On lower limits and equivalences for distribution tails of randomly stopped sums [1]

D. Denisov,[2] S. Foss,[3] and D. Korshunov[4]

Eurandom, Heriot-Watt University and Sobolev Institute of Mathematics

Abstract

For a distribution $F^{*\tau}$ of a random sum $S_\tau = \xi_1 + \ldots + \xi_\tau$ of i.i.d. random variables with a common distribution F on the half-line $[0, \infty)$, we study the limits of the ratios of tails $\overline{F^{*\tau}}(x)/\overline{F}(x)$ as $x \to \infty$ (here τ is an independent counting random variable). We also consider applications of obtained results to random walks, compound Poisson distributions, infinitely divisible laws, and sub-critical branching processes.

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1. Introduction. Let ξ_1, ξ_2, $\ldots$, be independent identically distributed nonnegative random variables. We assume that their common distribution F on the half-line $[0, \infty)$ has an unbounded support, that is, $\overline{F}(x) \equiv F(x, \infty) > 0$ for all x. Put $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2, \ldots$.

Let τ be a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$ and has finite mean. Denote by $F^{*\tau}$ the distribution of a randomly stopped sum $S_\tau = \xi_1 + \ldots + \xi_\tau$.

In this paper we discuss how does the tail behaviour of $F^{*\tau}$ relate to that of F and, in particular, under what conditions

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad = \quad \mathbf{E}\tau. \tag{1}$$

Relations on lower limits and on limits of ratios of from (1) have been first discussed by Rudin [21]. Theorem 2^* of that paper states (for an integer p) the following

Theorem 1. *Let there exists a positive $p \in [1, \infty)$ such that $\mathbf{E}\xi^p = \infty$, but $\mathbf{E}\tau^p < \infty$. Then* (1) *holds.*

Rudin's studies were motivated by the paper [7] of Chover, Ney, and Wainger who considered, in particular, the problem of existence of a limit for the ratio

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \qquad \text{as } x \to \infty. \tag{2}$$

From Theorem 1, it follows that, if F and τ satisfy its conditions and if a limit of (2) exists, then that limit must be equal $\mathbf{E}\tau$.

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[2]Address: Eurandom, P.O. Box 513 - 5600 MB Eindhoven, The Netherlands. E-mail address: Denisov@eurandom.tue.nl

[3]Address: School of MACS, Heriot-Watt University, Edinburgh EH14 4AS, UK; and Sobolev Institute of Mathematics, 4 Koptyuga Pr., Novosibirsk 630090, Russia. E-mail address: S.Foss@ma.hw.ac.uk

[4]Address: Sobolev Institute of Mathematics, 4 Koptyuga pr., Novosibirsk 630090, Russia. E-mail address: Korshunov@math.nsc.ru

Rudin proved Theorem 1 via probability generating functions techniques. Below we give an alternative and a more direct proof of Theorem 1 in the case of any positive p (i.e. not necessarily integer). Our method is based on truncation arguments; in this way, we propose a general scheme (see Theorem 4 below) which may be applied also to distributions with all finite moments.

The condition $\mathbf{E}\xi^p = \infty$ rules out a lot of distributions of interest, say, in the theory of subexponential distributions. For example, log-normal and Weibull-type distributions have all moments finite. Our first result presents a natural moment condition on stopping time τ guaranteeing relation (1) for the whole class of heavy-tailed distributions. It is intuitively clear that, for that, τ should be light-tailed.

Recall that a random variable ξ has *a light-tailed* distribution F on $[0,\infty)$ if $\mathbf{E}e^{\gamma\xi} < \infty$ with some $\gamma > 0$. Otherwise F is called *a heavy-tailed* distribution; this happens if and only if $\mathbf{E}e^{\gamma\xi} = \infty$ for all $\gamma > 0$.

