Asymptotics for the maximum of a modulated random walk with heavy-tailed increments

Serguei Foss and Stan Zachary

Abstract

We consider asymptotics for the maximum of a modulated random walk whose increments $\xi_n^{X_n}$ are heavy-tailed. Of particular interest is the case where the modulating process X is regenerative. Here we study also the maximum of the recursion given by $W_0 = 0$ and, for $n \geq 1$, $W_n = \max(0, W_{n-1} + \xi_n^{X_n})$.

1 Introduction

Let $S_n = \sum_1^n \xi_i$ be a sum of i.i.d. random variables (r.v.s) with a negative finite mean $\mathbf{E}\xi_1 = -a < 0$. The common distribution of the random variables ξ_n is assumed to be right heavy-tailed (i.e. $\mathbf{E}\exp(\lambda\xi_1) = \infty$ for all $\lambda > 0$). Moreover, the second tail of this distribution is assumed to be subexponential (see Section 2 for definitions). Then the classical result (see, e.g., [12]) states that, as $y \to \infty$,

$$\mathbf{P}(\sup_n S_n > y) \sim \frac{1}{a} \int_y^\infty \mathbf{P}(\xi_1 > t) dt. \tag{1}$$

We consider here a more general random walk

$$S_n = \sum_1^n \xi_i^{X_i} \tag{2}$$

whose increments $\xi_n^{X_n}$ are modulated by an independent sequence $X = \{X_n\}_{n\geq 1}$ (see Section 2 for more precise definitions and notation). We assume that $S_n \to -\infty$ a.s. and find conditions which are sufficient for the probability of the "rare" event $\mathbf{P}(\sup_n S_n > y)$ to behave asymptotically (as $y \to \infty$) similarly to (1). The results obtained may be applied to the study of complex stochastic models with modulated input.

Particular cases, with X a finite Markov chain, were considered in [2] and [1]. S. Asmussen ([4]) proposed an approach for getting the asymptotics for $\mathbf{P}(\sup_n S_n > y)$ on the basis of a regenerative structure: if the maximum of the partial sums over a typical cycle behaves asymptotically as the end-to-end sum, and these asymptotics are subexponential, then the result (1) stays the same. In [5], the authors assumed that X is countably-valued, a certain dependence between the X_n and the ξ_n^x was allowed, and some homogeneity in x of the distributions of the random variables ξ_n^x was required. By the use of matrix-analytic methods, they found the asymptotics for the stationary distribution of a Markov chain with increments $\xi_n^{X_n}$.

In [7], upper and lower bounds were found for the asymptotics of $\mathbf{P}(R > y)$, as $y \to \infty$, where R is the stationary response time in a tandem queue. Then, in [9], the asymptotics for the stationary waiting time W in the second queue were studied. Note that W may be represented as the limit of a recursion

$$W_n = \max(0, W_{n-1} + \xi_n^{X_n})$$

where $X = \{X_n\}$ forms a Harris ergodic Markov chain. In [6], the exact asymptotics for $\mathbf{P}(R > y)$ were found. The proof is based on ideas similar to that of Lemma 2 of the present paper.

Finally, nice overviews on the current state of large deviations theory in the presence of heavy tails were given in [10] and in recent new books [8] and [3].

We state our main results in Section 2. We consider in particular the case where the modulating process X is regenerative, where we give also an instructive example and counterexample. The latter shows our conditions on the tail of the distribution of the regeneration time to be best possible—in a sense made clear there. We study also the queueing theory recursion given by $W_0 = 0$ and, for $n \geq 1$, $W_n = \max(0, W_{n-1} + \xi_n^{X_n})$.

In Section 3 we collect together some useful known results, most of which are required for our proofs. These are given in Section 4. Perhaps the key result of the entire paper is Lemma 2 of that section, which develops an idea found also in [6].

2 The main results

Let $(\mathcal{X}, \mathcal{B})$ be a measurable space and $X = \{X_n\}_{n\geq 1}$ an $\mathcal{X}$-valued discrete-time random process. Let $P : \mathcal{X} \times \mathcal{B}_0 \to [0, 1]$ (where $\mathcal{B}_0$ is the Borel σ-algebra on $\mathbb{R}$) be a function such that

(i) for every $x \in \mathcal{X}$, $P(x, \cdot)$ is a probability measure;

(ii) for every $B \in \mathcal{B}_0$, $P(\cdot, B)$ is a measurable function.

For each $x \in \mathcal{X}$, let F_x denote the distribution function of $P(x, \cdot)$. For each integer $n \geq 1$, introduce the family of real-valued random variables $\{\xi_n^x\}_{x\in\mathcal{X}}$. Assume that these families are mutually independent (in n), do not depend on the process X, and that, for each $x \in \mathcal{X}$ and each n, ξ_n^x has distribution function F_x. We define the *random walk* $\{S_n\}_{n\geq 0}$ *modulated by the process* X by $S_0 = 0$ and, for any $n = 1, 2, \ldots$,

$$S_n = \sum_{i=1}^{n} \xi_i^{X_i}.$$

Define also, for $n \geq 1$,

$$M_n = \max_{0 \leq i \leq n} S_i,$$

and let

$$M = \sup_{n \geq 0} S_n.$$

Further, for each $y > 0$, define

$$\mu(y) = \min\{n \geq 1 : \; S_n > y\}.$$

Note that $\mu(y) = \infty$ if and only if $M \leq y$.

We are interested the asymptotics of the upper-tail distribution of M under conditions which guarantee that the random walk S_n behaves sufficiently regularly and has a strictly negative drift, and where additionally the distribution functions F_x have, in some appropriate sense, heavy positive tails. More precisely, we wish to make statements, under such conditions, about the behaviour, for any $B \in \mathcal{B}$ and as $y \to \infty$, of $\mathbf{P}(M > y, \, X_{\mu(y)} \in B)$.

Motivated by queueing theory applications, we are also interested in the behaviour of the process $\{W_n\}_{n \geq 0}$ defined recursively by $W_0 = 0$ and, for $n \geq 1$,

$$W_n = \max(0, W_{n-1} + \xi_n^{X_n}). \tag{3}$$

We assume throughout that P is such that each probability measure $P(x, \cdot)$ (i.e. each distribution F_x) has a finite mean. We further assume throughout that there exist a distribution function F on $\mathbb{R}_+$ with finite mean, and a measurable function $c : \mathcal{X} \to \mathbb{R}_+$ such that

$$\overline{F}_x(y) \sim c(x)\overline{F}(y) \qquad \text{as } y \to \infty, \qquad \text{for all } x \in \mathcal{X}, \tag{4}$$

$$\sup_x \sup_{y \geq 0} \frac{\overline{F}_x(y)}{\overline{F}(y)} = L, \qquad \text{for some } L < \infty. \tag{5}$$

Here, for any distribution function H on $\mathbb{R}$, $\overline{H}$ denotes the tail distribution given by $\overline{H}(y) = 1 - H(y)$.

