# DOMINANCE THEOREMS AND ERGODIC PROPERTIES OF POLLING SYSTEMS[1]

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*We consider a class of polling systems with stationary ergodic input flow such that the control in a system obeys a certain regeneration property. For this class, necessary and sufficient conditions for the queue-length process to be bounded in probability are found. Under these conditions, we prove that a stationary regime exists and the queue-length process for a system that starts from the zero initial state converges to this regime. In the proof, we use some monotonicity properties of the models considered and some dominance theorems based on these properties.*

## 1. Introduction

In the paper, we consider polling systems consisting of $K$ message queues and one transmission device (server) that moves from one queue to another along a random route. Messages entering the system form a stationary ergodic input flow. The routing of the transmitter has a certain regeneration property. Namely, in the sequence of transmitter transitions and queue-to-queue switching times, independent and identically distributed parts can be selected. These parts are called cycles.

In one visit of the transmitter to the $k$th queue, $f_k(x)$ messages are transmitted, where $x$ is the length of this queue at the time instant of the transmitter arrival. The conditions imposed on the transmission policy of $f_k$ messages are stated in Sec. 2; analogous conditions are also presented in [1, 2].

In the paper, some monotonicity properties of the models considered are proved. Based on them, necessary and sufficient conditions for the queue-length process to be bounded in probability are proved. Under these conditions, we prove that a stationary regime exists and the queue-length process for a system that starts from the zero initial state converges to this regime.

Let $\lambda$ be the intensity of the input flow, $p_k$ be the probability that an input message is sent to the $k$th queue, $\sigma$ be the mean time of the message transmission, $F_k = \lim_{x\to\infty} f_k(x)$, $C_k$ be the mean number of visits to the $k$th queue over a cycle, $W$ be the mean total switching time over a cycle (for precise definitions, see Sec. 2). Under the condition

$$\lambda \left[\sigma + \max_k \frac{p_k}{F_k C_k} W\right] < 1, \tag{1}$$

we show the so-called coupling-convergence to the stationary regime of the queue-length process (see Sec. 6). If, conversely, the traffic on the left-hand side of (1) is strictly greater than 1, the queue length in the system infinitely grows in probability.

The study of ergodicity conditions for polling systems started quite recently. In almost all papers known to us (except [2]), models with Poissonic input flow and i.i.d. times of message transmission and switching are studied.

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For example, in [3–5], ergodicity of various polling models is studied, for which the server routing is controlled by a matrix of transition probabilities, and transmission policies depend on a random variable.

Markovian polling systems for which the server routing does not depend on the system state (as in the present paper) but the input calls are distributed on the unit circle according to a continuous law are studied in [6–9].

In [10–14], various Markovian polling systems are studied, for which the server routing depends on the system state and is determined by a "greedy" ("locally optimal") strategy.

Properties of stationary polling systems are also discussed in [15, 16].

In [2], a polling system with stationary ergodic input flow is considered, for which the server routing forms a Markov chain. The existence of a stationary regime is proved provided the condition below holds:

$$\lambda \left[ \sigma + \sum_{k=1}^{K} \frac{p_k}{F_k \pi_k} w \right] < 1,$$

where $\{\pi_k,\, 1 \le k \le K\}$ is the stationary distribution of the corresponding Markov chain, and $w$ is the mean station-to-station switching time in the stationary regime. One can easily see that the above condition is sufficient, but not necessary, for the ergodicity.

In the present paper, we prove that the condition (1) is sufficient for the queue-length process to be bounded in probability for a polling system with considerably more general conditions on the transmission policy than those of [2].

In Sec. 2, basic definitions and statements are presented. In Sec. 3, for a determinate version of a polling system, some monotonicity properties of system characteristics are established as functions of arrival times of messages in the input flow. A determinate polling system can be considered as a realization of a stochastic system described in Sec. 2 on one elementary event. The main result of Sec. 4 is the necessity and sufficiency of the condition (1) for the exhausting time of the system and the queue length to be bounded. In the proof, methods presented in [17] are used. The constants involved in (1) are computed in Sec. 5. Section 6 deals with the proof of the existence of a stationary regime. Possible generalizations of the model are contained in Sec. 7. In the Appendix, some auxiliary statements are presented; in particular, the result on "random subadditive" subsequences that generalizes the results of [18, 19].

## 2. Description of systems and basic statements

We consider a polling system consisting of a finite number $K$ of queues with an arbitrarily large number of waiting positions in each queue and one transmission server. Messages enter the system in a common input flow. The server moves from one queue to another according to a random route.

**Input flow.** A message with number $n$, $n = 0, \pm 1, \pm 2, \ldots$, arrives at a time instant $T_n = T_{n-1} + \tau_n$, $T_0 = 0$, and is sent to the queue with number $\mu_n$; $\sigma_n$ is its transition time. Denote $\xi_n = (\tau_n, \mu_n, \sigma_n)$, $n \in \mathbb{Z}$. The sequence $\{\xi_n\}$ is assumed to be stationary and metric-transitive. Let $\mathbf{E}\tau_1 = \lambda^{-1} < \infty$ and $\neq 0$; $\mathbf{E}\sigma_1 = \sigma < \infty$; $\mathbf{P}(\mu_1 = k) = p_k > 0$ for all $k = 1, \ldots, K$, $\sum\limits_{k=1}^{K} p_k = 1$.

**Server routing.** We assume that a sequence of pairs of random variables $\{\nu_j, w_j\}_{j=-\infty}^{\infty}$ is given, where the random variable $\nu_j$ takes the values $1, \ldots, K$ and equals the number of the queue visited $j$th by the server, and $w_j \ge 0$ is the time required for the switching from the queue $\nu_j$ to the queue $\nu_{j+1}$. We assume that the sequence $\{\nu_j, w_j\}$ can be divided into i.i.d. parts of random lengths (cycles); namely, there exists an increasing sequence of integer random variables $\{j_i\}_{i=-\infty}^{\infty}$ such that random vectors

$$\eta_i = \big(\ell_i; \nu_{j_i+1}, \ldots, \nu_{j_{i+1}}; w_{j_i+1}, \ldots, w_{j_{i+1}}\big), \quad i \in \mathbb{Z},$$

are i.i.d. Here $\ell_i = j_{i+1} - j_i$ is the number of queues visited by the server during the $i$th cycle. Without loss of generality, we assume that the cycles start with the visit to a certain fixed queue, say, number 1, i.e., $\nu_{j_i+1} = 1$ a.s. Let $\psi_i = w_{j_i+1} + \ldots + w_{j_{i+1}}$, $W = \mathbf{E}\psi_1 < \infty$ be the mean switching time over a cycle, $C_k = \mathbf{E}(I(\nu_{j_1+1} = k) + \ldots + I(\nu_{j_2} = k)) < \infty$ be the mean number of visits to the $k$th queue over a "typical" cycle, $k = 1, \ldots, K$. We also assume the sequences $\{\eta_i\}$ and $\{\xi_n\}$ to be independent.

Note that the assumption on the "cyclic" character of the sequence $\{\nu_j\}$ is valid, e.g., if this sequence forms a homogeneous Markov chain. As cycles, parts of the chain trajectory between sequential returns to a fixed station can be considered.

Cycles can be formed in different ways, say, can be determinate.

Consider a marked point random process, the points of which are the starting instants of the cycles and the distance between points of which equals the total switching time during the corresponding cycle. We call this process the *server route in the empty system.* We denote by $\Psi$ the marked point process with points $\Psi_i$ and marks $\eta_i$, $i \in \mathbb{Z}$, for which $\Psi_0 = 0$, $\Psi_i = \Psi_{i-1} + \psi_i$ is the end instant of the $i$th cycle if the server moves in the empty system.

Consider also the stationary (in continuous time) version of the process $\Psi$ (we denote it by $\Psi^{(1)}$), i.e., the stationary point process with the same marks and same distances between points as those of $\Psi$. Assign the number 0 to the first positive point of this process. Then $\Psi^{(1)}$ is a stationary ergodic point process with marks $\eta_i$, $i \in \mathbb{Z}$, with point numeration such that $\Psi_0^{(1)}$ is the first positive point of this process (jump over $t = 0$). Let $\Psi^{(-n)}$, $n \geq 0$, be a stationary ergodic point process obtained from $\Psi^{(1)}$ by shifting each point of it by the random variable $\sum\limits_{j=-n}^{0} \sigma_j$ to the left and then by renumbering it such that $\Psi_0^{(-n)}$ is its first positive point.

*Remark 1.* It is easily seen that the distributions of the processes $\Psi^{(-n)}$ $(n \geq -1)$ coincide and these processes do not depend on the input flow. For shifting the points of the process by a constant, this follows from the stationarity, and for shifting by a random variable independent of the process, one should consider conditional distributions with respect to this random variable.