Theorem 2. *Let F be a heavy-tailed distribution and τ have a light-tailed distribution. Then* (1) *holds.*

Proof of Theorem 2 is based on a new technical tool (see Lemma 2) and significantly differs from a proof of Theorem 1 in [15] where a particular case $\tau = 2$ was considered. Theorem 2 is restricted to the case of light-tailed τ, but here extends Rudin's result to the class of all heavy-tailed distributions. The reasons for the restriction to $\mathbf{E}e^{\gamma\tau} < \infty$ come from the proof of Theorem 2 but in fact are rather natural; the tail of τ should be lighter than the tail of any heavy-tailed distribution. Indeed, if $\xi_1 \geq 1$ then $\overline{F^{*\tau}}(x) \geq \mathbf{P}\{\tau > x\}$. This shows that the tail of $F^{*\tau}$ is at least as heavy as that of τ. Note that, in Theorem 1, in some sense, the tail of $F^{*\tau}$ is heavier than the tail of τ.

Theorem 2 may be applied in various areas where randomly stopped sums do appear – see Sections 8–11 (random walks, compound Poisson distributions, infinitely divisible laws, and branching processes) and, e.g., [17] for further examples.

For any distribution on $[0,\infty)$, let

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(dx) \in (0,\infty], \quad \gamma \in \mathbf{R},$$

and

$$\widehat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0,\infty].$$

Note that the moment-generating function $\varphi(\gamma)$ is monotone continuous in the interval $(-\infty,\widehat{\gamma})$, and $\varphi(\widehat{\gamma}) = \lim_{\gamma\uparrow\widehat{\gamma}} \varphi(\gamma) \in [1,\infty]$.

Theorem 3. *Let $\varphi(\widehat{\gamma}) < \infty$ and $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^\tau < \infty$ for some $\varepsilon > 0$. Assume that*

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad \to \quad c \quad \text{as } x \to \infty,$$

where $c \in (0,\infty]$. Then $c = \mathbf{E}(\tau\varphi^{\tau-1}(\widehat{\gamma}))$.

For (comments on) earlier partial results in the case $\tau = 2$, see, e.g., papers [6–8, 10, 15, 19, 20, 22] and further references therein.

2. Preliminary result. We start with the following

Theorem 4. *Let there exist a non-decreasing concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\mathbf{E}e^{h(\xi)} < \infty \quad \text{and} \quad \mathbf{E}\xi e^{h(\xi)} = \infty. \tag{3}$$

For any $n \geq 1$, *put* $A_n = \mathbf{E}e^{h(\xi_1+\ldots+\xi_n)}$. *If* F *is heavy-tailed and if*

$$\mathbf{E}\tau A_{\tau-1} \quad < \quad \infty, \tag{4}$$

then (1) *holds.*

Proof. First we restate Theorem 1* of Rudin [21] in Lemma 1 below in terms of probability distributions and stopping times.

Lemma 1. *For any distribution* F *on* $[0,\infty)$ *with unbounded support and any independent counting random variable* τ,

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad \geq \quad \mathbf{E}\tau.$$

It follows from Lemma 1 that it is sufficient to prove the upper bound in Theorem 4. Assume the contrary, i.e. there exist $\delta > 0$ and x_0 such that

$$\overline{F^{*\tau}}(x) \quad \geq \quad (\mathbf{E}\tau+\delta)\overline{F}(x) \qquad \text{for all } x > x_0. \tag{5}$$

For any positive $b > 0$, consider a concave function

$$h_b(x) \quad \equiv \quad \min\{h(x), bx\}. \tag{6}$$

Since F is heavy-tailed, $h(x) = o(x)$ as $x \to \infty$. Therefore, for any fixed b, there exists x_0 such that $h_b(x) = h(x)$ for all $x > x_0$. Hence, by the condition (3),

$$\mathbf{E}e^{h_b(\xi)} < \infty \quad \text{ and } \quad \mathbf{E}\xi e^{h_b(\xi)} = \infty. \tag{7}$$