The following two conditions on the pair (X, P) will be satisfied in all our results, either by hypothesis or as a consequence of more fundamental modelling assumptions. (We show below that these conditions may arise naturally in the case where the process X is regenerative, but they may also arise in other contexts.)

(C1) There exists some probability distribution π on $(\mathcal{X}, \mathcal{B})$ such that, for some positive integer d,

$$\frac{\mathbf{P}(X_n \in \cdot) + \ldots + \mathbf{P}(X_{n+d-1} \in \cdot)}{d} \to \pi(\cdot), \qquad \text{as } n \to \infty, \tag{6}$$

in total variation norm. Here define also

$$C(B) = \int_B c(x)\pi(dx), \qquad B \in \mathcal{B}, \tag{7}$$

and put $C = C(\mathcal{X})$.

(C2) The pair (X, P) is such that

$$\lim_{n\to\infty} \frac{S_n}{n} = -a, \qquad \text{a.s., for some } a > 0. \tag{8}$$

We need to recall the following definitions. For any distribution function H on $\mathbb{R}$, the integrated, or second-tail, distribution H^s is given by

$$\overline{H}^s(y) = \min\left(1, \int_y^\infty \overline{H}(t)dt\right).$$

A distribution function H on $\mathbb{R}_+$ is *long-tailed* if $\overline{H}(y) > 0$ for all y and, for any fixed $z > 0$,

$$\frac{\overline{H}(y+z)}{\overline{H}(y)} \to 1 \qquad \text{as } y \to \infty.$$

A distribution function H on $\mathbb{R}_+$ is *subexponential* if $\overline{H}(y) > 0$ for all y and

$$\frac{\overline{H}^{*2}(y)}{\overline{H}(y)} \to 2 \qquad \text{as } y \to \infty.$$

(where H^{*2} is the convolution of H with itself). It is well known that any subexponential distribution is long-tailed.

We now have Theorems 1 and 2 below.

Theorem 1. *Assume that the conditions (C1) and (C2) hold and that F^s is long-tailed. Then*

$$\liminf_{y\to\infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} \geq \frac{C(B)}{a}, \qquad \textit{for all } B \in \mathcal{B}. \tag{9}$$

Theorem 2. *Assume that the conditions (C1) and (C2) hold, that F^s is subexponential, and that there exists a distribution function G with negative mean*

$$m(G) \equiv \int_{-\infty}^{\infty} t\, dG(t) < 0 \tag{10}$$

such that $\overline{F}_x(y) \leq \overline{G}(y)$ for all x and y. Then

$$\lim_{y\to\infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} = \frac{C(B)}{a}, \qquad \textit{for all } B \in \mathcal{B}. \tag{11}$$

Remark 1. In an obvious sense, the best candidate for the distribution function G in Theorem 2 is the right-continuous version of $\tilde{G}(y) = \sup_x F_x(y)$.

As mentioned above, the conditions (C1) and (C2) are frequently satisfied in applications. Perhaps the most common instance occurs in the case where X is a *regenerative process*. Here there exists an increasing integer-valued random sequence $0 \le T_0 < T_1 < T_2 < \dots$ a.s., such that, if $\tau_0 = T_0$, $\tau_n = T_n - T_{n-1}$, $n \ge 1$, then

$$\begin{aligned} Z_0 &= \{\tau_0; X_1, \dots, X_{T_0}\}, \\ Z_n &= \{\tau_n; X_{T_{n-1}+1}, \dots, X_{T_n}\}, \qquad n \ge 1, \end{aligned}$$

are mutually independent for $n \ge 0$ and identically distributed for $n \ge 1$. Here τ_0 is the length of the 0th cycle, $\tau \equiv \tau_1$ the length of the first cycle, etc; in particular, if $\tau_0 = 0$, then the 0th cycle is empty.

Assume also, for X regenerative as above, that $\mathbf{E}\tau$ is finite. Then the condition (C1) is automatically satisfied with

$$d = \mathrm{GCD}\{n : \ \mathbf{P}(\tau = n) > 0\}$$

and the probability measure π given by

$$\pi(B) = \mathbf{E}\left(\sum_{i=T_0+1}^{T_1} \mathbf{I}(X_i \in B)\right), \qquad B \in \mathcal{B}, \tag{12}$$

where $\mathbf{I}$ is the indicator function. Suppose also that the modulated random walk S_n is constructed as above, that

$$\int_{\mathcal{X}} \mathbf{E}(|\xi_1^x|)\pi(dx) < \infty, \tag{13}$$

and that

$$\int_{\mathcal{X}} \mathbf{E}\xi_1^x \, \pi(dx) = -a, \qquad \text{for some } a > 0. \tag{14}$$

Then, it is an elementary exercise, using the Strong Law of Large Numbers, to show that the condition (C2) is satisfied with a as given by (14).

Example 1. For a particular example of such a random walk modulated by a regenerative process, consider a stable tandem queue $GI/GI/1 \to GI/1$ which is defined by three mutually independent sequences $\{t_n\}$, $\{\sigma_n^{(1)}\}$, and $\{\sigma_n^{(2)}\}$ of i.i.d. random variables with $\mathbf{E}t_1 > \max\{\mathbf{E}\sigma_1^{(1)}, \mathbf{E}\sigma_1^{(2)}\}$. Here the t_n are the inter-arrival times at the first queue, while the $\sigma_n^{(1)}$ and the $\sigma_n^{(2)}$ are the service times at the first and second queues respectively. Let $\{X_n\}$ be the sequence of inter-departure times from the first queue. Then this sequence is regenerative (the regeneration indices corresponding to those customers who arrive to find the first queue empty), and the distribution of X_n converges in the total variation norm to a stationary distribution with mean $\mathbf{E}t_1$. Consider the sequence $\xi_n^{X_n} = \sigma_n^{(2)} - X_n$. Under natural conditions (see Theorems 3–5 below), we can show that the tail distribution of the supremum of a modulated random walk with increments $\xi_n^{X_n}$ asymptotically coincides with that of the stationary waiting time in the second queue.

In the case of a regenerative process as above we obtain, in Theorems 3 and 4 below, the conclusion (11) of Theorem 2 under weaker conditions than that given by (10). In each case the cost is that of suitable conditions on the distributions of the cycle times τ_0 and τ. Both the theorems are adapted to typical queueing theory applications.