We say that the process $\Psi$ ($\Psi^{(-n)}$) controls the polling system if the server starts working at $t = -\infty$, and in the empty system it moves from one queue to another according to this routing, i.e., the $i$th cycle ends at instant $\Psi_i$ ($\Psi_i^{(-n)}$). If, however, there are messages in the system which wait for the transmission, then each cycle is extended by the total transmission time during this cycle. In Sec. 4, to each routing introduced above we relate a polling system controlled by this routing.

**Message transmission policy.** If in the $k$th queue there are $x$ messages waiting at the instant of the server arrival, then $f_k(x) \leq x$ of them are transmitted and then the server switches to another queue of its route. The transmitted messages leave the system. The function $f_k : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying the conditions $f_k(0) = 0$, $1 \leq f_k(x) \leq x$ for all $x \geq 1$, $k = 1, \ldots, K$, is called the message transmission policy of the queue $k$. We consider the following three classes of message transmission policies:

$$\begin{aligned} \widehat{A} &= \Big\{ f : \lim_{x\to\infty} f(x) = F \leq \infty \text{ exists} \Big\}, \\ A &= \Big\{ f : \lim_{x\to\infty} f(x) = F \leq \infty \text{ exists}, \ \sup_x f(x) = F \Big\}, \\ M &= \Big\{ f : \ f(x) \leq f(x+1) \leq f(x) + 1 \text{ for all } x \in \mathbb{Z}^+ \Big\}. \end{aligned}$$

The class $M$ is called the class of monotone transmission policies. Obviously, $M \subset A \subset \widehat{A}$.

Denote by $Q_n$ the total queue length at an instant $T_n$ in the polling system controlled by $\Psi$ with a "cut-off" input flow, i.e., the messages with numbers $1, \ldots, n$ only enter the system at time instants $T_1 \leq \ldots \leq T_n$, respectively. Denote by $Q^k_{[-n,0]}(t)$ the queue length at the $k$th station at an instant $t$ in the system controlled by $\Psi^{(-n)}$, which is entered by messages with numbers $-n, \ldots, 0$ only (at time instants $T_{-n} \leq \ldots \leq T_0 = 0$ respectively). Denote by $W^k_{[-n,0]}(t)$ the residual time of the message transmission in the $k$th queue at a time instant $t$ (in the same system). Put $W^k_{[-n,0]}(t) = 0$ if at an instant $t$ the messages of the $k$th queue are not transmitted.

All above-mentioned processes are assumed to be continuous on the right. The initial conditions are assumed to be zero, i.e., before the first message of the input flow arrives, the system considered is empty.

Denote

$$F_k = \lim_{x\to\infty} f_k(x) \le \infty, \qquad \rho = \lambda \left[\sigma + \max_{1\le k\le K} \frac{p_k}{F_k C_k} W\right].$$

In the paper, the statements below are proved.

For systems with transmission policies from class $\widehat{A}$:

**Theorem 1.** *If* $\rho < 1$, *then* $Q_n$ *is bounded in probability, i.e.,*

$$\lim_{x\to\infty} \sup_n \mathbf{P}(Q_n > x) = 0.$$

For systems with policies from class $A$:

**Theorem 2.** *If* $\rho > 1$, *then* $Q_n \xrightarrow{\text{P}} \infty$.

Finally, for systems with policies from class $M$:

**Theorem 3.** *If* $\rho < 1$, *then a process* $\{Q^k(t), W^k(t);\ k = 1,\dots,K;\ t \ge 0\}$ *and a.s. finite random variables* $\overline{N}$, $t_0$ *exist such that*

1. $Q^k_{[-n,0]}(t), W^k_{[-n,0]}(t)\} \overset{\text{a.s.}}{=} \{Q^k(t), W^k(t)\}$ *for all* $n \ge \overline{N}$, $k = 1,\dots,K$, $t \ge 0$;
2. $Q^k_{[-n,0]}(t) = W^k_{[-n,0]}(t) = Q^k(t) = W^k(t) = 0$ *a.s. for all* $k = 1,\dots,K$, $t \ge t_0$, $n \ge 0$.

*If* $\rho > 1$, *then* $Q_{[-n,0]}(0) = \sum_{k=1}^{K} Q^k_{[-n,0]}(0) \xrightarrow{\text{P}} \infty$.

Thus, for the polling systems under consideration with policies from class $A$, the condition $\rho < 1$ is necessary and sufficient (up to the equality) for the total queue length to be bounded in probability. For systems with policies from $M$, this condition ensures that a stationary regime exists and the queue-length process converges to it.

*Remark 2.* The above statements can be naturally extended to the class of models for which
(1) the messages arrive in batches;
(2) transmission policies depend on a random variable;
(3) some (in particular, all) transmission policies have “exhaustive” character.
In more detail, these extensions are discussed in Sec. 7.

In Sec. 3, monotonicity properties of a determinate polling system are obtained. We also show that the class $M$ is in a certain sense “dense” in $A$; namely, any function from $A$ can be pointwise upper and lower bounded by functions from $M$ with the same limit as $x \to \infty$. The proof of Theorems 1 and 2 is presented in Sec. 4. We show that these statements hold for a system with a monotone transmission policy; here the “density” of $M$ in $A$ is used. Theorem 3 is proved in Sec. 6.

## 3. Monotonicity properties

In this section, we consider the determinate case of a polling system, where all control sequences are nonrandom. We assume that the input flow is “cut off,” i.e., only $N$ messages enter the system. Assume that the server starts working with the queue $\nu_1$ at a time instant $t^0$. Assume also that, starting from $t^0$, the server switches between queues even if there are no messages in the system.

A message transmission policy is assumed to be a set of $K$ sequences of functions $f = \{f(k),\ k = 1,\dots,K\}$, where a sequence $f(k) = \{f^j_k\}_{j=1}^{\infty}$ determines the (message transmission) policy in the queue $k$. If the server, arriving at the queue $k$ for the $j$th time, finds $x$ messages in the queue, it transmits $f^j_k(x) \ge 0$ messages in the FIFO order and then switches to the next queue of the route. The transmission of a message being completed, the message leaves the system.

*Remark 3.* In this section, we admit that the policies in one queue may depend on the number of the server visit to this queue (see Sec. 7).

Let for all $k \in \{1,\dots,K\}$, $j = 1, 2, \dots$, the following conditions hold:

$(M_1)$ If $x \le y$ for $x, y \in \mathbb{Z}^+$, then $f_k^j(x) \le f_k^j(y)$;
$(M_2)$ If $x \le y$ for $x, y \in \mathbb{Z}^+$, then $x - f_k^j(x) \le y - f_k^j(y)$.

The conditions $(M_1)$ and $(M_2)$ are presented in [1, 2] and are equivalent to the condition $f_k^j(x) \le f_k^j(x+1) \le f_k^j(x) + 1$ that determines the class $M$ of monotone policies. The conditions $(M_1)$–$(M_2)$ hold, for example, for the following functions: $f_k^j(x) \equiv x$, $f_k^j(x) = \min(x, \ell_j)$ for some $\ell_j \in \mathbb{N}$.

We assume that messages enter the system at time instants $T_1 \le \ldots \le T_N$. Before $T_1$ the system is empty. The message with number $n$ arrives at instant $T_n$, is sent to the queue with number $\mu_n$, and has the transmission time $\sigma_n$.

Put $\nu = \{\nu_j\}_{j=1}^{\infty}$, $w = \{w_j\}_{j=1}^{\infty}$, $\mu = \{\mu_n\}_{n=1}^{N}$, $\sigma = \{\sigma_n\}_{n=1}^{N}$, $T = \{T_n\}_{n=1}^{N}$. Denote such a system by $\Sigma = (N, T, \mu, \nu, w, \sigma, f)$.

Let $\widetilde{\Sigma} = (N, \widetilde{T}, \mu, \nu, w, \sigma, \widetilde{f})$ be a polling system differing from $\Sigma$ by instants of message arrivals and message transmission policies only.

**Monotonicity properties.** For a system $\Sigma$, introduce the following notations: $U_j$ is the instant of the server arrival at the queue $\nu_j$; $V_j$ is the instant where the transmission of messages from the queue $\nu_j$ is completed and switching to the queue $\nu_{j+1}$ starts, $U_{j+1} = V_j + w_j$, $j = 1, 2, \ldots$; $X_k(u)$ is the number of input-flow messages that have entered the queue $k$ by a time instant $u$; $W(u)$ is the total switching time of the server over the time interval $(t^0, u)$; $R_k(u)$ is the total working time of the server at the $k$th queue over the same time interval, $R(u) = \sum\limits_{k=1}^{K} R_k(u)$; $S_k(u)$ is the number of messages from the $k$th queue which have been completely transmitted by a time instant $u$, $S(u) = \sum\limits_{k=1}^{K} S_k(u)$.