For any x, we have the convergence $h_b(x) \downarrow 0$ as $b \downarrow 0$. Then, for any fixed n,

$$A_{n,b} \equiv \mathbf{E}e^{h_b(\xi_1+\ldots+\xi_n)} \quad \downarrow \quad 1 \qquad \text{as } b \downarrow 0.$$

This and the condition (4) imply that there exists b such that

$$\mathbf{E}\tau A_{\tau-1,b} \quad \leq \quad \mathbf{E}\tau + \delta/2. \tag{8}$$

For any random variable ζ and positive t, put $\zeta^{[t]} = \min\{\zeta, t\}$. Then

$$\begin{aligned}
\frac{\mathbf{E}(\xi_1^{[t]}+\ldots+\xi_\tau^{[t]})e^{h_b(\xi_1+\ldots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \quad &= \quad \sum_{n=1}^{\infty} \frac{\mathbf{E}(\xi_1^{[t]}+\ldots+\xi_n^{[t]})e^{h_b(\xi_1+\ldots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau=n\} \\
&= \quad \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1+\ldots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau=n\} \\
&\leq \quad \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)+h_b(\xi_2+\ldots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau=n\},
\end{aligned}$$

by concavity of the function h_b. Hence,

$$\begin{aligned}
\frac{\mathbf{E}(\xi_1^{[t]}+\ldots+\xi_\tau^{[t]})e^{h_b(\xi_1+\ldots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \quad &\leq \quad \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}\mathbf{E}e^{h_b(\xi_2+\ldots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau=n\} \\
&= \quad \sum_{n=1}^{\infty} nA_{n-1,b}\mathbf{P}\{\tau=n\} \\
&\leq \quad \mathbf{E}\tau + \delta/2, \qquad (9)
\end{aligned}$$

by (8).

On the other hand, since $(\xi_1 + \ldots + \xi_\tau)^{[t]} \le \xi_1^{[t]} + \ldots + \xi_\tau^{[t]}$,

$$\begin{aligned} \frac{\mathbf{E}(\xi_1^{[t]} + \ldots + \xi_\tau^{[t]}) e^{h_b(\xi_1+\ldots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]} e^{h_b(\xi_1)}} &\ge \frac{\mathbf{E}(\xi_1 + \ldots + \xi_\tau)^{[t]} e^{h_b(\xi_1+\ldots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]} e^{h_b(\xi_1)}} \\ &= \frac{\int_0^\infty x^{[t]} e^{h_b(x)} F^{*\tau}(dx)}{\int_0^\infty x^{[t]} e^{h_b(x)} F(dx)}. \end{aligned} \tag{10}$$

The right side, after integration by parts, is equal to

$$\frac{\int_0^\infty \overline{F^{*\tau}}(x) d(x^{[t]} e^{h_b(x)})}{\int_0^\infty \overline{F}(x) d(x^{[t]} e^{h_b(x)})}.$$

Since $\mathbf{E}\xi_1 e^{h_b(\xi_1)} = \infty$, both integrals (the divident and the divisor) in the latter fraction tend to infinity as $t \to \infty$. For the *non-decreasing* function $h_b(x)$, together with the assumption (5) it implies that

$$\liminf_{t\to\infty} \frac{\int_0^\infty \overline{F^{*\tau}}(x) d(x^{[t]} e^{h_b(x)})}{\int_0^\infty \overline{F}(x) d(x^{[t]} e^{h_b(x)})} = \liminf_{t\to\infty} \frac{\int_{x_0}^\infty \overline{F^{*\tau}}(x) d(x^{[t]} e^{h_b(x)})}{\int_{x_0}^\infty \overline{F}(x) d(x^{[t]} e^{h_b(x)})} \quad \ge \quad \mathbf{E}\tau + \delta.$$

Substituting this into (10) we get a contradiction to (9) for sufficiently large t. The proof is complete.

3. Proof of Theorem 1. Let an integer k be such that $p - 1 \le k < p$. Without loss of generality, we can assume that $\mathbf{E}\xi^k < \infty$.