Theorem 3. *Assume that X is regenerative with $\mathbf{E}\tau < \infty$, that the conditions* (13) *and* (14) *hold (with π as given by* (12)*), and that F^s is subexponential. Assume also that*

$$\mathbf{P}(b\tau_0 > y) = o(\overline{F}^s(y)), \quad \mathbf{P}(b\tau > y) = o(\overline{F}(y)), \quad \textit{as } y \to \infty, \tag{15}$$

for all $b > 0$. Then the conclusion (11) *of Theorem 2 again follows.*

Remark 2. As already discussed, the assumptions of Theorem 3 ensure that the earlier conditions (C1) and (C2) are satisfied.

Remark 3. It will follow from the proof of Theorem 3 that it is enough to assume that the condition (15) holds for a certain sufficiently large b.

Remark 4. The condition (15) holds for all $b > 0$ if there exists some $\lambda > 0$ such that both $\mathbf{E}\exp(\lambda\tau)$ and $\mathbf{E}\exp(\lambda\tau_0)$ are finite.

Under the conditions of Theorem 4 we relax the requirement that the condition (15) hold for all $b > 0$. This requirement may fail to be satisfied in some examples where the conditions of Theorem 4 are, however, quite natural—see, e.g., Example 1.

Theorem 4. *Assume that X is regenerative with $\mathbf{E}\tau < \infty$, that the conditions* (13) *and* (14) *hold, and that F^s is subexponential. Assume also that there exists a family $\{G_x\}_{x\in\mathcal{X}}$ of distribution functions on $\mathbb{R}_+$ such that $G_x(y)$ is measurable in x for all y,*

$$\overline{F}_x(y) \le \overline{G}_x(y) \quad \textit{for all } x \textit{ and for all } y, \tag{16}$$

and, for each x, G_x may be represented as a distribution function of a difference of two independent r.v.s

$$G_x(y) = \mathbf{P}(\zeta - b^x \le y), \tag{17}$$

where the distribution of ζ does not depend on x,

$$\limsup_{y\to\infty} \frac{\mathbf{P}(\zeta > y)}{\overline{F}(y)} < \infty, \tag{18}$$

b^x is non-negative a.s., and

$$\mathbf{E}_\pi b^X \equiv \int \mathbf{E} b^x \, \pi(dx) > \mathbf{E}\zeta. \tag{19}$$

Assume further that the condition (15) *holds for some $b > \mathbf{E}\zeta$. Then the conclusion* (11) *of Theorem 2 follows once more.*

Remark 5. The assumptions on X in Theorem 4 again ensure that the earlier condition (C1) is satisfied, while it follows from (16), (17) and (19) that the condition (14), and so the condition (C2), is satisfied.

Remark 6. It is easy to see that the asymptotics for $\mathbf{P}(M > y)$ may be quite different from (11) if the assumption (15) fails. For instance, assume that the remaining conditions of Theorem 4 hold with $\mathcal{X} = \mathcal{R}$, $F_x = G_x$ for all x, $\zeta \geq 1$ a.s., and $b^x \equiv 0$ for all $x \neq 0$. Assume also that $X_0 = 0$, $T_0 = 0$, $\tau \equiv \tau_1 = \min\{n > 0\colon X_n = 0\}$. The condition (19) here becomes $\mathbf{E}b^0 > \mathbf{E}\zeta\mathbf{E}\tau$. Then

$$M \geq \max(\tau_1 - 1, \tau_1 - b^0_{\tau_1} + \tau_2 - 1, \ldots) \equiv M^*.$$

If the second tail for $\mathbf{P}(\tau > t)$ is subexponential (and so also long-tailed), then, by (1),

$$\mathbf{P}(M^* > y) \sim \frac{1}{\mathbf{E}(b^0 - \tau)} \int_y^\infty \mathbf{P}(\tau > t)dt.$$

Finally, again in the case where X is regenerative, we consider the process $\{W_n\}_{n\geq 0}$ defined earlier by (3). In the special case where $F_x = F$ for all $x \in \mathcal{X}$ (so that $\{\xi_n\}$ is an i.i.d. sequence), it is well known that the distributions of W_n and S_n coincide. However, this is not generally the case in the present setting.

Theorem 5. *Assume that X is regenerative with $\mathbf{E}\tau < \infty$. Then, under the conditions of either Theorem 2, Theorem 3 or Theorem 4,*

$$\lim_{y\to\infty} \frac{1}{\overline{F}^s(y)} \limsup_{n\to\infty} \mathbf{P}(W_n > y) = \lim_{y\to\infty} \frac{1}{\overline{F}^s(y)} \liminf_{n\to\infty} \mathbf{P}(W_n > y) = \frac{C}{a}. \tag{20}$$

Remark 7. In fact, under the conditions of Theorem 3 or 4, the condition on τ_0 (in (15)) is not required for Theorem 5.

3 Useful Properties

We recall some known properties of distributions. For any distribution function G on $\mathbb{R}$ let

$$m(G) \equiv \int_{-\infty}^{\infty} t\, dG(t)$$

denote its mean. Further, we make the convention that, for distribution functions G and H, we write $\overline{H}(y) \sim 0 \cdot \overline{G}(y)$ if $\overline{H}(y) = o(\overline{G}(y))$ as $y \to \infty$.

Property 1. *Suppose that distribution functions G and H are such that $m(G)$ is finite, $m(H) = -h$ for some $h > 0$, and $\overline{H}(y) = o(\overline{G}(y))$ as $y \to \infty$. Then, for any $\varepsilon > 0$ we can find a distribution function H_ε such that $\overline{H}(y) \leq \overline{H}_\varepsilon(y)$ for all y, $m(H_\varepsilon) \leq -h/2$ and $\overline{H}_\varepsilon(y) = \varepsilon\overline{G}(y)$ for all sufficiently large y.*

Property 2. *Suppose that distribution functions G and H are such that G^s exists, and that $\overline{H}(y) \sim c\overline{G}(y)$ as $y \to \infty$ for some $c \geq 0$. Then H^s exists and $\overline{H}^s(y) \sim c\overline{G}^s(y)$ as $y \to \infty$.*

Property 3. *Suppose that a distribution function G is such that its second tail distribution G^s is long-tailed. Then*

$$\overline{G}(y) = o(\overline{G}^s(y)) \quad \text{as} \quad y \to \infty. \tag{21}$$

Further, for any $g > 0$ and any sequence $\{\alpha_n\}$ such that $\alpha_n \to \alpha$ as $n \to \infty$,

$$\lim_{y\to\infty} \frac{1}{\overline{G}^s(y)} \sum_{n=k}^{\infty} \alpha_n \overline{G}(y + l + ng) = \frac{\alpha}{g} \qquad \text{for all } k \text{ and for all } l. \tag{22}$$