The corresponding parameters for the system $\widetilde{\Sigma}$ are denoted by $\widetilde{U}_j$, etc. The parameters introduced have the following properties: $U_1 = \widetilde{U}_1 = t^0$, $W(u) + R(u) = u - t^0$, $W(U_j) = \widetilde{W}(\widetilde{U}_j) = \sum\limits_{i=1}^{j-1} w_i$, and if $S_k(V_i) \ge \widetilde{S}_k(\widetilde{V}_j)$ for some $i, j, k$, then $R_k(V_i) \ge \widetilde{R}_k(\widetilde{V}_j)$ since the messages are transmitted in the FIFO order. Furthermore, $W(u)$ and $R_k(u)$ do not decrease in $u$.

Let $\tau$ $(\widetilde{\tau})$ be the end instant of the transmission of the last of $N$ messages that entered the system $\Sigma$ $(\widetilde{\Sigma})$, i.e., $\tau = \inf(u : S(u) = N)$.

We use the notation $T \le \widetilde{T}$ if $T_n \le \widetilde{T}_n$ for all $1 \le n \le N$, and $f \ge \widetilde{f}$ if $f_k^j(x) \ge \widetilde{f}_k^j(x)$ for all $k = 1, \ldots, K$, $j = 1, 2, \ldots$; $x \in \mathbb{Z}^+$. The following statement is valid.

**Theorem 4.** *If $T \le \widetilde{T}$, $f \ge \widetilde{f}$, and the transmission policies $f$ and $\widetilde{f}$ satisfy the conditions* $(M_1)$–$(M_2)$, *then*

1. $U_j \ge \widetilde{U}_j$, $V_j \ge \widetilde{V}_j$, $j = 1, 2, \ldots$;
2. $S_k(U_j) \ge \widetilde{S}_k(\widetilde{U}_j)$, $k = 1, \ldots, K$, $j \in \mathbb{N}$;
3. $S_k(V_j) \ge \widetilde{S}_k(\widetilde{V}_j)$, $k = 1, \ldots, K$, $j \in \mathbb{N}$;
4. $\tau \le \widetilde{\tau}$.

The proof of the statements 1–3 of the theorem is performed by induction on $j$.

Let $j = 1$. Denote $\nu_i = \ell$. Since $U_1 = \widetilde{U}_1 = t^0$, the queue lengths satisfy the inequality

$$x = X_\ell(U_1) \stackrel{\text{def}}{=} \sum_{n=1}^{N} I(T_n \le U_1) I(\mu_n = \ell) \ge \sum_{n=1}^{N} I(\widetilde{T}_n \le U_1) I(\mu_n = \ell) = \widetilde{X}_\ell(\widetilde{U}_1) = y.$$

By $(M_1)$,

$$S_\ell(V_1) \stackrel{\text{def}}{=} f_\ell^1(x) \ge \widetilde{f}_\ell^1(x) \ge \widetilde{f}_\ell^1(y) \stackrel{\text{def}}{=} \widetilde{S}_\ell(\widetilde{V}_1).$$

Therefore, $R_\ell(V_1) \ge \widetilde{R}_\ell(\widetilde{V}_1)$, and for the end time instants of the transmission of the first group of messages, the relations

$$V_1 = U_1 + R_\ell(V_1) \ge \widetilde{U}_1 + \widetilde{R}_\ell(\widetilde{V}_1) = \widetilde{V}_1$$

hold. For $i \neq \ell$, evidently, $S_i(V_1) = \widetilde{S}_i(\widetilde{V}_1) = 0$.

Next, assume that the statements 1–3 of Theorem 4 are valid for the start and end time instants of the server visit to the queue $\nu_{j-1}$, $j \geq 2$. Let us show their validity for the instants $U_j, V_j$. We have

$$U_j = V_{j-1} + w_{j-1} \geq \widetilde{V}_{j-1} + w_{j-1} = \widetilde{U}_j,$$
$$S_k(U_j) = S_k(V_{j-1}) \geq \widetilde{S}_k(\widetilde{V}_{j-1}) = \widetilde{S}_k(\widetilde{U}_j).$$

Denote $\nu_j = \ell$. If $S_\ell(U_j) = 0$, then, repeating the arguments of the induction basis, we obtain the desired statement. Consider the case $S_\ell(U_j) > 0$ and assume that the server visits the queue $\ell$ for the $m$th time, $m > 1$.

Denote by $x = X_\ell(U_j) - S_\ell(U_j)$ and $y = \widetilde{X}_\ell(\widetilde{U}_j) - \widetilde{S}_\ell(\widetilde{U}_j)$ the lengths of the queue $\nu_j = \ell$ at the instant of the server visit to this queue in corresponding systems. Note that $X_\ell(U_j) \geq \widetilde{X}_\ell(\widetilde{U}_j)$ since $U_j \geq \widetilde{U}_j$. Two cases are possible, $x \geq y$ and $x < y$.

In the first case, $f_\ell^m(x) \geq f_\ell^m(y) \geq \widetilde{f}_\ell^m(y)$. Therefore, we also have

$$S_\ell(V_j) = S_\ell(U_j) + f_\ell^m(x) \geq \widetilde{S}_\ell(\widetilde{U}_j) + \widetilde{f}_\ell^m(y) = \widetilde{S}_\ell(\widetilde{V}_j).$$

In the second case, $x < y$ implies that

$$z \stackrel{\text{def}}{=} S_\ell(U_j) - \widetilde{S}_\ell(\widetilde{U}_j) \geq y - x > 0.$$

But, according to $(M_2)$, $f_\ell^m(y) - f_\ell^m(x) \leq y - x \leq z$, i.e., $f_\ell^m(x) \geq f_\ell^m(y) - z \geq \widetilde{f}_\ell^m(y) - z$. Finally,

$$S_\ell(V_j) = S_\ell(U_j) + f_\ell^m(x) \geq \widetilde{S}_\ell(\widetilde{U}_j) + z + \widetilde{f}_\ell^m(y) - z = \widetilde{S}_\ell(\widetilde{V}_j),$$

i.e., in both cases $S_\ell(V_j) \geq \widetilde{S}_\ell(\widetilde{V}_j)$ and, hence, $R_\ell(V_j) \geq \widetilde{R}_\ell(\widetilde{V}_j)$. Then

$$\begin{aligned} V_j &= t^0 + W(U_j) + R(V_j) = t^0 + W(U_j) + \sum_{i \neq \ell} R_i(U_j) + R_\ell(V_j) \\ &\geq t^0 + \widetilde{W}(\widetilde{U}_j) + \sum_{i \neq \ell} \widetilde{R}_i(\widetilde{U}_j) + \widetilde{R}_\ell(\widetilde{V}_j) \stackrel{\text{def}}{=} \widetilde{V}_j. \end{aligned}$$

The last inequality completes the induction step, and it remains to show that $\widetilde{\tau} \geq \tau$. Let $\tau = V_j \geq \widetilde{V}_j$ for some $j$, $S(V_j) = N = \widetilde{S}(\widetilde{\tau})$, $\widetilde{R}(\widetilde{\tau}) = R(V_j)$. Then

$$\widetilde{\tau} = t^0 + \underset{\cdot}{W}(\widetilde{\tau}) + \widetilde{R}(\widetilde{\tau}) \geq t^0 + \widetilde{W}(\widetilde{V}_j) + \widetilde{R}(\widetilde{\tau}) = t^0 + W(V_j) + R(V_j) = \tau. \ \triangle$$

**Corollary 1.** *If, under the conditions of Theorem* 4, *for any* $k, j$, *only one of two functions* $f_k^j$, $\widetilde{f}_k^j$ *obeys the monotonicity properties* $(M_1)$ *and* $(M_2)$, *i.e., belongs to* $M$, *then the assertion of Theorem* 4 *remains valid.*

Proof. It suffices to show that two inequalities hold:
(a) if $x \geq y$, then $f_k^j(x) \geq \widetilde{f}_k^j(y)$;
(b) if $x < y$, then $\widetilde{f}_k^j(y) - f_k^j(x) \leq y - x$.
Indeed, let, say, $\widetilde{f}_j^k \in M$. Then, in case (a), $f_k^j(x) \geq \widetilde{f}_k^j(x) \geq \widetilde{f}_k^j(y)$; in case (b), $\widetilde{f}_k^j(y) - f_k^j(x) \leq \widetilde{f}_k^j(y) - \widetilde{f}_k^j(x) \leq y - x$. $\triangle$

Consider three polling systems (with the same input flow and server route) $\Sigma^f$, $\Sigma^g$, $\Sigma^h$ which differ in the sets of message transmission policies only, where for all $k, j, x$,

$$f_k^j(x) \leq g_k^j(x) \leq h_k^j(x),$$

and the sets of functions $f$ and $h$ satisfy the monotonicity properties $(M_1)$ and $(M_2)$. Under these conditions, Theorem 4 and Corollary 1 imply the statement below.

