THE MAXIMUM ON A RANDOM TIME INTERVAL OF A RANDOM WALK WITH LONG-TAILED INCREMENTS AND NEGATIVE DRIFT

By Serguei Foss and Stan Zachary

Heriot-Watt University

We study the asymptotics for the maximum on a random time interval of a random walk with a long-tailed distribution of its increments and negative drift. We extend to a general stopping time a result by Asmussen (1998), simplify its proof, and give some converses.

1 Introduction

Random walks with long-tailed increments have many important applications in insurance, finance, queueing networks, storage processes, and the study of extreme events in nature and elsewhere. See, for example, Embrechts *et al.* (1997), Asmussen (1998, 1999) and Greiner *et al.* (1999) for some background. In this paper we study the distribution of the maximum of such a random walk over a random time interval.

Let F be the distribution function of the increments of a random walk $\{S_n\}_{n\geq 0}$ with $S_0 = 0$. Suppose that this distribution has a finite negative mean and that F is long-tailed in the positive direction (see below for this and other definitions). Of interest is the asymptotic distribution of the maximum of $\{S_n\}$ over the interval $[0, \sigma]$ defined by some stopping time σ. Some results for the case where σ is independent of $\{S_n\}$ are known (again see below). However, relatively little is known for other stopping times. Asmussen (1998) gives the expected result for the case $\sigma = \tau$, where

$$\tau = \min\{n \geq 1 : S_n \leq 0\} \tag{1}$$

(see also Heath *et al.* (1997) and Greiner *et al.* (1999)). This result requires the further condition that the distribution function F has a right tail which belongs to the class $\mathcal{S}^*$ introduced by Klüppelberg (1988) (we shall simply write $F \in \mathcal{S}^*$). In the present paper we extend Asmussen's result to a general stopping time σ. In doing so we also simplify the derivation of the original result, and we show that the condition $F \in \mathcal{S}^*$ is necessary as well as sufficient for it to hold. We also give a useful characterisation of the class $\mathcal{S}^*$. Finally, as a corollary of our results, we give a probabilistic proof of the known result that any distribution function $G \in \mathcal{S}^*$ is subexponential.

Thus, let $\{\xi_n\}_{n\geq 1}$ be a sequence of independent identically distributed random variables with distribution function F. We assume throughout that

$$\mathbf{E}\xi_n = -m < 0. \tag{NEG}$$

[0] *American Mathematical Society 1991 subject classifications.* Primary 60G70; secondary 60K30, 60K25

Key words and phrases. ruin probability, long-tailed distribution, subexponential distribution.

Short title. Maximum on a random time interval.

We further assume throughout that the distribution function F is *long-tailed* (LT), that is, that

$$\text{(LT)} \qquad \overline{F}(x) > 0 \text{ for all } x, \qquad \lim_{x\to\infty} \frac{\overline{F}(x-h)}{\overline{F}(x)} = 1, \quad \text{for all fixed } h > 0.$$

Here, for any distribution function G on $\mathbb{R}$, $\overline{G}$ denotes the tail distribution given by $\overline{G}(x) = 1 - G(x)$. Define the random walk $\{S_n\}_{n\geq 0}$ by $S_0 = 0$, $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$. For $n \geq 0$, let $M_n = \max_{0\leq i\leq n} S_i$, and let $M = \sup_{n\geq 0} S_n$. Similarly, for any stopping time σ (with respect to any filtration $\{\mathcal{F}_n\}_{n\geq 1}$ such that, for each n, ξ_n is measurable with respect to $\mathcal{F}_n$ and ξ_{n+1} is independent of $\mathcal{F}_n$), let $M_\sigma = \max_{0\leq i\leq \sigma} S_i$. We are interested in the asymptotic distribution of M_σ for a general stopping time σ (which need not be a.s. finite). In particular we are interested in obtaining conditions under which

$$\lim_{x\to\infty} \frac{\mathbf{P}(M_\sigma > x)}{\overline{F}(x)} = \mathbf{E}\sigma. \tag{2}$$

We require first some further definitions. For any distribution function G on $\mathbb{R}$ define the integrated, or *second-tail*, distribution function G^s by $\overline{G^s}(x) = \min(1, \int_x^\infty \overline{G}(t)\,dt)$. A distribution function G on $\mathbb{R}_+$ is *subexponential* if and only if $\overline{G}(x) > 0$ for all x and $\lim_{x\to\infty} \overline{G^{*2}}(x)/\overline{G}(x) = 2$ (where G^{*2} is the convolution of G with itself). More generally, a distribution function G on $\mathbb{R}$ is subexponential if and only if G^+ is subexponential, where $G^+ = G\mathbf{I}_{\mathbb{R}_+}$ and $\mathbf{I}_{\mathbb{R}_+}$ is the indicator function of $\mathbb{R}_+$. It is known that the subexponentiality of a distribution depends only on its (right) tail, and that a subexponential distribution is long-tailed. When F is subexponential, it is elementary that the result (2) holds for any a.s. constant σ. (The condition (NEG) is not required here. See, for example, Embrechts *et al.* (1997), or Sigman (1999).) In the case where F^s is subexponential, the asymptotic distribution of M is known—in particular $\mathbf{P}(M > x) = O(\overline{F^s}(x))$ as $x \to \infty$ (see Veraverbeke (1977), Embrechts and Veraverbeke (1982), and, for a simpler treatment, Embrechts *et al.* (1997)).

A distribution function G on $\mathbb{R}$ belongs to the class $\mathcal{S}^*$ if and only if $\overline{G}(x) > 0$ for all x and

$$\int_0^x \overline{G}(x-y)\overline{G}(y)\,dy \sim 2m_{G^+}\overline{G}(x), \quad \text{as } x \to \infty, \tag{3}$$

where $m_G{}^+ = \int_0^\infty \overline{G}(x)\,dx$ is the mean of G^+. It is again known that the property $G \in \mathcal{S}^*$ depends only on the tail of G, and that if $G \in \mathcal{S}^*$, then both G and G^s are subexponential—see Klüppelberg (1988).