Property 4. *Suppose that distribution functions G and H are such that $\overline{H}(y) \sim c\overline{G}(y)$ as $y \to \infty$ for some $c > 0$. Then if G is subexponential, H is subexponential, while if G^s subexponential, H^s is subexponential and $\overline{H}^s(y) \sim c\overline{G}^s(y)$ as $y \to \infty$.*

Property 5. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be n mutually independent r.v.s and G a subexponential distribution such that, for $i = 1, 2, \ldots, n$, $\mathbf{P}(\xi_i > y) \sim c_i\overline{G}(y)$ as $y \to \infty$, where $c_1, c_2, \ldots, c_n \geq 0$. Then*

$$\mathbf{P}(\xi_1 + \xi_2 + \ldots + \xi_n > y) \sim (c_1 + c_2 + \ldots + c_n)\overline{G}(y) \qquad \text{as } y \to \infty.$$

Property 6. *Let $\{\xi_n\}_{n\geq 1}$ be an i.i.d. sequence of nonnegative random variables with subexponential distribution G. For any n, put*

$$\alpha_n = \sup_{y\geq 0} \frac{\mathbf{P}(\xi_1 + \ldots + \xi_n > y)}{\mathbf{P}(\xi_1 > y)} \equiv \sup_{y\geq 0} \frac{\overline{G^{*n}}(y)}{\overline{G}(y)}.$$

Then, for any $u > 0$ one can choose $k > 0$ such that $\alpha_n \leq k(1+u)^n$ for all n.

Property 7 (Veraverbeke's Theorem). *Let $\{\xi_n\}_{n\geq 1}$ be an i.i.d. sequence of random variables with distribution function G and a negative mean $-g = \mathbf{E}\xi_1 < 0$. Suppose that the second-tail distribution G^s is subexponential. Set $S'_n = \sum_{i=1}^n \xi_i$, and $M' = \max(0, \sup_n S'_n)$. Then, as $y \to \infty$,*

$$\mathbf{P}(M' > y) \sim \mathbf{P}\left(\bigcup_{n\geq 1}\{\xi_n > y + ng\}\right) \sim \sum_{n\geq 1} \mathbf{P}(\xi_n > y + ng) \sim \frac{1}{g}\overline{G}^s(y). \tag{23}$$

Thus, under the conditions of Veraverbeke's Theorem, the supremum M' is large if and only if one of summands is large. The following three properties are all corollaries of Veraverbeke's Theorem. In particular Property 9 follows easily on using also Property 1 above.

Property 8. *Under the conditions of Veraverbeke's Theorem above, for any $\tilde{g} \in (0, g)$,*

$$\sum_{n=1}^{\infty} \mathbf{P}(M'_n \le y, S'_n \in (-n\tilde{g}, y], S'_{n+1} > y) = o(\overline{G}^s(y)) \qquad \text{as } y \to \infty,$$

where, for each n, $M'_n = \max(0, \max_{1 \le i \le n} S'_i)$.

Property 9. *Let $\{\xi_n\}_{n \ge 1}$ be an i.i.d. sequence of random variables with distribution function H and negative mean $\mathbf{E}\xi_1 < 0$. Suppose that $\overline{H}(y) = o(\overline{G}(y))$, as $y \to \infty$, for some distribution function G whose second-tail distribution G^s is subexponential. Set $S'_n = \sum_{i=1}^n \xi_i$ and $M' = \max(0, \sup_n S'_n)$. Then*

$$\mathbf{P}(M' > y) = o(\overline{G}^s(y)) \qquad \text{as } y \to \infty.$$

Property 10. *Let $\{\xi_n\}_{n \ge 1}$ be an i.i.d. sequence of random variables with distribution function H and negative mean $\mathbf{E}\xi_1 < 0$. Suppose that $\overline{H}(y) \sim c(\overline{G}(y))$, as $y \to \infty$, for some $c \ge 0$ and some distribution function G whose second-tail distribution G^s is subexponential. Let τ' be an independent positive integer-valued random variable. Then*

$$\mathbf{P}\left(\max_{1 \le n \le \tau'} \sum_{i=1}^{n} \xi_i > y\right) = o(\overline{G}^s(y)) \qquad \text{as } y \to \infty.$$

4 Proofs

Proof of Theorem 1. We prove the theorem in the case where the constant d of the condition (C1) is equal to 1. The modification required for the general case is quite obvious. By the Strong Law of Large Numbers, for any $\varepsilon \in (0, a)$, we can choose $R \equiv R(\varepsilon)$ such that

$$\mathbf{P}\left(S_n \in [-R - n(a+\varepsilon), R - n(a-\varepsilon)] \quad \text{for all } n = 0, 1, 2, \ldots\right) \ge 1 - \varepsilon.$$

Put

$$D_n = \{S_i \in [-R - i(a+\varepsilon), R - i(a-\varepsilon)] \quad \text{for all } i = 0, 1, 2, \ldots, n\}$$

and $D \equiv D_\infty$. Since $D_\infty \subseteq D_n$ for all n, $\mathbf{P}(D_n) \ge 1 - \varepsilon$.

Now, for all sufficiently large $y > 0$,

$$
\begin{aligned}
&\mathbf{P}(M > y, X_{\mu(y)} \in B) \\
&= \sum_{n=0}^{\infty} \mathbf{P}(M_n \le y, S_{n+1} > y, X_{n+1} \in B) \\
&\ge \sum_{n=0}^{\infty} \mathbf{P}(D_n, S_{n+1} > y, X_{n+1} \in B) \\
&\ge \sum_{n=0}^{\infty} \int_B \mathbf{P}(D_n, X_{n+1} \in dx) \overline{F}_x(y + R + n(a+\varepsilon)) \\
&\ge \sum_{n=0}^{\infty} \left[\int_B \mathbf{P}(X_{n+1} \in dx) \overline{F}_x(y + R + n(a+\varepsilon)) - \mathbf{P}(\overline{D}) L \overline{F}(y + R + n(a+\varepsilon)) \right] \\
&\ge \sum_{n=0}^{\infty} \left[\int_B \pi(dx) \overline{F}_x(y + R + n(a+\varepsilon)) - \left(\mathbf{P}(\overline{D}) + \delta_{n+1}\right) L \overline{F}(y + R + n(a+\varepsilon)) \right],
\end{aligned}
$$

where, for each n, $\delta_n = \sup_B |\mathbf{P}(X_n \in B) - \pi(B)|$ is the distance in total variation between the distributions of X_n and π. The condition (C1) implies that $\delta_n \to 0$ as $n \to \infty$. Since F^s is long-tailed, it now follows from (4), (5) and (22) that, for all x,

$$
\lim_{y\to\infty} \frac{1}{\overline{F}^s(y)} \sum_{n=0}^{\infty} \overline{F}_x(y + R + n(a+\varepsilon)) = \frac{c(x)}{a+\varepsilon}
$$

and that

$$
\begin{aligned}
\limsup_{y\to\infty} \frac{1}{\overline{F}^s(y)} \sum_{n=0}^{\infty} \overline{F}_x(y + R + n(a+\varepsilon)) &\le \lim_{y\to\infty} \frac{L}{\overline{F}^s(y)} \sum_{n=0}^{\infty} \overline{F}(y + R + n(a+\varepsilon)) \\
&= \frac{L}{a+\varepsilon}
\end{aligned} \tag{24}
$$