Consider a concave non-decreasing function $h(x) = (p-1)\ln x$. Then $\mathbf{E}e^{h(\xi_1)} < \infty$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Thus,

$$\begin{aligned} A_n \equiv \mathbf{E}e^{h(\xi_1+\ldots+\xi_n)} &= \mathbf{E}(\xi_1 + \ldots + \xi_n)^{p-1} \\ &\le (\mathbf{E}(\xi_1 + \ldots + \xi_n)^k)^{(p-1)/k}, \end{aligned}$$

since $(p-1)/k \le 1$. Further,

$$\begin{aligned} \mathbf{E}(\xi_1 + \ldots + \xi_n)^k &= \sum_{i_1,\ldots,i_k=1}^{n} \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) \\ &\le cn^k, \end{aligned}$$

where

$$c \equiv \sup_{1\le i_1,\ldots,i_k\le n} \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) \quad < \quad \infty,$$

due to $\mathbf{E}\xi^k < \infty$. Hence, $A_n \le c^{(p-1)/k} n^{p-1}$ for all n. Therefore, we get $\mathbf{E}\tau A_{\tau-1} \le c^{(p-1)/k}\mathbf{E}\tau^p < \infty$. All conditions of Theorem 4 are met and the proof is complete.

4. Characterization of heavy-tailed distributions. In the sequel we need the following existence result which generalises a lemma by Rudin [21, page 989] onto the whole class of heavy-tailed distributions. Fix any $\delta \in (0, 1]$.

Lemma 2. *If a random variable $\xi \ge 0$ has a heavy-tailed distribution, then there exists a monotone concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\mathbf{E}e^{h(\xi)} \le 1 + \delta$ and $\mathbf{E}\xi e^{h(\xi)} = \infty$.*

Proof. Without loss of generality assume that $\xi > 0$ a.s. We will construct a piecewise linear function $h(x)$. For that we introduce two sequences, $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$, and let

$$h(x) = h(x_{n-1}) + \varepsilon_n(x - x_{n-1}) \quad \text{if } x \in (x_{n-1}, x_n],\ n \geq 1.$$

This function is monotone, since $\varepsilon_n > 0$. Moreover, this function is concave, due to the monotonicity of ε_n.

Put $x_0 = 0$ and $h(0) = 0$. Since ξ is heavy-tailed, we can choose $x_1 \geq 2^1$ so that

$$\mathbf{E}\{e^{\xi}; \xi \in (x_0, x_1]\} + e^{x_1}\overline{F}(x_1) \quad > \quad e^{h(x_0)} + \delta = \overline{F}(0) + \delta.$$

Choose $\varepsilon_1 > 0$ so that

$$\mathbf{E}\{e^{\varepsilon_1 \xi}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 x_1}\overline{F}(x_1) \quad = \quad e^{h(x_0)}\overline{F}(0) + \delta/2,$$

which is equivalent to say that

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)}\overline{F}(x_1) \quad = \quad e^{h(x_0)}\overline{F}(0) + \delta/2,$$

By induction we construct an increasing sequence x_n and a decreasing sequence $\varepsilon_n > 0$ such that $x_n \geq 2^n$ and

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)}\overline{F}(x_n) \quad = \quad e^{h(x_{n-1})}\overline{F}(x_{n-1}) + \delta/2^n$$

for any $n \geq 2$. For $n = 1$ this is already done. Make the induction hypothesis for some $n \geq 2$. Due to heavy-tailedness, there exists $x_{n+1} \geq 2^{n+1}$ so large that

$$\mathbf{E}\{e^{\varepsilon_n(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_n(x_{n+1} - x_n)}\overline{F}(x_{n+1}) \quad > \quad 1 + \delta.$$

Note that

$$\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$

as a function of ε_{n+1} is continuously decreasing to $\overline{F}(x_n)$ as $\varepsilon_{n+1} \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, \varepsilon_n)$ so that

$$\begin{aligned}\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} &+ e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1}) \\ &= \overline{F}(x_n) + \delta/(2^{n+1}e^{h(x_n)}).\end{aligned}$$

By definition of $h(x)$ this is equivalent to the following equality:

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1}) \quad = \quad e^{h(x_n)}\overline{F}(x_n) + \delta/2^{n+1}.$$

Our induction hypothesis now holds with $n + 1$ in place of n as required.