Consider first the case where the stopping time σ is independent of the sequence $\{\xi_n\}$. Here, under the further condition that the distribution function F is subexponential, the result (2) is well known to hold for any stopping time σ such that

$$\mathbf{E}\exp\lambda\sigma < \infty, \quad \text{for some } \lambda > 0. \tag{4}$$

In this particular case the condition (NEG) is not required (see, for example, Embrechts *et al.* (1997) and the references therein.) The condition (4) may be dropped by suitably strengthening the subexponentiality condition on F (see Borovkov and Borovkov (2001), Korshunov (2001)).

The first results for a stopping time σ which is not independent of the sequence $\{\xi_n\}$ are given by Heath *et al.* (1997, Proposition 2.1) and, under more general conditions, by Asmussen (1998, Theorem 2.1)—see also Greiner *et al.* (1999, Theorem 3.3). Asmussen shows that if, in addition to our present conditions (NEG) and (LT), we have $F \in \mathcal{S}^*$, then the result (2) holds with $\sigma = \tau$, where τ is as given by (1). (Asmussen omits to state formally the necessity of some condition of the form $F \in \mathcal{S}^*$. However, this is rectified in the more recent paper by Asmussen *et al.* (2001). See also Asmussen, Foss and Korshunov (2002) for further extensions.)

The main result of the present paper is Theorem 1, which shows that, again under the condition $F \in \mathcal{S}^*$, the result (2) holds for a general stopping time σ. The theorem also shows that, for a wide class of stopping times σ, including $\sigma \equiv \tau$, the condition $F \in \mathcal{S}^*$ is necessary as well as sufficient for this result. In proving Theorem 1 we of necessity simplify the derivation of Asmussen's original result, which was quite tricky (as was that of Greiner *et al.* (1999)). The proof requires Theorem 2, which gives a characterisation of the class $\mathcal{S}^*$. One half of this theorem is due to Asmussen *et al.* (2001). Finally, as already remarked and as a very simple corollary of Theorem 1, we give a probabilistic proof of the result referred to above that any $G \in \mathcal{S}^*$ is subexponential.

2 Results

We state first our main result, which is Theorem 1. We give also Corollaries 1 and 2. We then proceed to the proofs, which are via Theorem 2 and a sequence of lemmas. A further necessary lemma, which is a fairly routine application of some results from renewal theory, is relegated to the Appendix. As remarked above, the proof of part (i) of Theorem 1, in the case $\sigma = \tau$, follows the general approach of Asmussen (1998) with some simplification of the argument. Recall that the conditions (NEG) and (LT) are assumed to hold throughout.

Theorem 1.

(i) Suppose that $F \in \mathcal{S}^*$. Let $\sigma \leq \infty$ be any stopping time. Then

$$\lim_{x\to\infty} \frac{\boldsymbol{P}(M_\sigma > x)}{\overline{F}(x)} = \boldsymbol{E}\sigma. \tag{5}$$

(ii) Suppose that the condition (5) *holds for some stopping time σ such that $\boldsymbol{P}(\sigma > 0) > 0$, $\boldsymbol{P}(S_\sigma \leq 0) = 1$ and $\boldsymbol{E}\sigma < \infty$. Then $F \in \mathcal{S}^*$.*

Remark 1. Suppose again that the conditions of Theorem 1 (i) hold. In the case $\mathbf{E}\sigma < \infty$, it follows from from that result (and the well-known result, from (LT), that $\overline{F}(x) = o(\overline{F^s}(x))$ as $x \to \infty$) that $\mathbf{P}(M_\sigma > x) = o(\overline{F^s}(x))$ as $x \to \infty$ (in contrast to the case $\sigma = \infty$ a.s.). In the case $\sigma < \infty$ a.s., $\mathbf{E}\sigma = \infty$, we have both $\overline{F}(x) = o(\mathbf{P}(M_\sigma > x))$ (by Theorem 1 (i)) and, again, $\mathbf{P}(M_\sigma > x) = o(\overline{F^s}(x))$, in each case as $x \to \infty$. We give a proof of the latter result in the Appendix. In this case a rich variety of behaviour is possible.

Corollary 1 is due to Klüppelberg (1988). However, we use the results of this paper to give a simple probabilistic proof.

Corollary 1 (Klüppelberg). *Suppose that a distribution function G on $\mathbb{R}$ belongs to $\mathcal{S}^*$. Then G is subexponential.*

Corollary 2. *Suppose that, under the conditions of Theorem 1 (i), the stopping time σ is additionally independent of the sequence $\{\xi_n\}$. Then also*

$$\lim_{x\to\infty} \frac{\boldsymbol{P}(S_\sigma > x)}{\overline{F}(x)} = \boldsymbol{E}\sigma.$$

Remark 2. The results of both Theorem 1 (i) and Corollary 2 continue to hold even when the condition (NEG) is dropped, provided that the distribution of the random variables ξ_n remains finite and a further condition is imposed on the tail of the distribution of the stopping time σ. Since (NEG) is assumed throughout the body of the paper, we give the details as Corollary 3 in the Appendix.

For several of our results and their proofs we require a function $h : \mathbb{R}_+ \to \mathbb{R}_+$, which, since F is long-tailed, may be chosen such that

$$h(x) \le x/2 \quad \text{for all } x, \tag{6}$$

$$h(x) \to \infty \quad \text{as } x \to \infty, \tag{7}$$

$$\frac{\overline{F}(x - h(x))}{\overline{F}(x)} \to 1 \quad \text{as } x \to \infty, \tag{8}$$

$$\text{there exists } x_0 \text{ with } h(x+t) \le h(x) + t \text{ for all } x \ge x_0,\ t \ge 0. \tag{9}$$

Now let π be the distribution of M $(= \sup_{n\ge0} S_n)$. Theorem 2 below is in part due to Asmussen *et al.* (2001), and provides a useful characterisation of the class $\mathcal{S}^*$. In particular, by taking the function g in the statements (a) and (b) of the theorem to be the indicator function of the interval $[0, c]$, we obtain the equivalence of the condition $F \in \mathcal{S}^*$ to a local limit result for $\pi((x, x+c])$ for *any*, and hence for *all*, $c > 0$.

Let $\mathcal{G}$ be the class of functions on $\mathbb{R}_+$ which are directly Riemann integrable (see, for example, Feller (1971, p. 362)) and nonnegative. Let $\mathcal{G}^*$ be the subclass of $\mathcal{G}$ consisting of those functions which are additionally bounded away from zero on some interval of nonzero width.