**Corollary 2.** *Time instants of exhausting of the systems* $\Sigma^f, \Sigma^g, \Sigma^h$ *satisfy the inequality* $\tau^f \geq \tau^g \geq \tau^h$.

The validity of Corollary 2 is obvious due to Theorem 4 and Corollary 1. A direct proof can also be found in [1].

**Corollary 3.** *If, in the system* $\Sigma^g$, *for any* $k, j$, *one has* $g_k^j \in A$, *then functions* $f_k^j, h_k^j \in M$ *can be found such that*

(a) $f_k^j(x) \leq g_k^j(x) \leq h_k^j(x)$,

(b) $G_k^j = \lim g_k^j(x) = \lim f_k^j(x) = \lim h_k^j(x)$ *as* $x \to \infty$,

*and for the systems* $\Sigma^f, \Sigma^g, \Sigma^h$ *with the same input flow and server route, Corollary 2 holds.*

Proof. It suffices to prove that for any function $g \in A$ there exist two functions $f, h \in M$ with the same limit $G = \lim g(x) = \lim f(x) = \lim h(x)$ at infinity such that $f(x) \leq g(x) \leq h(x)$ for all $x \geq 0$. As these "bounds," one can take, for instance, $h(x) = \min\{x, G\}$ and $f$ recursively constructed, $f(1) = 1$, $f(x+1) = \min\left\{f(x) + 1, \inf_{y \geq x+1} g(y)\right\}$. △

Corollary 3 asserts that the exhausting time of the system $\Sigma^g$ with all transmission policies from the class $A$ can be upper and lower "constrained" by the corresponding times for $\Sigma^f$ and $\Sigma^h$ with monotone policies that have the same limits at infinity as those of $\Sigma^g$.

*Remark 4.* If, under the conditions of Corollary 3, $g_k^j \in \widehat{A}$, then Corollary 3 remains valid if one replaces (b) by (b′) as follows:

(b′) $G_k^j = \lim g_k^j(x) = \lim f_k^j(x) \leq \lim h_k^j(x)$ as $x \to \infty$.

As a function $h_k^j(x) \in M$, one can take $\min\left\{x, \sup_y g_k^j(y)\right\}$.

## 4. Stochastic polling system

In this section, we study the behavior of polling systems under the assumptions and notations of Sec. 2. The plan of reasoning and proofs is close to [17] with, however, certain significant distinctions. Thus, for instance, the homogeneity property (Property 3 below) in [17] is assumed to hold a.s.

Let $T = \{T_n\}_{n=-\infty}^{\infty}$ be the input flow of a system, $T_0 = 0$. We assume that message transmission policies of each queue belong to the class $M$. For each server route $\Psi$, $\Psi^{(1)}$, $\Psi^{(-n)}$ $(n \geq 0)$ introduced in Sec. 2, consider a polling system controlled by this route. We call a cycle nonempty if at least one message is transmitted during it.

Introduce the following notations: $X_{[m,n]}(T)$ is the end instant of the last nonempty cycle in the system controlled by $\Psi$ with a cut-off flow, i.e., messages with numbers $m, \ldots, n$ only enter the system at time instants $T_m \leq \ldots \leq T_n$ respectively; $\widehat{X}_{[m,n]}(T)$ is the end instant of the last nonempty cycle for the system controlled by $\Psi^{(1)}$ with the same input flow; $\widetilde{X}_{[-k,\ell]}(T)$ is the same moment for the system controlled by $\Psi^{(-k)}$ $(k \geq 0)$ which is entered by messages with numbers $-k, -k+1, \ldots, \ell$ only.

The parameters introduced have the following properties.

**Property 1** (causality). *For all* $m \leq n$, $-k \leq \ell$, *the inequalities below hold a.s.*

$$X_{[m,n]}(T) \geq T_n, \qquad \widehat{X}_{[m,n]}(T) \geq T_n, \qquad \widetilde{X}_{[-k,\ell]}(T) \geq T_\ell.$$

**Property 2** (outer monotonicity). *If* $T \leq T'$ *a.s., then*

$$X_{[m,n]}(T) \leq X_{[m,n]}(T'), \qquad \widehat{X}_{[m,n]}(T) \leq \widehat{X}_{[m,n]}(T'), \qquad \widetilde{X}_{[-k,\ell]}(T) \leq \widetilde{X}_{[-k,\ell]}(T').$$

The proof of Property 2 follows directly from Theorem 4, under the conditions of which we now find ourselves. Let us prove, for example, the first inequality. Consider a realization of all control sequences of random variables on one elementary event.

Let $\tau$ and $\tau'$ be the instants of the end of transmission of all messages that have arrived in the systems under comparison ($\Sigma$ and $\Sigma'$). By Theorem 4, $\tau \leq \tau'$. Assume that $\tau$ belongs to the $i$th cycle in the server operation in the system $\Sigma$ and $\tau'$ belongs to the $i'$th cycle in $\Sigma'$. Note that the server routes in $\Sigma$ and $\Sigma'$ coincide up to the instant $T_m$, and also after $\tau'$ since the same service $R = \sigma_m + \ldots + \sigma_n$ is made, i.e., in both systems the $i'$th cycle ends at the instant $\Psi_{i'} + R$, and the $i$th cycle $\Sigma$ ends at $\Psi_i + R$. Therefore, $i \le i'$. Then, evidently, $\tau \le \Psi_i + R \overset{\text{def}}{=} X_{[m,n]}(T) \le \Psi_{i'} + R \overset{\text{def}}{=} X_{[m,n]}(T')$. $\triangle$

**Property 3** (homogeneity). *For all* $c \in \mathbb{R}$, $m \le n$, $-k \le \ell$,

$$\widehat{X}_{[m,n]}(c+T) \overset{\mathrm{D}}{=} \widehat{X}_{[m,n]}(T) + c, \qquad \widetilde{X}_{[-k,\ell]}(c+T) \overset{\mathrm{D}}{=} \widetilde{X}_{[-k,\ell]}(T) + c$$

*(equal in distribution).*

Proof. It suffices to note that the distributions of processes $\Psi^{(1)}$ and $\Psi^{(1)} - c$ coincide. $\triangle$

**Property 4** (separability). *If* $\widetilde{X}_{[-k,-k+m]}(T) \le T_{-k+m+1}$ *a.s. for some* $k, m, l$ *such that* $-k+m \le 0$, $-k+m < \ell$, *then*

$$\widetilde{X}_{[-k,\ell]}(T) = \widetilde{X}_{[-k+m+1,\ell]}(T).$$

Proof. Note that after the instant $\widetilde{X}_{[-k,-k+m]}(T)$ the server moves in both systems according to the route

$$\Psi^{(-k)} + \sum_{j=-k}^{-k+m-1} \sigma_j \overset{\text{def}}{=} \Psi^{(-k+m)},$$

and, starting from this instant, both systems operate a.s. identically. $\triangle$

In what follows, the numbers 1–4 over equalities and inequalities denote references to the properties proved above.

Introduce the notations $Z_{[m,n]}(T) \overset{\text{def}}{=} X_{[m,n]}(T) - T_n$, $\widehat{Z}_{[m,n]}(T) \overset{\text{def}}{=} \widehat{X}_{[m,n]}(T) - T_n$, $\widetilde{Z}_{-k,\ell}(T) \overset{\text{def}}{=} \widetilde{X}_{[-k,\ell]}(T) - T_\ell$.

**Lemma 1** (inner monotonicity of $\widetilde{X}$ and $\widetilde{Z}$). *For all integer* $-k \le \ell$,

$$\begin{aligned} \widetilde{X}_{[-k-1,\ell]}(T) &\ge \widetilde{X}_{[-k,\ell]}(T) \quad a.s., \\ \widetilde{Z}_{[-k-1,\ell]}(T) &\ge \widetilde{Z}_{[-k,\ell]}(T) \quad a.s. \end{aligned}$$

Proof. Consider the input flow $T'$ with points

$$T'_j = \begin{cases} T_j - \widetilde{Z}_{-k-1}(T) & \text{for } j \le -k-1, \\ T_j & \text{for } j \ge -k. \end{cases}$$

It is clear that $\widetilde{X}_{[-k,\ell]}(T') = \widetilde{X}_{[-k,\ell]}(T)$ since the instants of message arrivals for these systems are the same. Furthermore, $\widetilde{X}_{[-k-1,\ell]}(T') \overset{4}{=} \widetilde{X}_{[-k,\ell]}(T')$, and $\widetilde{X}_{[-k-1,\ell]}(T') \overset{2}{\le} \widetilde{X}_{[-k-1,\ell]}(T)$ a.s. $\triangle$

Define the shift transformations acting on functions that are measurable with respect to $\sigma$-algebras generated by the input flow and the server route $\Psi$. As in Sec. 2, a random vector $\eta_i$ characterizes the server route during the $i$th cycle for the process $\Psi$, $\psi_i = \Psi_i - \Psi_{i-1}$, $\Psi_0 = 0$. If

$$S = h(\ldots, \xi_n, \xi_{n+1}, \ldots; \ldots, \eta_m, \eta_{m+1}, \ldots),$$

then

$$\begin{aligned} \theta_\xi \circ S &\overset{\text{def}}{=} h(\ldots, \xi_{n+1}, \xi_{n+2}, \ldots; \ldots, \eta_m, \eta_{m+1}, \ldots), \\ \theta_\Psi \circ S &\overset{\text{def}}{=} h(\ldots, \xi_n, \xi_{n+1}, \ldots; \ldots, \eta_{m+1}, \eta_{m+2}, \ldots). \end{aligned}$$

Let $\theta_\xi^n$, $\theta_\Psi^n$ be iterations of transformations $\theta_\xi$, $\theta_\Psi$.