Next,

$$\begin{aligned}\mathbf{E}e^{h(\xi)} &= \sum_{n=1}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} \\ &\leq \sum_{n=1}^{\infty} \Big(e^{h(x_{n-1})}\overline{F}(x_{n-1}) - e^{h(x_n)}\overline{F}(x_n) + \delta/2^n\Big) \\ &= \overline{F}(0) + \delta = 1 + \delta.\end{aligned}$$

On the other hand, since $x_k \ge 2^k$,

$$\begin{aligned}\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} &= \sum_{k=n+1}^{\infty} \mathbf{E}\{\xi e^{h(\xi)}; \xi \in (x_{k-1}, x_k]\} \\ &\ge 2^n \sum_{k=n+1}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_{k-1}, x_k]\} \\ &\ge 2^n \sum_{k=n+1}^{\infty} \Big(e^{h(x_{k-1})}\overline{F}(x_{k-1}) - e^{h(x_k)}\overline{F}(x_k) + \delta/2^k\Big).\end{aligned}$$

Then, for any n,

$$\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} \ \ge\ 2^n e^{h(x_n)}\overline{F}(x_n) + \delta \ge \delta,$$

which implies $\mathbf{E}\xi e^{h(\xi)} = \infty$. Note also that necessarily $\lim_{n\to\infty} \varepsilon_n = 0$; otherwise $\liminf\limits_{x\to\infty} h(x)/x > 0$ and ξ is light tailed. The proof of the lemma is complete.

5. Proof of Theorem 2. Since τ has a light-tailed distribution,

$$\mathbf{E}\tau(1+\varepsilon)^{\tau-1} \ <\ \infty$$

for some sufficiently small $\varepsilon > 0$. By Lemma 2, there exists a concave increasing function h, $h(0) = 0$, such that $\mathbf{E}e^{h(\xi_1)} \le 1 + \varepsilon$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Then by concavity

$$A_n \equiv \mathbf{E}e^{h(\xi_1+\ldots+\xi_n)} \ \le\ \mathbf{E}e^{h(\xi_1)+\ldots+h(\xi_n)} \le (1+\varepsilon)^n.$$

Combining altogether, we get $\mathbf{E}\tau A_{\tau-1} < \infty$. All conditions of Theorem 4 are met and the proof is complete.

6. Fractional exponential moments. One can go further and obtain various results on lower limits and equivalencies for heavy-tailed distributions F which have all finite power moments (like Weibull and log-normal distributions). For instance, the following result takes place (see [9] for the proof):

Let there exist α, $0 < \alpha < 1$, such that $\mathbf{E}e^{c\xi^\alpha} = \infty$ for all $c > 0$. If $\mathbf{E}e^{\delta\tau^\alpha} < \infty$ for some $\delta > 0$, then (1) *holds.*

7. Tail equivalence for randomly stopped sums. The following auxiliary lemma compares the tail behavior of the convolution tail and that of the exponentially transformed distribution.

Lemma 3. *Let the distribution F and the number $\gamma \ge 0$ be such that $\varphi(\gamma) < \infty$. Let the distribution G be the result of the exponential change of measure with parameter γ, i.e., $G(du) = e^{\gamma u}F(du)/\varphi(\gamma)$. Let τ be an independent stopping time such that $\mathbf{E}\varphi^\tau(\gamma) < \infty$ and ν have the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\gamma)\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^\tau(\gamma)$. Then*

$$\liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \ \ge\ \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}$$

and

$$\limsup_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \ \le\ \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \limsup_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