Theorem 2. *For any function $g : \mathbb{R}_+ \to \mathbb{R}$ consider the property*

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^\infty \pi(x + dt) g(t) = \frac{1}{m} \int_0^\infty g(t)\, dt. \tag{10}$$

(a) If $F \in \mathcal{S}^*$, then (10) *holds for all directly Riemann integrable functions g on $\mathbb{R}_+$.*

(b) If (10) *holds for any given $g \in \mathcal{G}^*$, then $F \in \mathcal{S}^*$.*

Proof. The result (a) is Corollary 1 of Asmussen *et al.* (2001). We prove (b).

Note first that, for any $g \in \mathcal{G}$,

$$\liminf_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^\infty \pi(x+dt)g(t) \geq \frac{1}{m}\int_0^\infty g(t)\,dt. \tag{11}$$

This follows routinely from the lower bound (30) on the distribution π of M given by Lemma 6 in the Appendix. (In particular we may initially take g to be zero outside a finite interval, and then use Fatou's Lemma to obtain the general result.)

Now suppose that (10) holds with g given by $g_0 \in \mathcal{G}^*$. Let $\mathbf{I}_A$ denote the indicator function of any $A \subset \mathbb{R}_+$. Then, for all sufficiently small $c > 0$, we can find $\varepsilon > 0$, $b \geq 0$, and $g_2 \in \mathcal{G}$ such that $g_0 \equiv g_1 + g_2$ where $g_1 \equiv \varepsilon \mathbf{I}_{[b,b+c]} \in \mathcal{G}^*$. Now

$$\begin{aligned}\limsup_{x\to\infty} &\frac{1}{\overline{F}(x)} \int_0^\infty \pi(x+dt)g_1(t)\\ &\leq \lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^\infty \pi(x+dt)g_0(t) - \liminf_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^\infty \pi(x+dt)g_2(t).\end{aligned}$$

It therefore follows from (10) with g given by g_0 and from (11) with g given by each of g_1 and g_2, that (10) also holds with g given by g_1. Since F is long-tailed, the property (10) is preserved under any finite shift of the function g. Thus, finally, we obtain that (10) holds for all g of the form $\mathbf{I}_{[0,c]}$ for all sufficiently small $c > 0$, and so, by additivity, for all $c > 0$.

Now fix any $c > 0$. Let the sequences of random variables $\{\psi_n\}_{n\geq 1}$, $\{T_n\}_{n\geq 1}$, the random variable ν and the constant p be as defined in the Appendix. From (10) with $g \equiv \mathbf{I}_{[0,c]}$, a variation of the argument at the end of the proof of Lemma 6 gives

$$\begin{aligned}\frac{c}{m} &= \lim_{x\to\infty} \frac{\mathbf{P}(M \in (x, x+c])}{\overline{F}(x)}\\ &\geq \mathbf{P}(\nu = 2) \limsup_{x\to\infty} \frac{\mathbf{P}(T_2 \in (x, x+c])}{\overline{F}(x)} + \liminf_{x\to\infty} \sum_{n\geq 1,\, n\neq 2} \mathbf{P}(\nu = n) \frac{\mathbf{P}(T_n \in (x, x+c])}{\overline{F}(x)}\\ &\geq \mathbf{P}(\nu = 2) \limsup_{x\to\infty} \frac{\mathbf{P}(T_2 \in (x, x+c])}{\overline{F}(x)} + \sum_{n\geq 1,\, n\neq 2} \mathbf{P}(\nu = n) \liminf_{x\to\infty} \frac{\mathbf{P}(T_n \in (x, x+c])}{\overline{F}(x)},\end{aligned}$$

where the last line above follows from Fatou's lemma Thus, from the lower bounds given by (28) and (29) in the Appendix (and the calculation leading to (30)),

$$\limsup_{x\to\infty} \frac{\mathbf{P}(T_2 \in (x, x+c])}{\overline{F}(x)} \leq \frac{2pc}{(1-p)m},$$

and hence, again using (29),

$$\lim_{x\to\infty} \frac{\mathbf{P}(T_2 \in (x, x+c])}{\overline{F}(x)} = \frac{2pc}{(1-p)m}, \tag{12}$$

(where, as usual, the above equation includes the assertion that the limit exists). Now, for the function h defined above (in fact we do not require the condition (9)) for this proof), it follows from (6) that

$$\mathbf{P}(\psi_1 + \psi_2 \in (x, x+c],\ \psi_1 \leq h(x),\ \psi_2 \leq h(x)) = 0.$$

Thus, from (7), (8) and (28),

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \mathbf{P}(\psi_1 + \psi_2 \in (x, x+c],\, \psi_i \le h(x)) = \frac{pc}{(1-p)m}, \qquad i = 1, 2,$$

and so, from (12),

$$\mathbf{P}(\psi_1 + \psi_2 \in (x, x+c],\, \psi_1 > h(x),\, \psi_2 > h(x)) = o(\overline{F}(x)), \qquad \text{as } x \to \infty. \tag{13}$$

Now it is convenient here to take h such that $x - 2h(x)$ is an integer multiple $n(x)$ of c for all x. Let also $d = p/(1-p)m$. Then, as $x \to \infty$,

$$\begin{aligned}
\mathbf{P}(\psi_1 + \psi_2 \in (x, x+c],\, \psi_1 > h(x),\, \psi_2 > h(x)) &\ge \int_{h(x)}^{x-h(x)} \mathbf{P}(\psi_1 \in dt)\mathbf{P}(\psi_2 \in (x-t, x-t+c]) \\
&\sim dc \int_{h(x)}^{x-h(x)} \mathbf{P}(\psi_1 \in dt)\overline{F}(x-t) \qquad (14) \\
&= dc \sum_{k=1}^{n(x)} \int_{h(x)+(k-1)c}^{h(x)+kc} \mathbf{P}(\psi_1 \in dt)\overline{F}(x-t) \\
&= (1+o(1))d^2 c \int_{h(x)}^{x-h(x)} \overline{F}(t)\overline{F}(x-t)\, dt, \qquad (15)
\end{aligned}$$

where (14) follows from (28), and (15) follows from (28) and (LT). Thus, using also (13), we obtain

$$\int_{h(x)}^{x-h(x)} \overline{F}(t)\overline{F}(x-t)\, dt = o(\overline{F}(x)), \qquad \text{as } x \to \infty. \tag{16}$$