**Lemma 2** (subadditivity property for $X$ and $Z$). *For arbitrary $m \le \ell < n$, for any $T$, inequalities*

$$\begin{aligned} X_{[m,n]}(T) &\le X_{[m,\ell]}(T) + \theta_\xi^\ell \circ \theta_\Psi^{\rho[m,\ell]} \circ X_{[1,n-\ell]}(T), \\ Z_{[m,n]}(T) &\le Z_{[m,\ell]}(T) + \theta_\xi^\ell \circ \theta_\Psi^{\rho[m,\ell]} \circ Z_{[1,n-\ell]}(T) \end{aligned}$$

*hold a.s., where* $\rho[m,\ell] = \min\left\{k:\ \Psi_k \ge X_{[m,\ell]}(T) - \sum_{j=m}^{\ell} \sigma_j\right\}$.

Proof. Consider the input flow $T^1 = \{T_j^1\}$,

$$T_j^1 = \begin{cases} T_j & \text{for } j \le \ell, \\ T_j + Z_{[m,\ell]}(T) & \text{for } j > \ell. \end{cases}$$

Then

$$X_{[m,n]}(T) \overset{2}{\le} X_{[m,n]}(T^1) = X_{[m,\ell]}(T) + \theta_\xi^\ell \circ \theta_\Psi^{\rho[m,\ell]} \circ X_{[1,n-\ell]}(T). \tag{2}$$

The subadditivity property for $Z$ follows from (2). $\triangle$

**Lemma 3** (subadditivity property for $\widehat{X}$ and $\widehat{Z}$). *For arbitrary $m \le \ell < n$, for any $T$, inequalities*

$$\begin{aligned} \widehat{X}_{[m,n]}(T) &\le \widehat{X}_{[m,\ell]}(T) + \theta_\xi^\ell \circ \theta_\Psi^{\widehat{\rho}[m,\ell]} \circ X_{[1,n-\ell]}(T), \\ \widehat{Z}_{[m,n]}(T) &\le \widehat{Z}_{[m,\ell]}(T) + \theta_\xi^\ell \circ \theta_\Psi^{\widehat{\rho}[m,\ell]} \circ Z_{[1,n-\ell]}(T) \end{aligned}$$

*hold a.s., where* $\widehat{\rho}[m,\ell] = \min\left\{k:\ \Psi_k^{(1)} \ge \widehat{X}_{[m,\ell]}(T) - \sum_{j=m}^{\ell} \sigma_j\right\}$.

The proof of this statement is similar to that of Lemma 2.

**Lemma 4** (the law of large numbers). *A finite constant $\gamma \ge 0$ exists such that*

$$\begin{aligned} &\frac{Z_{[1,n]}}{n} \xrightarrow{\mathrm{P}} \gamma, && \frac{\mathbf{E}Z_{[1,n]}}{n} \to \gamma, \\ &\frac{Z_{[-n,-1]}}{n} \xrightarrow{\mathrm{P}} \gamma, && \frac{\mathbf{E}Z_{[-n,-1]}}{n} \to \gamma, \\ &\frac{\widehat{Z}_{[1,n]}}{n} \xrightarrow{\mathrm{P}} \gamma, && \frac{\widehat{Z}_{[-n,-1]}}{n} \xrightarrow{\mathrm{P}} \gamma \end{aligned}$$

*as $n \to \infty$.*

Proof. The statement of the lemma for the process $Z$ follows from the results presented in the Appendix (on random subadditive subsequences). Let us prove that the lemma is valid for $\widehat{Z}$. Let us show that

$$\lim_{n\to\infty} \mathbf{P}\left(\left|\frac{\widehat{Z}_{[1,n+M]}(T)}{n} - \gamma\right| \ge \varepsilon\right) = 0$$

for some constant $M$ and an arbitrary $\varepsilon > 0$.

(a) By the subadditivity property,

$$\widehat{Z}_{[1,M+n]}(T) \le \widehat{Z}_{[1,M]}(T) + \theta_\xi^M \circ \theta_\Psi^{\widehat{\rho}[1,M]} \circ Z_{[1,n]}(T).$$

Therefore, for any $\varepsilon > 0$ and $\delta < \varepsilon$,

$$\mathbf{P}\left(\frac{\widehat{Z}_{[1,n+M]}(T)}{n} \ge \gamma + \varepsilon\right) \le \mathbf{P}\left(\frac{\widehat{Z}_{[1,M]}(T)}{n} \ge \delta\right) + \mathbf{P}\left(\frac{Z_{[1,n]}(T)}{n} \ge \gamma + \varepsilon - \delta\right) \to 0$$

since $\widehat{Z}_{[1,M]}(T)$ has nonsingular distribution and $Z$ obeys the law of large numbers (LLN).

(b) To prove the second part of the statement, we have to ascertain some property of "solidarity" of processes $Z$ and $\widehat{Z}$.

**Lemma 5.** *For any $x > 0$, $x_0 < x$, and any $M \in \mathbb{N}$, an event $A_M$ and a random variable $\phi_M$ with nonsingular distribution exist such that*

$$\mathbf{P}\Big(Z_{[1,n+1]} \geq x\Big) \leq \mathbf{P}\Big(\overline{A_M}\Big) + \mathbf{P}\Big(\widehat{Z}_{[1,M+n]} \geq x - x_0\Big) + \mathbf{P}\big(\phi_M \geq x_0\big),$$
$$\mathbf{P}\Big(\overline{A_M}\Big) \to 0 \quad \textit{as} \quad M \to \infty.$$

Indeed, let $M > 1$ be an arbitrary fixed number, $T^1$ be the input flow with points

$$T_j^1 = \begin{cases} -\infty & \text{for } j \leq M-1, \\ T_j & \text{for } j > M. \end{cases}$$

Let us supply, for convenience, all processes with an additional argument to indicate the server route that controls the system. We have

$$\widehat{X}_{[1,M+n]}(T, \Psi^{(1)}) \overset{2}{\geq} \widehat{X}_{[1,M+n]}(T^1, \Psi^{(1)}) \overset{\mathrm{D}}{=} \widehat{X}_{[M,M+n]}\left(T^1, \Psi^{(1)} - \sum_{j=1}^{M-1} \sigma_j\right).$$

Let $\widetilde{\Psi} = \Psi^{(1)} - \sum\limits_{j=1}^{M-1} \sigma_j$, $\widetilde{\Psi}_0$ be the first positive point of this process, the event $A_M = \{\widetilde{\Psi}_0 < T_{M-1}\}$. Denote by $T^2$ the input flow with points $T_j^2 = \widetilde{\Psi}_0 + T_j - T_{M-1}$. Then

$$\widehat{X}_{[M,M+n]}\Big(T^1, \widetilde{\Psi}\Big) I\{A_M\} \overset{2}{\geq} \widehat{X}_{[M,M+n]}\Big(T^2, \widetilde{\Psi}\Big) I\{A_M\} \overset{\mathrm{D}}{=} \Big(\widetilde{\Psi}_0 + \theta_\xi^{M-1} \circ X_{[1,n+1]}(T)\Big) I\{A_M\}.$$

Denoting $\phi_M = T_{M-1} - \widetilde{\Psi}_0$, we obtain

$$\big(Z_{[1,n+1]}(T) - \phi_M\big) I\{A_M\} \overset{\mathrm{D}}{\leq} \widehat{Z}_{[1,M+n]}(T) I\{A_M\}. \tag{3}$$

By (3), we conclude that for any $x$, $x_0 < x$,

$$\begin{aligned} \mathbf{P}\Big(Z_{[1,n+1]} \geq x\Big) &\leq \mathbf{P}\Big(\overline{A_M}\Big) + \mathbf{P}\Big(Z_{[1,n+1]} \times I\{A_M\} \geq x\Big) \\ &\leq \mathbf{P}\Big(\overline{A_M}\Big) + \mathbf{P}\Big(\widehat{Z}_{[1,M+n]} \geq x - x_0\Big) + \mathbf{P}\big(\phi_M \geq x_0\big), \end{aligned}$$

which completes the proof of Lemma 5.