Proof. Put

$$\widehat{c} \quad \equiv \quad \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

By Lemma 1, $\widehat{c} \in [\mathbf{E}\tau, \infty]$. For any fixed $c \in (0, \widehat{c})$, there exists $x_0 > 0$ such that, for any $x > x_0$,

$$\overline{F^{*\tau}}(x) \quad \geq \quad c\overline{F}(x). \tag{11}$$

By the total probability law,

$$\begin{aligned}
\overline{G^{*\nu}}(x) &= \sum_{k=1}^{\infty} \mathbf{P}\{\nu = k\}\overline{G^{*k}}(x) \\
&= \sum_{k=1}^{\infty} \frac{\varphi^k(\gamma)\mathbf{P}\{\tau = k\}}{\mathbf{E}\varphi^\tau(\gamma)} \int_x^\infty e^{\gamma y} \frac{F^{*k}(dy)}{\varphi^k(\gamma)} \\
&= \frac{1}{\mathbf{E}\varphi^\tau(\gamma)} \sum_{k=1}^{\infty} \mathbf{P}\{\tau = k\} \int_x^\infty e^{\gamma y} F^{*k}(dy).
\end{aligned}$$

After integration by parts, the latter sum is equal to

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbf{P}\{\tau = k\}\Big[e^{\gamma x}\overline{F^{*k}}(x) + \int_x^\infty \overline{F^{*k}}(y) de^{\gamma y}\Big] \\
&\qquad\qquad = e^{\gamma x}\overline{F^{*\tau}}(x) + \int_x^\infty \overline{F^{*\tau}}(y) de^{\gamma y}.
\end{aligned}$$

Using also (11) we get, for $x > x_0$,

$$\begin{aligned}
\overline{G^{*\nu}}(x) &\geq \frac{c}{\mathbf{E}\varphi^\tau(\gamma)}\Big[e^{\gamma x}\overline{F}(x) + \int_x^\infty \overline{F}(y) de^{\gamma y}\Big] \\
&= \frac{c}{\mathbf{E}\varphi^\tau(\gamma)} \int_x^\infty e^{\gamma y} F(dy) = \frac{c}{\mathbf{E}\varphi^{\tau-1}(\gamma)}\overline{G}(x).
\end{aligned}$$

Letting $c \uparrow \widehat{c}$, we obtain the first conclusion of the lemma. The proof of the second conclusion follows similarly.

Lemma 4. *If $0 < \widehat{\gamma} < \infty$, $\varphi(\widehat{\gamma}) < \infty$, and $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^\tau < \infty$ for some $\varepsilon > 0$, then*

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad \leq \quad \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma})$$

and

$$\limsup_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad \geq \quad \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma}).$$

Proof. We apply the exponential change of measure with parameter $\widehat{\gamma}$ and consider the distribution $G(du) = e^{\widehat{\gamma}u}F(du)/\varphi(\widehat{\gamma})$ and the stopping time ν with the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\widehat{\gamma})\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^\tau(\widehat{\gamma})$. Again from the definition of $\widehat{\gamma}$, the distribution G is heavy-tailed. The distribution of ν is light-tailed, because $\mathbf{E}e^{\kappa\nu} < \infty$ with $\kappa = \ln(\varphi(\widehat{\gamma}) + \varepsilon) - \ln\varphi(\widehat{\gamma}) > 0$. Hence,

$$\limsup_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \quad = \quad \mathbf{E}\nu,$$

by Theorem 2. The result now follows from Lemma 3 with $\gamma = \widehat{\gamma}$, since $\mathbf{E}\nu = \mathbf{E}\tau\varphi^\tau(\widehat{\gamma})/\mathbf{E}\varphi^\tau(\widehat{\gamma})$.

Proof of Theorem 3. In the case where F is heavy-tailed, we have $\widehat{\gamma} = 0$ and $\varphi(\widehat{\gamma}) = 1$. By Theorem 2, $c = \mathbf{E}\tau$ as required.