From (8), $\overline{F}(x-t) = (1+o(1))\overline{F}(x)$ as $x \to \infty$, uniformly in $t \in [0, h(x)]$. Hence, from (7),

$$\int_0^{h(x)} \overline{F}(t)\overline{F}(x-t)\, dt \sim \overline{F}(x) \int_0^{h(x)} \overline{F}(t)\, dt \sim m_{F^+}\overline{F}(x), \qquad \text{as } x \to \infty,$$

and so the condition (16) is equivalent to the condition (3) that $F \in \mathcal{S}^*$. □

Remark 3. Note that a trivial extension of the above proof gives directly (i.e. without reference to part (a) of the theorem) the result that if (10) holds for any given $g \in \mathcal{G}^*$ then it does so for all directly Riemann integrable functions g.

For any $x \ge 0$, define the stopping time

$$\mu(x) = \min\{n : S_n > x\}.$$

(Thus $\{\mu(x) \le n\} = \{M_n > x\}$.) For any stopping time σ and any $x \ge 0$, define

$$A_{\sigma,1}(x) = \{\mu(x) \le \sigma,\, S_{\mu(x)-1} \le h(x)\}$$
$$A_{\sigma,2}(x) = \{\mu(x) \le \sigma,\, S_{\mu(x)-1} > h(x)\},$$

where h is as given by (6)–(9). Define also

$$\delta_\sigma(x) = \sup_{y \ge x} \frac{\mathbf{P}(A_{\sigma,2}(y))}{\overline{F}(y)}.$$

Lemma 1 below is, in an obvious sense, very close to what we require for the proof of Theorem 1. The loose ends are tied up by Lemmas 3, 4 and 5, while Lemma 2 is necessary for the proof of Lemma 3.

Lemma 1. *Let σ be a stopping time such that $\boldsymbol{E}\sigma < \infty$. Then*

$$\lim_{x\to\infty} \frac{\boldsymbol{P}(A_{\sigma,1}(x))}{\overline{F}(x)} = \boldsymbol{E}\sigma.$$

Proof. For any $x \geq 0$,

$$\begin{aligned}
\mathbf{P}(A_{\sigma,1}(x)) &= \sum_{n=0}^{\infty} \int_{-\infty}^{h(x)} \mathbf{P}(\sigma > n, M_n \leq x, S_n \in dy, S_{n+1} > x) \\
&\leq \sum_{n=0}^{\infty} \int_{-\infty}^{h(x)} \mathbf{P}(\sigma > n, M_n \leq x, S_n \in dy)\overline{F}(x - h(x)) \\
&\leq \sum_{n=0}^{\infty} \mathbf{P}(\sigma > n)\overline{F}(x - h(x)) \\
&= (1 + o(1))\mathbf{E}\sigma\overline{F}(x) \quad \text{as } x \to \infty.
\end{aligned}$$

We now establish the lower bound. For any positive integer N,

$$\begin{aligned}
\mathbf{P}(A_{\sigma,1}(x)) &\geq \sum_{n=0}^{N} \int_{-h(x)}^{h(x)} \mathbf{P}(\sigma > n, M_n \leq h(x), S_n \in dy, S_{n+1} > x) \\
&\geq \sum_{n=0}^{N} \mathbf{P}(\sigma > n, M_n \leq h(x), S_n \in [-h(x), h(x)])\overline{F}(x + h(x)) \\
&= (1 + o(1))\overline{F}(x) \sum_{n=0}^{N} \mathbf{P}(\sigma > n) \quad \text{as } x \to \infty,
\end{aligned}$$

by (7) and (8). Let $N \to \infty$ to obtain

$$\mathbf{P}(A_{\sigma,1}(x)) \geq (1 + o(1))\mathbf{E}\sigma\overline{F}(x) \quad \text{as } x \to \infty.$$

□

Now define

$$m^- = \int_0^{\infty} F(-y)\, dy.$$

For any $t, x > 0$, define

$$\mathcal{D}(-t, -x) = \mathbf{E}(\#\{n \geq 0 : S_n > -t,\ S_{n+1} \leq -t,\ \min_{0\leq k\leq n} S_k > -t - x\})$$

to be the expected number of downcrossings by $\{S_n\}$ of $-t$ before the random walk first reaches $-t - x$. Define also $\mathcal{D}(-t) = \mathcal{D}(-t, -\infty)$.

Lemma 2.

$$\lim_{t,x\to\infty} \mathcal{D}(-t, -x) = \frac{m^-}{m}.$$

Proof. Asmussen (1998, Lemma 2.4) shows that

$$\lim_{t\to\infty} \mathcal{D}(-t) = \frac{m^-}{m}. \tag{17}$$

For completeness we repeat his proof: let R given by $R(A) = \sum_{n=0}^{\infty} \mathbf{P}(S_n \in A)$ be the renewal measure associated with the random walk $\{S_n\}$. Then there exist constants a_1 and a_2 such that $R[y, y+x] \le a_1 + a_2 x$ for all $x \ge 0$ and for all y. We have

$$\mathcal{D}(-t) = \int_{-t}^{\infty} R(dy)F(-t-y) = \int_{0}^{\infty} R(dz-t)F(-z).$$

When F is nonlattice, $R(dz-t)$ converges vaguely to Lebesgue measure with density $1/m$ as $t \to \infty$. It follows that

$$\lim_{t\to\infty} \mathcal{D}(-t) = \frac{1}{m}\int_0^{\infty} F(-z)\,dz = \frac{m^-}{m}.$$

The usual straightforward modifications are required to establish also (17) in the lattice case.

Now further, there exists $K < \infty$ with

$$\sup_{t>0} \mathcal{D}(-t) = K. \tag{18}$$

Thus, by the strong Markov property,

$$\mathcal{D}(-t,-x) \le \mathcal{D}(-t) \le \mathcal{D}(-t,-x) + K\mathbf{P}(M > x)$$

and so the required result follows from (17) and since $\mathbf{P}(M > x) \to 0$ as $x \to \infty$. □

Lemma 3. $\lim_{x\to\infty} \delta_\tau(x) = 0$ *if and only if* $F \in \mathcal{S}^*$.