Using Lemma 5, we conclude that for any $\varepsilon, \varepsilon_1, n, M$,

$$\mathbf{P}\left(\frac{Z_{[1,n+1]}}{n} \geq \gamma - \varepsilon\right) \leq \mathbf{P}\Big(\overline{A_M}\Big) + \mathbf{P}\left(\frac{\widehat{Z}_{[1,M+n]}}{n} \geq \gamma - \varepsilon - \varepsilon_1\right) + \mathbf{P}\left(\frac{\phi_M}{n} \geq \varepsilon_1\right).$$

Since $\widetilde{\Psi}_0$ and $\phi_M$ are random variables with nonsingular distribution and $T_{M-1}$ infinitely grows with $M$, then for any $\delta > 0$ there exist $M = M(\delta)$, $\varepsilon_1 = \varepsilon_1(\delta) > 0$ such that $\mathbf{P}\big(\overline{A_M}\big) < \delta$ and, for all $n$ sufficiently large, $\mathbf{P}(\Phi_M/n \geq \varepsilon_1) < \delta$. Then

$$\liminf_n \mathbf{P}\left(\frac{\widehat{Z}_{[1,M+n]}}{n} \geq \gamma - \varepsilon - \varepsilon_1\right) \geq \liminf_n \mathbf{P}\left(\frac{Z_{[1,n+1]}}{n} \geq \gamma - \varepsilon\right) - 2\delta = 1 - 2\delta,$$

which, together with (a), completes the proof of the LLN. $\triangle$

**Corollary 4.** *Let $\gamma$ be the constant introduced in Lemma* 4. *Then*

$$X_{[1,n]}/n \xrightarrow{\mathrm{P}} \gamma + \lambda^{-1}, \qquad \widehat{X}_{[1,n]}/n \xrightarrow{\mathrm{P}} \gamma + \lambda^{-1}.$$

Let $A$ be the following event:

$$A = \left\{ \lim_n \widetilde{Z}_{[-n,0]} < \infty \right\} = \left\{ \widetilde{X} = \lim_n \widetilde{X}_{[-n,0]} < \infty \right\}$$

(the limits are considered in the sense of a.s. convergence, and the existence of limits is guaranteed by Lemma 1).

**Theorem 5** (the law of 0 and 1). *The probability of $A$ equals either* 0 *or* 1.

Proof. Consider a polling system where entering messages have numbers $-n, -n+1, \dots, 0, 1$, and the server moves along the route $\Psi^{(-n,1)} \overset{\text{def}}{=} \Psi^{(-n)} - \sigma_1$. Denote the end instant of the last nonempty cycle in such system by $X^0_{[-n,1]}(T)$. Consider the auxiliary input flow $T^1$ with points

$$T^1_j = \begin{cases} T_j & \text{for } j \le 0, \\ \max\left\{T_1, \widetilde{X}\right\} & \text{for } j \ge 1. \end{cases}$$

Then $\widetilde{X}_{[-n,1]}(T^1) = \widetilde{X}_{[-n,0]}(T) + \widehat{X}_{[1,1]}(T^1) - \max\{T_1, \widetilde{X}\}$, since $\Psi^{(-n)} + \sum\limits_{j=-n}^{0} \sigma_j \overset{\text{def}}{=} \Psi^{(1)}$, i.e., starting from the instant $\widetilde{X}_{[-n,0]}(T) \le \widetilde{X}$, the server moves identically in the systems with routes $\Psi^{(-n)}$ and $\Psi^{(1)}$. Denote $\phi = \widehat{X}_{[1,1]}(T^1) - \max\{T_1, \widetilde{X}\}$. Then a.s.

$$\widetilde{X}_{[-n,0]}(T) = \widetilde{X}_{[-n,1]}(T^1) - \phi \ge \widetilde{X}_{[n,1]}(T) - \phi \ge \widetilde{X}_{[-n,1]}(T - \sigma_1) - \phi = X^0_{[-n,1]}(T) - \sigma_1 - \phi. \tag{4}$$

The shift

$$\theta_\xi \circ \Psi^{(-n)} \equiv \Psi^{(-n+1,1)} \equiv \Psi^{(-n)} + \sigma_{-n} - \sigma_1$$

is ergodic (see, e.g., [20, Chapter 1, Sec. 7]), and

$$\theta_\xi \circ \widetilde{X}_{[-n-1,0]}(T) = X^0_{[-n,1]}(T).$$

The inequality (4) implies that $A \subseteq \theta_\xi \circ A$. Since the shift $\theta_\xi$ is ergodic, the event $A$ has probability either 0 or 1. △

**Scaling.** For any $0 \le c < \infty$, denote by $cT$ the sequence consisting of the points $\{cT_i\}$, $i \in \mathbb{Z}$. It is obvious that, for all $n$, $Z_{[-n,-1]}(cT)$ decreases in $c$ and $X_{[1,n]}(cT)$ increases in $c$ a.s. Therefore, the lemma below is valid.

**Lemma 6.** *For any $c \ge 0$, a nonnegative constant $\gamma(c)$ exists such that $Z_{[-n,-1]}(cT)/n \xrightarrow{\mathrm{P}} \gamma(c)$ and $\gamma(c)$ decreases in $c$, whereas $\gamma(c) + c\lambda^{-1}$ increases in $c$.*

**Theorem 6.** *If $\lim \widetilde{X}_{[-n,0]}(T) = \infty$ a.s., then $\lambda\gamma(0) \ge 1$. If $\lambda\gamma(0) > 1$, then $\lim \widetilde{X}_{[-n,0]}(T) = \infty$ a.s.*

Proof. First, let us prove the second assertion. Let $Q$ be a point process whose points are all zero, $Q = 0 \times T$. For a fixed $n$, denote by $T^n$ the point process with points $T^n_j = T_{-n}$ for any $j$. Then

$$\widetilde{X}_{[-n,0]}(T) \overset{2}{\ge} \widetilde{X}_{[-n,0]}(T^n) = Y_{[-n,0]}(Q) + T_{-n},$$

where $Y_{[-n,0]}(Q)$ is the end instant of the last nonempty cycle in the system with $n+1$ messages and server route $\Psi^{(-n)} - T_{-n}$. This route is also stationary and does not depend on an input flow, and random variables $Y_{[-n,0]}(Q)$ and $\widehat{X}_{[-n,0]}(Q)$ are identically distributed. Hence,

$$\liminf \frac{\widetilde{X}_{[-n,0]}(T)}{n} \ge \gamma(0) - \lambda^{-1} > 0,$$

which completes the proof of the second assertion of the lemma.

Let us prove the first one. Let $\ell \geq 1$ be a fixed integer number. Consider the events

$$A_{n,\ell} = \left\{ \widetilde{X}_{[-n,0]}(T) \geq T_\ell \right\}, \qquad B_{n,\ell} = \left\{ \widehat{X}_{[-n,0]}(T) \geq T_\ell \right\}.$$

By the condition of the theorem, $\mathbf{P}(A_{n,\ell}) \to 1$ as $n \to \infty$ and, since the random variables $\widetilde{X}_{[-n,0]}(T)$ and $\widehat{X}_{[-n,0]}(T)$ are identically distributed, we also have

$$\mathbf{P}\big(B_{n,\ell}\big) \to 1, \qquad \mathbf{P}\big(\,\overline{B_{n,\ell}}\,\big) \to 0. \tag{5}$$

By the subadditivity property, for any integer $n > 0$,

$$\widehat{X}_{[-n,\ell]}(T) \leq \widehat{X}_{[-n,0]}(T) + \theta_\Psi^{\rho[-n,0]} \circ \sum_{i=1}^{\ell} X_i \quad \text{a.s.}, \tag{6}$$

where $X_1 = X_{[1,1]}, \ldots, X_i = \theta_\xi \circ \theta_\Psi \circ X_{i-1}$ are integrable and identically distributed.