(where the convergence to the limit above is of course independent of x). Hence, by the Bounded Convergence Theorem,

$$
\lim_{y\to\infty} \frac{1}{\overline{F}^s(y)} \sum_{n=0}^{\infty} \int_B \pi(dx) \overline{F}_x(y + R + n(a+\varepsilon)) = \frac{C(B)}{a+\varepsilon}. \tag{25}
$$

Also, again from (22),

$$
\lim_{y\to\infty} \frac{1}{\overline{F}^s(y)} \sum_{n=0}^{\infty} \delta_{n+1} \overline{F}(y + R + n(a+\varepsilon)) = 0.
$$

Thus, again using (24),

$$
\liminf_{y\to\infty} \frac{1}{\overline{F}^s(y)} \mathbf{P}(M > y, X_{\mu(y)} \in B) \ge \frac{C(B) - L\mathbf{P}(\overline{D})}{a+\varepsilon}.
$$

Now let $\varepsilon \to 0$ to obtain the required result. □

We now give two lemmas which are required for the remaining results.

Lemma 1. *Suppose that the conditions of Theorem 1 hold and that*

$$\limsup_{y\to\infty} \frac{\mathbf{P}(M > y)}{\overline{F}^s(y)} \le \frac{C}{a}. \tag{26}$$

Then the conclusion (11) *follows.*

Proof. From (26), for any $B \in \mathcal{X}$,

$$\begin{aligned}
\frac{C}{a} &\ge \limsup_{y\to\infty} \frac{\mathbf{P}(M > y)}{\overline{F}^s(y)} \\
&= \limsup_{y\to\infty} \left(\frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \frac{\mathbf{P}(M > y, X_{\mu(y)} \in \overline{B})}{\overline{F}^s(y)} \right) \\
&\ge \limsup_{y\to\infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \liminf_{y\to\infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in \overline{B})}{\overline{F}^s(y)} \\
&\ge \limsup_{y\to\infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \frac{C(\overline{B})}{a},
\end{aligned}$$

where the last inequality follows by Theorem 1. Since $C = C(B) + C(\overline{B})$, the conclusion (11) follows as required. □

In each of the proofs of Theorems 2, 3 and 4 we show that, for all ε satisfying $0 < \varepsilon < a$, there exists $R > 0$ (depending on ε) such that, if, for each $n = 1, 2, \ldots$,

$$D'_n \equiv \{S_j \le R - j(a - \varepsilon) \quad \text{for all } j = 1, \ldots, n-1;\ S_{n+i} - S_n \le R \quad \text{for all } i = 1, 2, \ldots\}, \tag{27}$$

then $\mathbf{P}(D'_n) > 1 - \varepsilon$ for all n. In each case we then require Lemma 2 below to complete the proof.

Lemma 2. *Suppose that F^s is subexponential, that there exist a sequence of i.i.d. random variables $\{\psi_n\}_{n\ge1}$ and a constant L_1 such that $\mathbf{E}\psi_1 < 0$ and*

$$\mathbf{P}(\psi_1 > y) \le L_1 \overline{F}(y) \tag{28}$$

for all $y \ge 0$, and that ψ_n is independent of D'_n for all $n \ge 1$. Suppose further that the condition (C1) is satisfied and that

$$\mathbf{P}(M > y) \le \mathbf{P}(M > y, M^\psi > y) + o(\overline{F}^s(y)) \quad \text{as } y \to \infty, \tag{29}$$

where $M^\psi = \max(0, \sup_n \sum_{i=1}^n \psi_i)$. Then the conclusion (11) *follows.*

Proof. As in the proof of Theorem 1, we assume that the constant d of the condition (C1) is equal to 1. We may further assume, without loss of generality, that the condition (28) is satisfied with equality for all sufficiently large y. (If this is not the case we can use Property 1 of Section 3 to replace $\{\psi_n\}_{n\geq 1}$ with i.i.d. sequence $\{\tilde{\psi}_n\}_{n\geq 1}$ satisfying all the conditions of the lemma and with also the required equality in (28).) It follows that the common distribution of the random variables ψ_n has a second-tail distribution which is subexponential. Thus, if $g = -\mathbf{E}(\psi_1)$ (so $g > 0$), it follows from the conditions of the lemma and Veraverbeke's Theorem that

$$\begin{aligned}
\mathbf{P}(M > y) &\leq \mathbf{P}(M > y, M^{\psi} > y) + o(\overline{F}^s(y)) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(M > y, \psi_n > y + ng) + o(\overline{F}^s(y)) \\
&\leq \Sigma_1 + \Sigma_2 + o(\overline{F}^s(y)),
\end{aligned} \tag{30}$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} \mathbf{P}(D'_n, M > y), \qquad \Sigma_2 = \sum_{n=1}^{\infty} \mathbf{P}(\overline{D}'_n, M > y, \psi_n > y + ng).$$

Since, for each n, ψ_n is independent of D'_n, we have, using (22),

$$\Sigma_2 \leq \sum_n \mathbf{P}(\overline{D}'_n)\mathbf{P}(\psi_n > y + ng) \leq (1 + o(1))\frac{\varepsilon L_1}{g}\overline{F}^s(y) \quad \text{as } y \to \infty. \tag{31}$$

We now consider Σ_1. Take $y > R$. For any n, the event

$$V_n \equiv D'_n \cap \{\xi_n^{X_n} \leq y - 2R + (n-1)(a - \varepsilon)\} \subseteq \{M \leq y\}. \tag{32}$$