In the case $\widehat{\gamma} \in (0, \infty)$ and $\varphi(\widehat{\gamma}) < \infty$, the desired conclusion follows from Lemma 4.

8. Supremum of a random walk. Let $\{\xi_n\}$ be a sequence of independent random variables with a common distribution F on $\mathbf{R}$ and $\mathbf{E}\xi_1 = -m < 0$. Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$. By the SLLN, $M = \sup_{n \ge 0} S_n$ is finite with probability 1.

Let F^I be the integrated-tail distribution on $\mathbf{R}^+$, that is,

$$\overline{F^I}(x) \quad \equiv \quad \min\Big(1, \int_x^\infty \overline{F}(y)dy\Big), \quad x > 0.$$

It is well-known (see, e.g. [1, 12, 13] and references therein) that if $F^I \in \mathscr{S}$, then

$$\mathbf{P}\{M > x\} \quad \sim \quad \frac{1}{m}\overline{F^I}(x) \quad \text{as } x \to \infty. \tag{12}$$

Korshunov [18] proved the converse: (12) implies $F^I \in \mathscr{S}$. Now we accompany this assertion by the following

Theorem 5. *Let F^I be long-tailed, that is, $\overline{F^I}(x+1) \sim \overline{F^I}(x)$ as $x \to \infty$. If, for some $c > 0$,*

$$\mathbf{P}\{M > x\} \quad \sim \quad c\overline{F^I}(x) \quad \textit{as } x \to \infty,$$

then $c = 1/m$ and F^I is subexponential.

Proof. Consider the defective stopping time

$$\eta = \inf\{n \ge 1 : \ S_n > 0\} \le \infty$$

and let $\{\psi_n\}$ be i.i.d. random variables with common distribution function

$$G(x) \equiv \mathbf{P}\{\psi_n \le x\} = \mathbf{P}\{S_\eta \le x \mid \eta < \infty\}.$$

It is well-known (see, e.g. Feller [14, Chapter 12]) that the distribution of the maximum M coincides with the distribution of the randomly stopped sum $\psi_1 + \cdots + \psi_\tau$, where the stopping time τ is independent of the sequence $\{\psi_n\}$ and is geometrically distributed with parameter $p = \mathbf{P}\{M > 0\} < 1$, i.e., $\mathbf{P}\{\tau = k\} = (1-p)p^k$ for $k = 0, 1, \ldots$. Equivalently,

$$\mathbf{P}\{M \in B\} \quad = \quad G^{*\tau}(B).$$

From Borovkov [4, Chapter 4, Theorem 10], if F^I is long-tailed, then

$$\overline{G}(x) \sim \frac{1-p}{pm}\overline{F^I}(x). \tag{13}$$

Then it follows from the theorem hypothesis that

$$\overline{G^{*\tau}}(x) \quad \sim \quad \frac{cpm}{1-p}\overline{G}(x) \quad \text{as } x \to \infty.$$

Therefore, by Theorem 3 with $\widehat{\gamma} = 0$, $c = \mathbf{E}\tau(1-p)/pm = 1/m$. Then it follows from [11] that F^I is subexponential. Now the proof is complete.

9. The compound Poisson distribution. Let F be a distribution on $\mathbf{R}_+$ and t a positive constant. Let G be the compound Poisson distribution

$$G \quad = \quad e^{-t}\sum_{n \ge 0}\frac{t^n}{n!}F^{*n}.$$

Considering τ in Theorem 3 with $\mathbf{P}\{\tau = n\} = t^n e^{-t}/n!$, we get

Theorem 6. *Let $\varphi(\widehat{\gamma}) < \infty$. If, for some $c > 0$, $\overline{G}(x) \sim c\overline{F}(x)$ as $x \to \infty$, then $c = te^{t(\varphi(\widehat{\gamma})-1)}$.*

Corollary 1. *The following statements are equivalent:*
(i) *F is subexponential;*
(ii) *G is subexponential;*
(iii) *$\overline{G}(x) \sim t\overline{F}(x)$ as $x \to \infty$;*
(iv) *F is heavy-tailed and $\overline{G}(x) \sim c\overline{F}(x)$ as $x \to \infty$, for some $c > 0$.*

Proof. Equivalence of (i), (ii), and (iii) was proved in [11, Theorem 3]. The implication (iv)⇒(iii) follows from Theorem 3 with $\widehat{\gamma} = 0$.