Introduce the process $T^n$ with points

$$T_j^n = \begin{cases} T_j & \text{for } j \leq 0, \\ \widehat{X}_{[-n,0]}(T) & \text{for } j \geq 1. \end{cases}$$

Then

$$\widehat{X}_{[-n,\ell]}(T) I(B_{n,\ell}) \leq \widehat{X}_{[-n,\ell]}(T^n) I(B_{n,\ell}) = \Big(\widehat{X}_{[-n,0]}(T) + Y_{[1,\ell]}(Q)\Big)\, I(B_{n,\ell}) \quad \text{a.s.}, \tag{7}$$

where $Y_{[1,\ell]}(Q) = \theta_\Psi^{\rho[-n,0]} \circ X_{[1,\ell]}(Q) \stackrel{\mathrm{D}}{=} X_{[1,\ell]}(Q)$. From (6) and (7), we obtain (a.s.)

$$\widehat{Z}_{[-n,\ell]}(T) - \widehat{Z}_{[-n,0]}(T) \leq \theta_\Psi^{\rho[-n,0]} \circ \sum_{i=1}^{\ell} X_i \times I\big(\,\overline{B_{n,\ell}}\,\big) + Y_{[1,\ell]}(Q) I(B_{n,\ell}) - T_\ell \equiv I_1 + I_2, \tag{8}$$

$\mathbf{E}(I_1) + \mathbf{E}(I_2) < \infty$. Due to the stationarity of the server routing,

$$\widetilde{Z}_{[-n,\ell]}(T) - \widetilde{Z}_{[-n,0]}(T) \stackrel{\mathrm{D}}{=} \widehat{Z}_{[-n,\ell]}(T) - \widehat{Z}_{[-n,0]}(T) \stackrel{\text{a.s.}}{\leq} I_1 + I_2. \tag{9}$$

Due to the stationarity of the input flow and the property of inner monotonicity,

$$\widetilde{Z}_{[-n,\ell]}(T) - \widetilde{Z}_{[-n,0]}(T) \stackrel{\mathrm{D}}{=} \widetilde{Z}_{[-n-\ell,0]}(T) - \widetilde{Z}_{[-n-\ell,-\ell]}(T) \stackrel{\text{a.s.}}{\geq} \widetilde{Z}_{[-n-\ell,0]}(T) - \widetilde{Z}_{[-n-2\ell,-\ell]}(T) \stackrel{\text{def}}{=} \phi_1. \tag{10}$$

Inequalities (9) and (10) imply that $\Phi_1 \stackrel{\mathrm{D}}{\leq} I_1 + I_2$, hence, $\mathbf{E}(\Phi_1^+) < \infty$.

Next, note that random variables

$$\widetilde{Z}_{[-n-\ell,0]}(T), \quad \widetilde{Z}_{[-n-2\ell,-\ell]}(T), \quad \widetilde{Z}_{[-n-3\ell,-2\ell]}(T), \quad \ldots$$

are identically distributed and form a stationary sequence. By Lemma 10 (see Appendix), $\mathbf{E}|\phi_1| < \infty$, $\mathbf{E}\phi_1 = 0$. Then

$$0 = \mathbf{E}\phi_1 \leq I_1 + I_2 = \ell \mathbf{E}\big(X_{[1,1]} I\big(\,\overline{B_{n,\ell}}\,\big)\big) + \mathbf{E}\big(Y_{[1,\ell]}(Q) I(B_{n,\ell})\big) - \ell\lambda^{-1}. \tag{11}$$

Letting $n$ tend to infinity and using (5), we obtain that the first component on the right-hand side of (11) tends to zero, and

$$0 \leq \mathbf{E}\big(Y_{[1,\ell]}(Q)\big) - \ell\lambda^{-1}.$$

Dividing both sides of this inequality by $\ell$, and letting $\ell$ tend to infinity, we obtain $0 \leq \gamma(0) - \lambda^{-1}$. △

**Boundedness of the queue-length process.**

**Property 5.** *The conditions below are equivalent:*

(a) $\sup_n \mathbf{P}\Big(\widehat{Z}_{[1,n]} > x\Big) = \sup_n \mathbf{P}\Big(\widetilde{Z}_{[-n,0]} > x\Big) \to 0$ *as* $x \to \infty$,

(b) $\sup_n \mathbf{P}\big(Z_{[1,n]} > x\big) \to 0$ *as* $x \to \infty$.

Indeed, (b) implies (a) by Lemma 3; the converse holds by Lemma 5.

Let us show that boundedness in probability of the process $Z_{[1,n]}(T)$ is equivalent to boundedness in probability of the total queue length $Q_n = Q_n(T)$ in the system at instant $T_n$. Let $Q_n = Q_n^1 + \ldots + Q_n^K$, $\sigma_{k,j}$ be the transmission time of the $j$th message in the $k$th queue, i.e., if

$$\begin{aligned} m_1 &= \min\{i \geq 1 : I(\mu_i = k) = 1\}, \\ m_j &= \min\{i > m_{j-1} : I(\mu_i = k) = 1\}, \end{aligned}$$

then $\sigma_{k,j} = \sigma_{m_j}$. Let $n_k = \max\{j : m_j \leq n\}$. Then, for any $k = 1, \ldots, K$,

$$\sum_{j=n_k-Q_n^k+2}^{n_k} \sigma_{k,j} \leq Z_{[1,n]} \leq \sum_{k=1}^{K} \sum_{j=n_k-Q_n+1}^{n_k} \sigma_{k,j} + \sum_{j=n_0+1}^{n_0+Q_n} \psi_j, \tag{12}$$

where $\psi_j = \Psi_j - \Psi_{j-1}$ is the total switching time during the $j$th cycle, $n_0(T, \Psi)$ is the largest number of a cycle completed up to $T_n$.

Since the input flow is stationary and ergodic, and random variables $\psi_j$ are i.i.d., (12) implies that $Q_n(T)$ and $Z_{[1,n]}(T)$ are bounded in probability simultaneously.

If the condition $\lambda_\gamma(0) < 1$ holds, $\lim \widetilde{Z}_{[-n,0]}(T) < \infty$ a.s. Then Property 5 and inequality (12) imply that $Q_n(T)$ is bounded in probability. In the case $\lambda_\gamma(0) > 1$, Theorem 6 yields $\lim \widetilde{Z}_{[-n,0]}(T) = \infty$ a.s.; therefore, $Z_{[1,n]}(T)$ and $Q_n(T)$ infinitely grow in probability.

**Systems with policies from the classes $A$, $\widehat{A}$.** Let us state some corollaries of Theorem 6. Consider a stochastic system $\Sigma^g$, where $g_k \in A$ is the transmission policy in the $k$th queue. According to Corollary 3, there exist functions $f_k, h_k \in M$ such that

$$\lim_{x\to\infty} f_k(x) = \lim_x g_k(x) = F_k = \lim_x h_k(x),$$

$f_k(x) \leq g_k(x) \leq h_k(x)$ for all $x \geq 0$. As will be shown in Sec. 5, for systems $\Sigma^f$, $\Sigma^h$, we have

$$\gamma(0) = \sigma + \max_k \left[\frac{p_k}{F_k C_k} W\right].$$

Then, for $Z^g_{[1,n]}(T)$, the corollary below holds.

**Corollary 5.** *If* $Z^g_{[1,n]}(T) \xrightarrow{\text{P}} \infty$, *then* $\lambda_\gamma(0) \geq 1$. *If* $\lambda_\gamma(0) > 1$, *then* $Z^g_{[1,n]} \xrightarrow{\text{P}} \infty$.

Proof. Indeed, if $Z^g_{[1,n]}(T) \xrightarrow{\text{P}} \infty$, then, by Corollary 3,

$$Z^f_{[1,n]}(T) \overset{\text{a.s.}}{\geq} Z^g_{[1,n]}(T) \xrightarrow{\text{P}} \infty.$$

Since Corollary 5 is valid for the system $\Sigma^f$, $\widetilde{Z}^f_{[-n,0]}(T) \xrightarrow{\text{P}} \infty$. Since the sequence $\widetilde{Z}^f_{[-n,0]}(T)$ is monotone in $n$, convergency to infinity holds a.s., and then Theorem 6 yields $\lambda_\gamma(0) \geq 1$. In the case $\lambda_\gamma(0) > 1$, $\widetilde{Z}^h_{[-n,0]}(T) \to \infty$ a.s.; hence,

$$Z^g_{[1,n]}(T) \overset{\text{a.s.}}{\geq} Z^h_{[1,n]}(T) \xrightarrow{\text{P}} \infty$$

due to Corollary 3 and Property 5. $\triangle$

The statement of Theorem 2 follows now directly from Corollary 5 and inequality (12) which, unlike preceding statements of this section, is not related with the assumptions on the monotonicity of transmission policies.

Consider the class $\widehat{A}$. For $g \in \widehat{A}$, generally speaking, it is impossible to construct a monotone function $h \geq g$ with the same limit at infinity, but the lower function can still be constructed (see Remark 4). Thus, the corollary below holds.

**Corollary 6.** *If* $Z^g_{[1,n]}(T) \xrightarrow{\mathrm{P}} \infty$, *then* $\lambda_\gamma(0) \geq 1$.

The proof is quite similar to that of Corollary 5. Since $Z^g_{[1,n]}(T)$ and $Q^g_n(T)$ are bounded in probability simultaneously, Corollary 5 implies Theorem 1.