To see this, note that, on the set V_n, $S_j \leq R - j(a - \varepsilon)$ for all $j < n$,

$$S_n = S_{n-1} + \xi_n^{X_n} \leq y - R$$

and, for $i = 1, 2, \ldots$,

$$S_{n+i} = S_n + (S_{n+i} - S_n) \leq y - R + R = y.$$

Thus, from (32),

$$\begin{aligned}
\Sigma_1 &\leq \sum_n \mathbf{P}(\xi_n^{X_n} > y - 2R + (n-1)(a - \varepsilon)) \\
&= \sum_n \int_{\mathcal{X}} \mathbf{P}(X_n \in dx)\overline{F}_x(y - 2R + (n-1)(a - \varepsilon)) \\
&\leq \sum_n \int_{\mathcal{X}} \pi(dx)\overline{F}_x(y - 2R + (n-1)(a - \varepsilon)) + L\sum_n \delta_n \overline{F}(y - 2R + (n-1)(a - \varepsilon)),
\end{aligned}$$

where, as in the the proof of Theorem 1, δ_n is the distance in total variation between the distributions of X_n and π, and so tends to 0 as $n \to \infty$. Exactly as in that proof, it now follows from (22) and the Bounded Convergence Theorem that

$$\limsup_{y\to\infty} \frac{\Sigma_1}{\overline{F}^s(y)} \le \frac{C}{a-\varepsilon}.$$

It now follows, on recalling (30) and (31) and letting $\varepsilon \to 0$, that the condition (26) of Lemma 1 is satisfied. The required conclusion (11) now follows from that lemma. □

Proof of Theorem 2. It follows from the condition (5) that, without loss of generality, we can assume that $\overline{G}(y) \le L\overline{F}(y)$ for all y. Let $\{\alpha_n\}_{n\ge1}$ be an i.i.d. sequence of random variables uniformly distributed on $(0,1)$ and independent of $X = \{X_n\}$. Construct the required family of random variables $\{\xi_n^x\}_{n\ge1}$ by defining, for each n, $\xi_n^x = F_x^{-1}(\alpha_n)$; for each n define also $\psi_n = G^{-1}(\alpha_n)$. Here, for any distribution function H, the quantile function H^{-1} is given by

$$H^{-1}(t) = \sup\{z :\ H(z) \le t\}.$$

Note that the pairs (ξ_n^x, ψ_n), $n \ge 1$, are independent in n, that the sequence $\{\psi_n\}_{n\ge1}$ is i.i.d. with $\mathbf{E}\psi_1 < 0$ (from (10)) and distribution function G, and that

$$\xi_n^x \le \psi_n \quad \text{a.s..} \tag{33}$$

Put $S_n^\psi = \sum_{j=1}^n \psi_j$ and

$$M^\psi = \max(0, \sup_n S_n^\psi).$$

From the SLLN for $\{\psi_n\}$ and from the condition (C2), for any $\varepsilon > 0$, there exists $R > 0$ such that, for any $n = 1, 2, \ldots$,

$$\mathbf{P}(S_j < R - j(a-\varepsilon),\ j = 1, 2, \ldots, n-1;\ S_{n+i}^\psi - S_n^\psi < R,\ i = 1, 2, \ldots) > 1 - \varepsilon.$$

Hence, from (33), $\mathbf{P}(D_n') > 1 - \varepsilon$ for all n, where each D_n' is as given by (27). Also from (33),

$$\mathbf{P}(M > y) = \mathbf{P}(M > y, M^\psi > y).$$

It is now easy to check that all the conditions of Lemma 2 are satisfied, with each ψ_n and D_n' as given here, and the required result now follows from that lemma. □

The following further two lemmas are also required in each of the proofs of Theorems 3 and 4 (where in each case X is regenerative).

Lemma 3. *Suppose that X is regenerative with $\mathbf{E}\tau < \infty$ and also that F^s is subexponential. Let $\{\{\eta_n^x\}_{x\in\mathcal{X}}\}_{n\ge1}$ be a sequence of families of random variables such that these families are independent and identically distributed in n and are further independent of X. Suppose further that there exists a constant $b > 0$ satisfying the condition* (15) *and such that*

$$\eta_1^x \le b \quad \textit{a.s.} \qquad \textit{for all } x, \tag{34}$$

and that

$$\int_{\mathcal{X}} \mathbf{E}\eta_1^x \, \pi(dx) < 0. \tag{35}$$

Define

$$M^\eta = \max\left(0, \sup_n \sum_{i=1}^n \eta_i^{X_i}\right).$$

Then

$$\mathbf{P}(M^\eta > y) = o(\overline{F}^s(y)) \qquad \text{as } y \to \infty.$$

Proof. Define

$$\beta_n = \sum_{i=T_{n-1}+1}^{T_n} \eta_i^{X_i}, \qquad n \geq 1.$$

Observe that $\{\beta_n\}_{n\geq1}$ is an i.i.d. sequence with, from (34) and (35),

$$\mathbf{E}\beta_1 < 0, \qquad\qquad \beta_n \leq b\tau_n, \quad n \geq 1.$$

Since also X is regenerative with $\mathbf{E}(\tau) < \infty$, we can choose $K > 0$ sufficiently large that if

$$\gamma_n = \max(\beta_n, b\tau_n - K), \qquad n \geq 1, \tag{36}$$

then $\{\gamma_n\}_{n\geq1}$ is an i.i.d. sequence with

$$\mathbf{E}\gamma_1 < 0, \qquad\qquad \gamma_n \leq b\tau_n, \quad n \geq 1. \tag{37}$$

Define also

$$M^\gamma = \max\left(0, \sup_{n\geq1} \sum_{i=1}^n \gamma_i\right).$$

It follows from (37), the assumed condition (15) (for b as given) and the extension of Veraverbeke's Theorem given by Property 9 of Section 3, that

$$\mathbf{P}(M^\gamma > y) = o(\overline{F}^s(y)), \quad \text{as } y \to \infty. \tag{38}$$

Now

$$\begin{aligned} M^\eta &\leq b\tau_0 + \sup(b\tau_1, \beta_1 + b\tau_2, \beta_1 + \beta_2 + b\tau_3, \ldots) \\ &\leq b\tau_0 + K + \sup(\gamma_1, \gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3, \ldots) \\ &\leq b\tau_0 + K + M^\gamma, \end{aligned}$$

where the second inequality above follows from (36). Further τ_0 and M^γ are independent. The required result now follows from (38), the assumed condition (15) and Property 5 of Section 3. $\square$

The following lemma combines the results of Lemmas 2 and 3 to provide a set of conditions for the regenerative case under which there follows the desired conclusion (11) of both Theorems 3 and 4.