Proof. The proof is in part an extended, and somewhat clarified, version of the argument of Asmussen (1998). Define the reflected random walk (workload process or Lindley queueing theory recursion) $\{W_n\}_{n\ge0}$ by

$$W_0 = 0, \qquad W_n = \max(0, W_{n-1} + \xi_n), \quad n \ge 1.$$

Note that, from (NEG), this is an ergodic Markov chain. Note also that $W_n = S_n$ for $n < \tau$ (where τ is as defined by (1)). It is further convenient to extend the sequence $\{\xi_n\}_{n\ge1}$ to the doubly-infinite sequence $\{\xi_n\}_{-\infty<n<\infty}$ of independent identically distributed random variables with distribution function F, and to define also the stationary version $\{W^n\}_{-\infty<n<\infty}$ of the above workload process (indexed on $(-\infty,\infty)$) by

$$W^n = \max(0, \sup_{j\ge0} \sum_{i=0}^{-j} \xi_{n+i}) \tag{19}$$

Note that $W^n = \max(0, W^{n-1} + \xi_n)$ for all n. It is well known that the common distribution of the W^n is given by π (as is clear from (19)).

For any $x > 0$, let

$$N^-(x) = \#\{n : 1 \le n < \tau,\, S_n > x,\, S_{n+1} \le x\}$$

be the number of downcrossings of x in $[0,\tau]$ by $\{S_n\}$ or $\{W_n\}$. Note that, by the ergodicity of the process $\{W^n\}$,

$$\begin{aligned}\frac{\mathbf{E}N^-(x)}{\mathbf{E}\tau} &= \mathbf{P}(W^0 > x,\, W^0+\xi_1 \le x)\\ &= \int_0^\infty \pi(x+dt)F(-t).\end{aligned}$$

From (NEG), the function g on $\mathbb{R}_+$ defined by $g(t) = F(-t)$ belongs to $\mathcal{G}^*$. Hence, by Theorem 2,

$$\mathbf{E}N^-(x) \sim \frac{m^-}{m}\mathbf{E}\tau\overline{F}(x) \text{ as } x\to\infty \quad \text{if and only if} \quad F\in\mathcal{S}^*. \tag{20}$$

We also have

$$\begin{aligned}\mathbf{E}[N^-(x)\mathbf{I}(A_{\tau,1}(x))] &= \mathbf{E}[\mathbf{I}(A_{\tau,1}(x))\mathbf{E}\{N^-(x)\mid S_{\mu(x)}\}]\\ &= \mathbf{E}[\mathbf{I}(A_{\tau,1}(x))\mathcal{D}(-(S_{\mu(x)}-x),-x)].\end{aligned}$$

Since, for any $u\in(0,h(x))$,

$$\begin{aligned}\mathbf{P}(S_{\mu(x)}-x > h(x)\mid A_{\tau,1}(x), S_{\mu(x)-1}\in du) &\ge \frac{\overline{F}(x+h(x))}{\overline{F}(x-h(x))}\\ &\to 1, \quad \text{as } x\to\infty,\end{aligned}$$

by (8), the overshoot $S_{\mu(x)}-x$ converges in distribution to ∞ as $x\to\infty$. Hence, again as $x\to\infty$,

$$\begin{aligned}\mathbf{E}[N^-(x)\mathbf{I}(A_{\tau,1}(x))] &\sim \mathbf{P}(A_{\tau,1}(x))\frac{m^-}{m}\\ &\sim \frac{m^-}{m}\mathbf{E}\tau\overline{F}(x).\end{aligned}\tag{21}$$

Here, the first line above follows by Lemma 2 (and the spatial homogeneity of the random walk $\{S_n\}$) and the second follows from Lemma 1. It now follows from (20) and (21) that

$$\mathbf{E}[N^-(x)\mathbf{I}(A_{\tau,2}(x))] = o(\overline{F}(x)) \text{ as } x\to\infty \quad \text{if and only if} \quad F\in\mathcal{S}^*.$$

Since also $1\le \mathbf{E}(N^-(x)\,|\,A_{\tau,2}(x))\le K$, where K is as defined by (18), the required result now follows. □

Lemma 4.

(i) Suppose that $\lim_{x\to\infty}\delta_\tau(x)=0$. *Let* σ *be any stopping time such that* $\mathbf{E}\sigma<\infty$. *Then* $\lim_{x\to\infty}\delta_\sigma(x)=0$.

(ii) Suppose that there exists a stopping time σ *such that* $\mathbf{P}(\sigma>0)>0$, $\mathbf{P}(S_\sigma\le 0)=1$ *and* $\lim_{x\to\infty}\delta_\sigma(x)=0$. *Then* $\lim_{x\to\infty}\delta_\tau(x)=0$.

Proof. To prove (i), note first that it follows from (9) (where x_0 is as defined there) that, for all $x\ge x_0$ and for all $t\ge 0$,

$$\begin{aligned}\mathbf{P}(\mu(x+t)\le\tau,\, S_{\mu(x+t)-1}>h(x)+t) &\le \mathbf{P}(\mu(x+t)\le\tau,\, S_{\mu(x+t)-1}>h(x+t))\\ &\le \delta_\tau(x)\overline{F}(x).\end{aligned}\tag{22}$$

Now define the sequence of stopping times $\{\tau_k\}_{k\geq 0}$ by

$$\tau_0 = 0, \qquad \tau_k = \min\{n : n > \tau_{k-1},\, S_n \leq S_{\tau_{k-1}}\}, \quad k \geq 1, \tag{23}$$

so that τ_k is the kth decreasing ladder time (and in particular $\tau_1 = \tau$). For all k, since $S_{\tau_k} \leq 0$, it follows from (22) and the temporal and spatial homogeneity of the random walk $\{S_n\}$ that

$$\mathbf{P}(\tau_k < \mu(x) \leq \tau_{k+1}, S_{\mu(x)-1} > h(x) \mid \sigma > \tau_k) \leq \delta_\tau(x)\overline{F}(x), \qquad \text{for all } x \geq x_0.$$