Some local aspects of this problem for heavy-tailed distributions were discussed in [2, Theorem 6].

10. Infinitely divisible laws. Let F be an infinitely divisible law on $[0, \infty)$. The Laplace transform of an infinitely divisible law F can be expressed as

$$\int_0^\infty e^{-\lambda x} F(dx) \quad = \quad e^{-a\lambda - \int_0^\infty (1-e^{-\lambda x})\nu(dx)}$$

(see, e.g. [14, Chapter XVII]). Here $a \geq 0$ is a constant and the Lévy measure ν is a Borel measure on $(0, \infty)$ with the properties $\mu = \nu(1, \infty) < \infty$ and $\int_0^1 x\nu(dx) < \infty$. Put $G(B) = \nu(B \cap (1, \infty))/\mu$.

The relations between the tail behaviour of measure F and the corresponding Lévy measure ν were considered in [11, 19]. The local analogue of that result was proved in [2]. We strength the corresponding result of [11] in the following way.

Theorem 7. *The following assertions are equivalent:*
(i) *F is subexponential;*
(ii) *G is subexponential;*
(iii) *$\overline{\nu}(x) \sim \overline{F}(x)$ as $x \to \infty$;*
(iv) *F is heavy-tailed and $\overline{\nu}(x) \sim c\overline{F}(x)$ as $x \to \infty$, for some $c > 0$.*

Proof. Equivalence of (i), (ii), and (iii) was proved in [11, Theorem 1].

It remains to prove the implication (iv)⇒(iii). It is pointed out in [11] that the distribution F admits the representation $F = F_1 * F_2$, where $\overline{F}_1(x) = O(e^{-\varepsilon x})$ for some $\varepsilon > 0$ and

$$F_2(B) \quad = \quad e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} G^{*n}(B).$$

Since F is heavy-tailed and F_1 is light-tailed, we get the equivalence $\overline{F}(x) \sim \overline{F}_2(x)$ as $x \to \infty$. Therefore, as $x \to \infty$,

$$\mu\overline{G}(x) \quad = \quad \overline{\nu}(x) \sim c\overline{F}(x) \sim c\overline{F}_2(x).$$

With necessity G is heavy-tailed, and $c = 1$ by Corollary 1.

11. Branching processes. In this section we consider the limit behaviour of sub-critical, age-dependent branching processes for which the Malthusian parameter does not exist.

Let $h(z)$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution F (see [3, Chapter IV], [16, Chapter VI] for background). We take the process to be sub-critical, i.e. $A \equiv h'(1) < 1$. Let $Z(t)$ denote the number of particles at time t. It is known (see, for example, [3, Chapter IV, Section 5] or [5]) that $\mathbf{E}Z(t)$ admits the representation

$$\mathbf{E}Z(t) \quad = \quad (1-A)\sum_{n=1}^{\infty} A^{n-1}\overline{F^{*n}}(t).$$

It was proved in [5] for sufficiently small values of A and then in [6, 7] for any $A < 1$ that $\mathbf{E}Z(t) \sim \overline{F}(t)/(1-A)$ as $t \to \infty$, provided F is subexponential. The local asymptotics were considered in [2].

Applying Theorem 3 with τ geometrically distributed and $\widehat{\gamma} = 0$, we deduce

Theorem 8. *Let F be heavy-tailed, and, for some $c > 0$, $\mathbf{E}Z(t) \sim c\overline{F}(t)$ as $t \to \infty$. Then $c = 1/(1-A)$ and F is subexponential.*

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