Lemma 4. *Suppose that X is regenerative with $\mathbf{E}\tau < \infty$ and also that F^s is subexponential. Suppose also that there exist a sequence of i.i.d. random variables $\{\psi_n\}_{n\geq 1}$ and a constant L_1 satisfying the conditions of Lemma 2, i.e. that*

$$\mathbf{E}(\psi_1) < 0, \qquad \mathbf{P}(\psi_1 > y) \leq L_1 \overline{F}(y) \quad \text{for all } y \geq 0, \tag{39}$$

and that

$$\psi_n \text{ is independent of } D'_n \text{ for all } n \geq 1 \tag{40}$$

(where D'_n is as given by (27)). Suppose further that there exists a sequence of families of random variables $\{\{\eta_n^x\}_{x\in\mathcal{X}}\}_{n\geq 1}$ and a constant $b > 0$ satisfying all the conditions of Lemma 3, and that

$$\xi_n^x \leq \psi_n + \eta_n^x, \qquad x \in \mathcal{X}, \quad n \geq 1. \tag{41}$$

Again define

$$M^\psi = \max\left(0, \sup_n \sum_{i=1}^n \psi_i\right), \qquad M^\eta = \max\left(0, \sup_n \sum_{i=1}^n \eta_i^{X_i}\right).$$

Finally, suppose that M^ψ and M^η are independent. Then the conclusion (11) *follows.*

Proof. As in the proof of Lemma 2 we may assume, without loss of generality, that $\mathbf{P}(\psi_1 > y) = L_1\overline{F}(y)$ for all sufficiently large y. It then follows from the conditions on the sequence $\{\psi_n\}$ and Veraverbeke's Theorem that

$$\mathbf{P}(M^\psi > y) \sim L_1 \overline{F}^s(y), \quad \text{as } y \to \infty, \tag{42}$$

while it follows from Lemma 3 that

$$\mathbf{P}(M^\eta > y) = o(\overline{F}^s(y)), \quad \text{as } y \to \infty. \tag{43}$$

From the condition (41) we have that

$$M \leq M^\psi + M^\eta. \tag{44}$$

Since also M^ψ and M^η are independent, it now follows from (42), (43), (44) and Property 5 of Section 3 that

$$\mathbf{P}(M > y) = \mathbf{P}(M > y, M^\psi > y) + o(\overline{F}^s(y)) \quad \text{as } y \to \infty. \tag{45}$$

Finally, since X is regenerative, the condition (C1), and so now all the conditions of Lemma 2, are satisfied and so the required conclusion (11) again follows from that lemma. □

Proof of Theorem 3. We construct the sequences $\{\psi_n\}_{n\geq 1}$ and $\{\{\eta_n^x\}_{x\in\mathcal{X}}\}_{n\geq 1}$ and the constants L_1 and b such that all the conditions of Lemma 4 are satisfied.

It follows from (5) that we can find a distribution function G on $\mathbb{R}$ such that

$$\overline{F}_x(y) \leq \overline{G}(y) \leq L\overline{F}(y), \tag{46}$$

for all y and for all $x \in \mathcal{X}$. As in the proof of Theorem 2, let $\{\alpha_n\}_{n\geq 1}$ be an i.i.d. sequence of random variables uniformly distributed on $(0,1)$ and independent of $X = \{X_n\}$. Again construct the required family of random variables $\{\xi_n^x\}_{n\geq 1}$ by defining, for each n, $\xi_n^x = F_x^{-1}(\alpha_n)$; for each n define also $\zeta_n = G^{-1}(\alpha_n)$. Then the pairs (ξ_n^x, ζ_n), $n \geq 1$, are independent in n, the sequence $\{\zeta_n\}_{n\geq 1}$ is i.i.d., and

$$\xi_n^{X_n} \leq \zeta_n \quad \text{a.s.,} \qquad \text{for all } n. \tag{47}$$

For $y_0 > 0$, define

$$u(y_0) = \mathbf{E}[\mathbf{I}(\zeta_1 > y_0)\zeta_1], \qquad v(y_0) = -\int_{\mathcal{X}} \mathbf{E}[\mathbf{I}(\zeta_1 \leq y_0)\xi_1^x]\pi(dx).$$

Observe that $u(y_0) \to 0$ as $y_0 \to \infty$ and, by the conditions (13) and (14), $v(y_0) \to a$ as $y_0 \to \infty$. Choose y_0 sufficiently large and $K > 0$ such that

$$u(y_0) < \mathbf{P}(\zeta_1 > y_0)K < v(y_0). \tag{48}$$

We define the required i.i.d. sequence $\{\psi_n\}_{n\geq 1}$ by

$$\psi_n = \mathbf{I}(\zeta_n > y_0)(\zeta_n - K).$$

It follows from the construction of this sequence, and in particular from (46), (48) and the definition of $u(y_0)$, that it satisfies the conditions (39) and (40) of Lemma 4 with $L_1 = L$. Define also, for each n and for each x,

$$\eta_n^x = \mathbf{I}(\zeta_n \leq y_0)\xi_n^x + \mathbf{I}(\zeta_n > y_0)K,$$

The random variables η_n^x are bounded above by $b = \max(y_0, K)$. Further, by (48) and the definition of $v(y_0)$,

$$\int_{\mathcal{X}} \mathbf{E}\eta_1^x \, \pi(dx) < 0.$$

It now follows, using also the condition (15) of the theorem, that the sequence $\{\{\eta_n^x\}_{x\in\mathcal{X}}\}_{n\geq 1}$ and b as given above satisfy the conditions of Lemma 3, and so also of Lemma 4.

The condition (41) follows on observing that, from (47), for all x and for all n,

$$\begin{aligned}\xi_n^x &= \mathbf{I}(\zeta_n > y_0)(\xi_n^x - K) + \mathbf{I}(\zeta_n \leq y_0)\xi_n^x + \mathbf{I}(\zeta_n > y_0)K \\ &\leq \psi_n + \eta_n^x.\end{aligned}$$

Finally, it is not difficult to see that the random variables M^ψ and M^η (defined as in the statement of Lemma 4) are independent (although the sequences $\{\psi_n\}$ and $\{\eta_n^{X_n}\}$ of which they are the maxima are *not* independent!). The required conclusion (11) now follows from Lemma 4. $\square$