Since also σ is a.s. finite, it follows that, for any $x \geq x_0$,

$$\begin{aligned}
\mathbf{P}(A_{\sigma,2}(x)) &= \sum_{k\geq 0} \mathbf{P}(\tau_k < \mu(x) \leq \tau_{k+1},\, \sigma \geq \mu(x),\, S_{\mu(x)-1} > h(x)) \\
&\leq \sum_{k\geq 0} \mathbf{P}(\tau_k < \mu(x) \leq \tau_{k+1},\, \sigma > \tau_k,\, S_{\mu(x)-1} > h(x)) \\
&= \sum_{k\geq 0} \mathbf{P}(\sigma > \tau_k)\mathbf{P}(\tau_k < \mu(x) \leq \tau_{k+1},\, S_{\mu(x)-1} > h(x) \,|\, \sigma > \tau_k) \\
&\leq \delta_\tau(x)\overline{F}(x) \sum_{k\geq 0} \mathbf{P}(\sigma > \tau_k) \\
&\leq \delta_\tau(x)\overline{F}(x) \sum_{k\geq 0} \mathbf{P}(\sigma > k) \\
&= \delta_\tau(x)\overline{F}(x)\mathbf{E}\sigma,
\end{aligned}$$

so that (i) now follows.

To prove (ii) note first that we may assume, without loss of generality, that $\mathbf{P}(\sigma > 0) = 1$ (for otherwise we may simply condition on the event $\{\sigma > 0\}$, which is assumed to have a nonzero probability). The given conditions on σ then imply that $\tau \leq \sigma$ a.s.. Thus, for all $x > 0$, $\mathbf{P}(A_{\tau,2}(x)) \leq \mathbf{P}(A_{\sigma,2}(x))$ and so the result follows immediately.

□

Lemma 5. *Let σ be any stopping time such that $\mathbf{E}\sigma < \infty$. Then*

$$\lim_{x\to\infty} \delta_\sigma(x) = 0 \quad \textit{if and only if} \quad \lim_{x\to\infty} \frac{\mathbf{P}(M_\sigma > x)}{\overline{F}(x)} = \mathbf{E}\sigma.$$

Proof. We have $\mathbf{P}(M_\sigma > x) = \mathbf{P}(A_{\sigma,1}(x)) + \mathbf{P}(A_{\sigma,2}(x))$, so that the result is immediate from Lemma 1. □

Proof of Theorem 1. In the case $\mathbf{E}(\sigma) < \infty$, the proofs of both (i) and (ii) are immediate from Lemmas 3, 4 and 5. The extension to the case $\mathbf{E}(\sigma) = \infty$ is by a simple truncation argument. □

Proof of Corollary 1. Given $G \in \mathcal{S}^*$, let $\{\phi_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with distribution function G^+. Take the sequence $\{\xi_n\}_{n\geq 1}$ of the present paper to be given by $\xi_n = \phi_n - b$ for all n, where b is chosen sufficiently large that these random variables each have a negative mean. Since the property that a distribution belongs to $\mathcal{S}^*$ is easily shown to be shift invariant, the common distribution function F of the random variables ξ_n belongs to $\mathcal{S}^*$. Thus also F is long-tailed.

To show that G is subexponential we apply Theorem 1 with $\sigma \equiv 2$ to obtain

$$\begin{aligned}\mathbf{P}(\phi_1+\phi_2>x) &= \mathbf{P}(\xi_1+\xi_2>x-2b)\\ &\le \mathbf{P}(M_2>x-2b)\\ &\sim 2\overline{F}(x-2b)\\ &\sim 2\overline{G^+}(x), \qquad \text{as } x\to\infty,\end{aligned} \tag{24}$$

where the last line above follows since $G \in \mathcal{S}^*$ implies that G^+ is long-tailed. Further,

$$\begin{aligned}\mathbf{P}(\phi_1+\phi_2>x) &\ge \mathbf{P}(\phi_1>x)+\mathbf{P}(\phi_2>x)-\mathbf{P}(\phi_1>x,\phi_2>x)\\ &= 2\overline{G^+}(x)-(\overline{G^+}(x))^2\\ &\sim 2\overline{G^+}(x), \qquad \text{as } x\to\infty.\end{aligned} \tag{25}$$

Hence, from (24) and (25), G is subexponential as required. □

Proof of Corollary 2. Under the conditions of Theorem 1 (i), it follows from Corollary 1 that F is subexponential. Hence, for any $n \ge 1$, $\mathbf{P}(S_n > x) \sim n\overline{F}(x)$ as $x \to \infty$. If the stopping time σ is independent of the sequence $\{\xi_n\}$ a simple truncation argument now gives

$$\liminf_{x\to\infty} \frac{\mathbf{P}(S_\sigma > x)}{\overline{F}(x)} \ge \mathbf{E}\sigma.$$

Since $\mathbf{P}(S_\sigma > x) \le \mathbf{P}(M_\sigma > x)$ for all x, the result now follows from Theorem 1 (i). □

Remark 4. Theorem 1 can also be used to give a (rather circuitous) probabilistic proof of the result that if $G \in \mathcal{S}^*$ then G^s is subexponential. By taking $\sigma \equiv \tau$ in the theorem, and using standard renewal theory, we may show that, for the shifted version F of G, defined as in the above proof,

$$\lim_{x\to\infty} \frac{1}{\overline{F^s}(x)}\mathbf{P}(M > x) = \frac{1}{m}.$$

That F^s, and so G^s, is subexponential now follows from the converse to Veraverbeke's Theorem proved by Korshunov (1997).

Acknowledgement

The authors are grateful to Claudia Klüppelberg for some very helpful discussions, to Denis Denisov for a very valuable simplification of the proof of Lemma 1, and to the referee for a very thorough reading of the paper and suggested improvements.

Appendix

We give here some auxiliary results relating to the successive ladder heights and to the maximum of the process $\{S_n\}$. We also prove the last statement of Remark 1.

