

Thus, $nL(-g^*(t^{(k)})) - t^{(k)} = t^{(k+1)} - t^{(k)} = o(\sqrt{n})$ and one can take $\widehat{t} = t^{(k)}$ and apply Theorem 7.1. □

For example, assume that $\delta = 1/4$. Then we take $k = 2$ and use the expansion $L(\lambda) = L(0) + \lambda L'(0) + o(\lambda) = \mu + \lambda\sigma^2 + o(\lambda)$. We get

$$t^{(2)} = nL(-g^*(n\mu)) = n(\mu - \sigma^2 g^*(n\mu)) + o(\sqrt{n}).$$

Thus, $H(t) = D(\widehat{t}\,) + o(1)$, where $\widehat{t} = n\mu - ng^*(n\mu)\sigma^2$. Now one can check that the expansion of $D(\widehat{t}\,)$ at the point $n\mu$ yields the asymptotics of Corollary 6.4 if $\delta = 1/3$.

8. Asymptotic behaviour of $\mathsf{E}\exp(-g(S_n))$ in the case of a Poisson renewal process

In this section we apply the general results of the previous sections to the special case of a Poisson process $X(t)$ with intensity $1/\mu$. Then ξ has the exponential distribution with mean μ, variance $\sigma^2 = \mu^2$ and the Laplace transform

$$\varphi(\lambda) = 1/(1 - \lambda\mu), \quad \lambda < 1/\mu.$$

We have $\alpha_- = 0$. The solution to the equation $d\ln\varphi(\lambda)/d\lambda = \alpha$ is equal to $\lambda(\alpha) = 1/\mu - 1/\alpha$ for $\alpha \in (0, \mu]$.

Let F be a Weibull distribution with parameter $\beta \in (0, 1)$. In this case, equation (5.2) has the form $\beta t^{\beta-1} = -\lambda(t/n)$ or, equivalently,

$$\beta t^\beta + t/\mu = n, \tag{8.1}$$

which is identical to equation (4.2) from [1]. Since

$$\Lambda(\alpha) = \int_\mu^\alpha \lambda(\theta)d\theta = \frac{\alpha - \mu}{\mu} + \ln\frac{\mu}{\alpha},$$

Theorem 5.1 gives the asymptotics

$$\begin{aligned} \mathsf{E}\exp(-S_n^\beta) \quad &\sim \quad \exp(-t^\beta - n\Lambda(t/n)) \quad = \quad \exp(-t^\beta + n(1 - t/n\mu))(t/n\mu)^n \\ &= \quad \exp(-(1-\beta)t^\beta + n\ln(1 - \beta t^\beta/n)) \qquad \text{as } n \to \infty, \end{aligned} \tag{8.2}$$

because $t/n\mu = 1 - \beta t^\beta/n$. Since

$$\ln(1 - \beta t^\beta/n) = -\beta t^\beta/n - (\beta^2\mu^{2\beta}/2 + o(1))n^{2\beta-2},$$

we may write the following estimate for $\mathsf{E}\exp(-S_n^\beta)$:

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-(1-\beta)t^\beta - \beta t^\beta - (\beta^2\mu/2 + o(1))(n\mu)^{2\beta-1}). \tag{8.3}$$

By Lemma 6.3, $t = n\mu - (\mu\beta + o(1))(n\mu)^\beta$ as $n \to \infty$. Hence, $t^\beta = (n\mu)^\beta - (\beta^2\mu + o(1))(n\mu)^{2\beta-1}$ and it follows from (8.3) that

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-(1-\beta)t^\beta - \beta(n\mu)^\beta + (\beta^2\mu(\beta - 1/2) + o(1))(n\mu)^{2\beta-1}). \tag{8.4}$$

Let us compare this with the asymptotics given in [1, Theorem 4.1]:

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-(1-\beta)t^\beta - \beta(n\mu)^\beta). \tag{8.5}$$

For $\beta > 1/2$ it turns out that the latter two asymptotics are different; (8.4) is heavier than (8.5).

Finally, we give the following correct version of Theorem 4.1 from [1].

Theorem 8.1. *Let* $\overline{F}(x) = \exp(-u(x) - x^\beta)$, *where* $\beta \in (0,1)$ *and* $u(x)$ *satisfies condition* (6.4). *Then*

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-u(t) - (1-\beta)t^\beta + n\ln(1-\beta t^\beta/n)) \qquad \text{as } n\to\infty,$$

where t *is the solution to equation* (8.1). *In addition, if*

$$u(x + O(x^\beta)) = u(x) + o(1) \quad \text{as } x\to\infty, \tag{8.6}$$

then

$$\mathsf{E}\exp(-S_n^\beta) \sim \exp(-u(n\mu) - (1-\beta)t^\beta + n\ln(1-\beta t^\beta/n)) \quad \text{as } n\to\infty.$$

Proof. The first assertion follows from (8.2) and Lemma 6.2.

For $g(x) = x^\beta$, we may take $g^*(x) = g'(x) = \beta x^{\beta-1}$. By Lemma 6.3, $n\mu - t = O(ng'(n\mu)) = O(g(n))$. This relation and condition (8.6) imply $u(t) = u(n\mu) + o(1)$ and, therefore, the second assertion of Theorem 8.1. □

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