Proof of Theorem 4. We again use Lemma 4. It follows from the conditions of the theorem that we may take b such that

$$\mathbf{E}\zeta < b < \mathbf{E}_\pi b^X \tag{49}$$

and satisfying (15). It follows also from (18) that there exists $L_1 > 0$ such that

$$\mathbf{P}(\zeta > y) \le L_1 \overline{F}(y) \tag{50}$$

for all y. Further, we may define random variables $\{\xi^x\}_{x\in\mathcal{X}}$, ζ, and $\{b^x\}_{x\in\mathcal{X}}$ in such a way that ζ and the family $\{b^x\}_{x\in\mathcal{X}}$ are independent and, for all x,

$$\xi^x \le \zeta - b^x \quad \text{a.s..} \tag{51}$$

For $n = 1, 2, \dots$, let $\{\xi_n^x, b_n^x, \zeta_n\}$ be i.i.d. copies of $\{\xi^x, b^x, \zeta\}$, such that these sequences are jointly independent of the process X. Define, for all n,

$$\psi_n = \zeta_n - b, \qquad\qquad \eta_n^x = b - b_n^x, \quad x \in \mathcal{X}. \tag{52}$$

Then it is easy to check, from (49)–(52) and the condition (15) and the independence assumption of the theorem, that all the conditions of Lemma 4 are satisfied. The sequences $\{\psi_n\}_{n\ge1}$ and $\{\{\eta_n^x\}_{x\in\mathcal{X}}\}_{n\ge1}$ and the constants L_1 and b of that lemma are as given here. We thus have the required result. □

Proof of Theorem 5. We again give the proof in the case $d = 1$. It follows straightforwardly from the regenerative structure of X, the condition $\mathbf{E}\tau < \infty$, and the condition (C2) that the random vectors

$$Y_0 = \{\tau_0; W_1, \dots, W_{T_0}\},$$
$$Y_n = \{\tau_n; W_{T_{n-1}+1}, \dots, W_{T_n}\}, \qquad n \ge 1,$$

form a Harris ergodic Markov chain (see, for example, [11]). Then, since $d = 1$, it is again straightforward that W_n converges in the total variation norm to a distribution on $\mathbb{R}_+$ which is independent of that of Y_0. Now let $\tilde{X} = \{\tilde{X}_n\}_{-\infty<n<\infty}$ be the corresponding stationary version of the process X indexed over the entire set of integers, and similarly extend the i.i.d. sequence of families $\{\{\xi_n^x\}_{x\in\mathcal{X}}\}_{n\ge1}$ to $\{\{\xi_n^x\}_{x\in\mathcal{X}}\}_{-\infty<n<\infty}$. Let $\{\tilde{W}_n\}_{n\ge0}$ (with $\tilde{W}_0 \equiv 0$ as usual) be the corresponding version of the process $\{W_n\}_{n\ge0}$. It follows from the recursion (3) that

$$\tilde{W}_n = \max\left(0, \xi_n^{\tilde{X}_n}, \xi_n^{\tilde{X}_n} + \xi_{n-1}^{\tilde{X}_{n-1}}, \dots, \xi_n^{\tilde{X}_n} + \dots + \xi_1^{\tilde{X}_1}\right)$$

which, by stationarity, has the same distribution as

$$\max\left(0, \xi_{-1}^{\tilde{X}_{-1}}, \xi_{-1}^{\tilde{X}_{-1}} + \xi_{-2}^{\tilde{X}_{-2}}, \dots, \xi_{-1}^{\tilde{X}_{-1}} + \dots + \xi_{-n}^{\tilde{X}_{-n}}\right).$$

Thus, for any y, $\lim_{n\to\infty} \mathbf{P}(W_n > y)$ and $\lim_{n\to\infty} \mathbf{P}(\tilde{W}_n > y)$ both exist and are equal to $\mathbf{P}(M^- > y)$ where

$$M^- = \sup\left(0, \xi_{-1}^{\tilde{X}_{-1}}, \xi_{-1}^{\tilde{X}_{-1}} + \xi_{-2}^{\tilde{X}_{-2}}, \dots\right).$$

The required result now follows from the application of Theorem 2, 3 or 4 as appropriate, in each case with $B = \mathcal{X}$, to the time-reversed version of the stationary process $\{\tilde{X}_n, \xi_n^{\tilde{X}_n}\}$. However, under the conditions of Theorem 3 or Theorem 4, we must also verify the required condition on τ_0^-, defined to be the time of the first regeneration at or after time 0 in the reversed process $X^- = \{X_n^-\}_{n \geq 0}$ given by $X_n^- = \tilde{X}_{-n}$. Standard renewal theory shows that the distribution of τ_0^- is given by

$$\mathbf{P}(\tau_0^- \geq n) = \frac{1}{\mathbf{E}(\tau)} \sum_{k=n+1}^{\infty} \mathbf{P}(\tau \geq k), \qquad n = 0, 1, \ldots$$

An easy calculation, analogous to that of the derivation of Property 2 of Section 3, now gives that, if $b > 0$ is such that $\mathbf{P}(b\tau > y) = o(\overline{F}(y))$ as $y \to \infty$, then $\mathbf{P}(b\tau_0^- > y) = o(\overline{F}^s(y))$ as $y \to \infty$. Thus, in each case, the required condition on τ_0^- follows from the assumed condition on τ.

The modifications for the case of general d are again routine. $\square$

References

[1] *G. Alsmeyer and M. Sgibnev* (1999). On the tail behaviour of the supremum of a random walk defined on a Markov chain. *Yokohama Math. J.* **46**, 139–159.

[2] *K. Arndt* (1980). Asymptotic properties of the distribution of the supremum of a random walk on a Markov chain. *Theory Probab. Appl.* **25**, 309–324.

[3] *S. Asmussen* (2000). *Ruin Probabilities.* World Scientific.

[4] *S. Asmussen* (1999). Semi-Markov queues with heavy tails. In *Semi-Markov Models and Applications*, eds J. Janssen and N Limnios. Kluwer.

[5] *S. Asmussen and J.R. Møller* (1999). Tail asymptotics for M/G/1 type queueing processes with subexponential increments *Queueing Systems* **33**, 153–176.

[6] *F. Baccelli and S. Foss* (2001). Moments and tails in monotone-separable stochastic networks. Submitted.

[7] *F. Baccelli, S. Schlegel, and V. Schmidt* (1999). Asymptotics of stochastic networks with subexponential service times. *Queueing Systems* **33**, 205–232.

[8] *P. Embrechts, C. Kluppelberg, and Th. Mikosch* (1997). *Modelling Extremal Events.* Springer Verlag.

[9] *T. Huang and K. Sigman* (1999). Steady-state asymptotics for tandem, split-match and other feedforward queues with heavy-tailed service. *Queueing Systems* **33**, 233–259.

[10] *K. Sigman* (1999). A primer on heavy-tailed distributions. *Queueing Systems* **33**, 261–275.

[11] *H. Thorisson* (2000). *Coupling, Stationarity, and Regeneration.* Springer Verlag.

[12] *N. Veraverbeke* (1977). Asymptotic behavior of Wiener– Hopf factors of a random walk. *Stoch. Process. Appl.* **5**, 27–37.