Define $\eta = \min\{n \geq 1 : S_n > 0\} \leq \infty$, and let

$$p = \mathbf{P}(\eta = \infty) = \mathbf{P}(M = 0). \tag{26}$$

Note that $0 < p < 1$. Let $\{\psi_n\}_{n \geq 1}$ be a sequence of i.i.d. copies of a positive random variable ψ such that, for all (measurable) $B \subseteq \mathbb{R}_+$,

$$\mathbf{P}(\psi \in B) = \mathbf{P}(S_\eta \in B \mid \eta < \infty).$$

Let ν be a random variable, independent of the above sequence, such that

$$\mathbf{P}(\nu = n) = p(1-p)^n, \qquad n = 0, 1, 2, \dots.$$

Then it is a standard result that

$$M =_D \sum_{i=1}^{\nu} \psi_i$$

(here $\sum_1^0 = 0$ by definition). For each $n \geq 1$, define also

$$T_n = \sum_{i=1}^{n} \psi_i. \tag{27}$$

Lemma 6. *Under the conditions* (NEG) *and* (LT), *for all $c > 0$ in the case where F is nonlattice, and for all positive multiples c of the span in the case where F is lattice,*

$$\lim_{x \to \infty} \frac{\boldsymbol{P}(\psi \in (x, x+c])}{\overline{F}(x)} = \frac{pc}{(1-p)m}, \tag{28}$$

$$\liminf_{x \to \infty} \frac{\boldsymbol{P}(T_n \in (x, x+c])}{\overline{F}(x)} \geq \frac{npc}{(1-p)m}, \qquad n \geq 2, \tag{29}$$

$$\liminf_{x \to \infty} \frac{\boldsymbol{P}(M \in (x, x+c])}{\overline{F}(x)} \geq \frac{c}{m}. \tag{30}$$

Proof. The results are reasonably well known. However, the lemma is adapted to the needs of the present paper, and, for completeness, we give also proofs here. We restrict ourselves to the nonlattice case.

Let $\hat{H}$ denote the "taboo" renewal measure on $\mathbb{R}_- = (-\infty, 0]$ given by, for $B \subseteq \mathbb{R}_-$,

$$\hat{H}(B) = \sum_{n=0}^{\infty} \mathbf{P}(S_n \in B, M_n = 0).$$

Note that

$$\lim_{x \to \infty} \hat{H}((-x, -x+c]) = \frac{pc}{m}, \qquad c > 0, \tag{31}$$

and there exists a constant $L < \infty$ such that

$$\hat{H}((-x, 0]) \leq \frac{px}{m} + L, \qquad x \geq 0. \tag{32}$$

Then, for any $c > 0$ and all $x \geq 0$,

$$\mathbf{P}(\psi \in (x, x+c]) = \int_0^\infty \hat{H}(-dt) \left[\overline{F}(x+t) - \overline{F}(x+t+c) \right]. \tag{33}$$

Since F satisfies (LT), we can choose a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the earlier conditions (6)–(8) and such that

$$h(x)\left[\overline{F}(x) - \overline{F}(x+h(x)+c)\right] = o(\overline{F}(x)), \qquad \text{as } x \to \infty.$$

Then, from (32),

$$\begin{aligned}
0 &\le \int_0^{h(x)} \hat{H}(-dt)\left[\overline{F}(x+t) - \overline{F}(x+t+c)\right] \\
&\le \left(\frac{p}{m}h(x) + L\right)\left[\overline{F}(x) - \overline{F}(x+h(x)+c)\right] \\
&= o(\overline{F}(x)) \qquad \text{as } x \to \infty. \qquad (34)
\end{aligned}$$

Further

$$\begin{aligned}
\frac{p}{m}\int_{h(x)}^{\infty}\left[\overline{F}(x+t) - \overline{F}(x+t+c)\right]\,dt &= \frac{p}{m}\int_{h(x)}^{h(x)+c}\overline{F}(x+t)\,dt \\
&\sim \frac{pc}{m}\overline{F}(x) \qquad \text{as } x \to \infty, \qquad (35)
\end{aligned}$$

since F satisfies (LT) and by the condition (8) on h. Finally, it follows from (31) that, given $\varepsilon > 0$, for all sufficiently large x, and hence $h(x)$,

$$\begin{aligned}
&\left|\int_{h(x)}^{\infty}\left(\hat{H}(-dt) - \frac{p}{m}\,dt\right)\left[\overline{F}(x+t) - \overline{F}(x+t+c)\right]\right| \\
&\qquad \le \varepsilon\sum_{k=0}^{\infty}\left[\overline{F}(x+h(x)+kc) - \overline{F}(x+h(x)+(k+2)c)\right] \\
&\qquad \le 2\varepsilon\overline{F}(x). \qquad (36)
\end{aligned}$$

The result (28) now follows from (33)–(36).

To show (29) note that, from (6)–(8),

$$\begin{aligned}
\mathbf{P}(T_2 \in (x, x+c]) &\ge \mathbf{P}(T_2 \in (x, x+c], \psi_1 \le h(x)) + \mathbf{P}(T_2 \in (x, x+c], \psi_2 \le h(x)) \\
&\sim 2\mathbf{P}(\psi \in (x, x+c]), \qquad \text{as } x \to \infty.
\end{aligned}$$

Hence the result (29) follows for $n = 2$ from (28); the result for general n now follows by induction arguments. Finally, the result (30) follows from Fatou's lemma and (29), since

$$\begin{aligned}
\liminf_{x\to\infty}\frac{\mathbf{P}(M \in (x, x+c])}{\overline{F}(x)} &= \liminf_{x\to\infty}\sum_{n\ge1}\mathbf{P}(\nu = n)\frac{\mathbf{P}(T_n \in (x, x+c])}{\overline{F}(x)} \\
&\ge \sum_{n\ge1}\mathbf{P}(\nu = n)\liminf_{x\to\infty}\frac{\mathbf{P}(T_n \in (x, x+c])}{\overline{F}(x)} \\
&\ge \frac{c}{m}.
\end{aligned}$$

□

We now prove the claim made in Remark 1. Suppose that the conditions of Theorem 1 (i) hold and that $\sigma < \infty$ a.s. Given $\varepsilon > 0$, we can find a positive integer K and $L > 0$ such that $\mathbf{P}(S_K > -L,\, \sigma \le K) \ge 1 - \varepsilon$. Then

$$\begin{aligned} \mathbf{P}(M > x,\, \sigma \le K) &\ge \mathbf{P}(S_K > -L,\, M > x,\, \sigma \le K) \\ &\ge (1-\varepsilon)\mathbf{P}(M > x + L) \qquad (37) \\ &= (1 + o(1))(1-\varepsilon)\mathbf{P}(M > x) \qquad \text{as } x \to \infty, \end{aligned}$$

where (37) above follows since the distribution of M is invariant under the obvious shift operation.

From the above, Theorem 1 (i), and Veraverbeke's Theorem (Veraverbeke (1977)),

$$\begin{aligned} \mathbf{P}(M > x) &\le \mathbf{P}(M_K > x) + \mathbf{P}(M > x,\, \sigma > K) \\ &\le (1 + o(1))(K\overline{F}(x) + \frac{\varepsilon}{m}\overline{F^s}(x)) \qquad \text{as } x \to \infty. \end{aligned}$$

Hence, since $\overline{F}(x) = o(\overline{F^s}(x))$ as $x \to \infty$, we have $\limsup_{x\to\infty} \mathbf{P}(M > x)/\overline{F^s}(x) \le \varepsilon/m$. Now let $\varepsilon \to 0$ to obtain the required result.

Finally, in the following further corollary to Theorem 1, we consider the extent to which Theorem 1 (i) and Corollary 2 remain true when the condition (NEG) is dropped.

Corollary 3. *Suppose that $F \in \mathcal{S}^*$ (which, by Corollary 1 implies that F is subexponential and so satifies* (LT)*), that the corresponding distribution of the random variables ξ_n has a finite mean, but that* (NEG) *does not necessarily hold. Let $\sigma < \infty$ be a stopping time such that, for some function h satisfying* (8),

$$\mathbf{P}(\sigma > h(x)) = o(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{38}$$

Then, again,

$$\lim_{x\to\infty} \frac{\mathbf{P}(M_\sigma > x)}{\overline{F}(x)} = \mathbf{E}\sigma. \tag{39}$$

If, additionally, σ is independent of the sequence $\{\xi_n\}$, then (39) *also holds with M_σ replaced by S_σ.*

Proof. Choose any $a > \mathbf{E}\xi$. For each $n \ge 0$, define $\tilde{S}_0 = 0$, $\tilde{S}_n = \sum_{i=1}^n (\xi_i - a)$ (with $\tilde{S}_0 = 0$) and $\tilde{M}_n = \max_{0\le i\le n} \tilde{S}_i$. Then, for each x,

$$\begin{aligned} \mathbf{P}(M_\sigma > x) &\le \mathbf{P}(\tilde{M}_\sigma + a\sigma > x) \\ &\le \mathbf{P}(a\sigma > ah(x)) + \mathbf{P}(\tilde{M}_\sigma > x - ah(x)) \\ &\sim \mathbf{E}\sigma\overline{F}(x - ah(x)) \qquad \text{as } x \to \infty \qquad (40) \\ &\sim \mathbf{E}\sigma\overline{F}(x) \qquad \text{as } x \to \infty. \qquad (41) \end{aligned}$$

where (40) follows from Theorem 1 (i) and (38), while (41) follows since we may clearly replace $h(x)$ by $ah(x)$ in (8). Hence the result (39) is established. The final assertion follows as in the proof of Corollary 2. □

Note that the condition (38) need not be unduly restrictive. For example, in the regularly varying case $F(x) = x^{\alpha}L(x)$ for some function L which is slowly varying at infinity and some $\alpha < -1$ (for a finite mean), for a function h to satisfy (8) it is sufficient that $h(x) = o(x)$ as $x \to \infty$. Hence the tail of the stopping time σ need only be slightly lighter than that of the random variables ξ_n.

References

[1] Asmussen, S. (1998). Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities. *Ann. Appl. Probab.*, **8**, 354–374.

[2] Asmussen, S. (1999). Semi-Markov queues with heavy tails. In *Semi-Markov Models and Applications*, eds J. Janssen and N Limnios. Kluwer.

[3] Asmussen, S., Foss, S.G., and Korshunov, D.A. (2002). Asymptotics for sums of random variables with local subexponential behaviour. To appear.

[4] Asmussen, S., Kalashnikov, V., Konstantinides, D., Klüppelberg, C. and Tsitiashvili, G. (2001). A local limit theorem for random walk maxima with heavy tails. To appear.

[5] Borovkov, A.A. and Borovkov, K.A. (2001). On large deviation probabilities for random walks. I. Regularly varying distribution tails. II. Regularly exponential distribution tails. *Theory Probab. Appl.*, **46**, 2, 209–232. (In Russian.)

[6] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events.* Springer Verlag.

[7] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Mathematics. Econom.*, **1**, 55–72.

[8] Feller, W. (1971). *An Introduction to Probability Theory and its Applications.* Wiley.

[9] Greiner, M., Jobmann, M. and Klüppelberg, C. (1999). Telecommunication traffic, queueing models, and subexponential distributions. *Queueing Systems*, **33**, 125–152.

[10] Heath, D., Resnick, S. and Samorodnitsky, G. (1997). Patterns of buffer overflow in a class of queues with long memory in the input stream. *Ann. Appl. Probab.*, **7**, 1021–1057.

[11] Klüppelberg, C. (1988). Subexponential distributions and integrated tails. *J. Appl. Prob.*, **35**, 325–347.

[12] Korshunov, D.A. (1997). On distribution tail of the maximum of a random walk. *Stoch. Proc. Appl.*, **72**, 1, 97–103.

[13] Korshunov, D.A. (2001). Large-deviation probabilities for maxima of sums of independent random variables with negative mean and subexponential distribution. *Theory Probab. Appl.*, **46**, 2, 387–397. (In Russian.)

[14] Sigman, K. (1999). A primer on heavy-tailed distributions. *Queueing Systems*, **33**, 261–275.

[15] Veraverbeke, N. (1977). Asymptotic behavior of Wiener-Hopf factors of a random walk. *Stoch. Process. Appl.*, **5**, 27–37.

S. Foss
Department of Actuarial
Mathematics and Statistics
Heriot-Watt University
Riccarton
Edinburgh
UK, EH14 4AS
Email: s.foss@ma.hw.ac.uk

S. Zachary
Department of Actuarial
Mathematics and Statistics
Heriot-Watt University
Riccarton
Edinburgh
UK, EH14 4AS
Email: s.zachary@ma.hw.ac.uk