## 5. Explicit form of the ergodicity condition

Consider a system which is entered by $N$ messages with numbers $1, \ldots, N$ at $t = 0$; other messages do not arrive; the server moves from queue to queue along a cyclic route $\Psi$. It is required to find $\gamma(0) \equiv \lim\limits_N Z_{[1,n]}/N$. The existence of this limit was proved in Sec. 4. Note that $Z_{[1,n]} = Z_N + \sum\limits_{n=1}^{N} \sigma_n$, where $Z_N$ is the end instant of the last nonempty cycle for the same system with zero message-transmission times. Then, $\gamma(0) \equiv \lim\limits_N Z_N/N + \sigma$.

Consider a determinate model of a polling system, which is a realization of the above-described system with zero transmission times on one elementary event.

We assume that some laws of large numbers hold, the validity of which is guaranteed by the conditions of Sec. 2. Also, let $f_k \in M$ for $k = 1, \ldots, K$.

Let $w_n$ be the total switching time over the $n$th cycle, $\ell_{k,n}$ be the number of visits to the $k$th queue ($k = 1, \ldots, K$) during the $n$th cycle, $L_k(n) = \sum\limits_{j=1}^{n} \ell_{k,j}$. Assume that (for any $k$)

$$\lim_n \frac{\sum\limits_{j=1}^{n} w_j}{n} = W > 0, \qquad \lim_n \frac{L_k(n)}{n} = \lim_n \frac{\sum\limits_{j=1}^{n} \ell_{k,j}}{n} = C_k > 0.$$

Let $N = N_1 + \ldots + N_K$ be the queue length at instant 0. Let $\lim\limits_N N_k/N = p_k > 0$ for any $k$.

**Bounded transmission policies.** Let $\lim\limits_x f_k(x) = F_k < \infty$ and, for $t \geq 0$, $Q_N(t)$ be the total queue length in the system at instant $t$, $Q_N(0) = N$.

Denote

$$\begin{aligned} x^0 &= \min\{x : f_k(y) = F_k \ \forall k, \ \forall y \geq x\}, \\ n_k &= \max\left\{n : \sum_{j=1}^{n} \ell_{k,j} F_k + x^0 \leq N_k\right\}. \end{aligned}$$

Since $n_k \to \infty$ as $N \to \infty$,

$$\lim_N \frac{\sum\limits_{j=1}^{n_k} \ell_{k,j} F_k + x^0}{n_k} = F_k C_k \quad \text{and} \quad \lim_N \frac{N_k}{n_k} = F_k C_k.$$

Therefore,

$$\frac{n_k}{N} = \frac{n_k}{N_k} \times \frac{N_k}{N} \to \frac{p_k}{F_k C_k} = \alpha_k > 0.$$

Let $\overline{n} = \max\{n_1, \ldots, n_K\}$. Then

$$Z_N = \sum_{j=1}^{\overline{n}} w_j + o(N) = \overline{n} W + o(N).$$

Since $\overline{n}/N \to \max\limits_{k \le K} \alpha_k$, we obtain $\lim\limits_N Z_N/N = \max\limits_k \alpha_k W$, i.e., for the case of bounded transmission policies we have proved the theorem below.

**Theorem 7.** *The constant* $\gamma(0)$ *introduced above is given by*

$$\gamma(0) = \sum_{k=1}^{K} p_k C_{\sigma,k} + \max_k \left\{ \frac{p_k}{F_k C_k} \right\} W = \sigma + \max_k \left\{ \frac{p_k}{F_k C_k} \right\} W,$$

*where* $\sigma = \mathbf{E}\sigma_n$ *is the mean message-transmission time.*

**Unbounded transmission policies.** Let, for some fixed $m \in \{1, \ldots, K-1\}$, $F_1, \ldots, F_m < \infty$, $F_{m+1} = \ldots = F_K = \infty$. Fix an arbitrary positive $L > 0$ and consider the system $\Sigma^L$ that differs from our system in message transmission policies in the queues $m+1, \ldots, K$ only, i.e., let $f_k^L(x) = \min\{L, f_k(x)\} \le f_k(x)$ for $m+1 \le k \le K$ and $f_k^L(x) = f_k(x)$ for $1 \le k \le m$. It is obvious that the limits $\lim\limits_x f_k^L(x) = F_k^L$ exist and are equal to $f_k$ for the first $m$ stations and to $L$ for the others. By Theorem 4, the end instant of the last nonempty cycle $Z_N^L \ge Z_N$ a.s., and by Theorem 7 just proved,

$$\limsup \frac{Z_N}{N} \le \lim_N \frac{Z_N^L}{N} = \max_k \frac{p_k}{F_k^L C_k} W = \max\left( \max_{k \le m} \frac{p_k}{F_k C_k}, \max_{k > m} \frac{p_k}{L C_k} \right) W. \qquad (13)$$

Consider the system $\Sigma'$ with the following message transmission policies: $f_k'(x) = x \ge f_k(x)$ for $m+1 \le k \le K$ and $f_k'(x) = f_k(x)$ for $1 \le k \le m$. By Theorem 4, $Z_N' \le Z_N$ a.s.

Let us find $\lim\limits_N Z_N'/N$. Repeating the construction from the first part of the section, we obtain that

(a) for $1 \le k \le m$, $n_k \to \infty$;

(b) for $m+1 \le k \le K$, one visit to the $k$th queue suffices, i.e., $n_k = \min\{n : \ell_{k,n} > 0\} < \infty$. In particular, $\max\{n_{m+1}, \ldots, n_K\}/N \to 0$.

Let $s = \max\{n_1, \ldots, n_m\}$. We have the representation as follows: $Z_N' = \sum\limits_{j=1}^{s} w_j + o(N) = sW + o(N)$. Therefore,

$$\liminf \frac{Z_N}{N} \ge \lim_N \frac{Z_N'}{N} = \max_{1 \le k \le m} \alpha_k W = \max_{1 \le k \le m} \frac{p_k}{F_k C_k} W. \qquad (14)$$

By (13) and (14), for any $L$,

$$\sigma + \max_{1 \le k \le m} \frac{p_k}{F_k C_k} W \le \gamma(0) \le \sigma + \max\left( \max_{1 \le k \le m} \frac{p_k}{F_k C_k}, \max_{m < k \le K} \left\{ \frac{p_k}{L C_k} \right\} \right) W.$$

Letting $L$ tend to infinity, we obtain that Theorem 7 holds for the case of unbounded transmission policies as well ($1/F_k = 0$ for $F_k = \infty$).

## 6. Existence of the stationary regime

Consider an arbitrary realization of a polling system described in Sec. 2 on one elementary event (determinate model). Renumber the server route $\Psi^{(1)}$ in the empty system as follows. Assign the number 0 to the time of the switching that is made by the server at instant $+0$; to the next switching time, assign the number 1, etc.; to the previous one, the number $-1$, etc. Thus, we obtain the sequence of switching times $\{w_i\}_{i=-\infty}^{\infty}$. Let, as before, $\Psi_0^{(1)} = \min\left\{\Psi_\ell^{(1)} : \Psi_\ell^{(1)} > 0\right\}$ be the end instant of the zero cycle, $\Psi_\ell^{(1)}$, $\ell \in \mathbb{Z}$, be the end instant of the $\ell$th cycle.

Introduce the following notations: $a_i$ is the starting instant of the $i$th switching time in the empty system, $a_{i+1} = a_i + w_i$; $E_\ell = \Psi_\ell^{(1)} - \Psi_{\ell-1}^{(1)}$ is the total switching time in the $\ell$th cycle.

For the point process $\Psi^{(-n)} = \Psi^{(1)} - \sum_{j=-n}^{0} \sigma_j$, keep the above-introduced numeration of switching times and cycles, i.e., the $i$th switching time for the process $\Psi^{(-n)}$ starts at the time instant $a_i^{(-n)} \stackrel{\text{def}}{=} a_i - \sum_{j=-n}^{0} \sigma_j$, the $\ell$th cycle ends at the instant $\Psi_\ell^{(-n)} \stackrel{\text{def}}{=} \Psi_\ell^{(1)} - \sum_{j=-n}^{0} \sigma_j$.

Consider a polling system where $n+1$ entering messages from the input flow have numbers $-n, -n+1, \ldots, 0$ (see Sec. 4) and the server moves along the route $\Psi^{(-n)}$. We call such a system a $[-n,0]$-model. The instant of the end of the transmission of all arrived messages in a $[-n,0]$ system is denoted by $\tau_{[-n,0]}$.

The model introduced has the following properties:

1. $\tau_{[-n,0]} < \infty$;
2. $\tau_{[-n,0]}$ coincides with one of the instants $a_i$, $i \geq 1$, say, with $a_{M_{[-n,0]}}$;
3. $\tau_{[-n,0]} \leq \widetilde{X}_{[-n,0]}$, where $\widetilde{X}_{[-n,0]}$ is the instant of the end of the last nonempty cycle, as in Sec. 4.

Denote by $\delta_{[-n,0]}$ the number of the last switching time in a $[-n,0]$-model before the instant $\tau_{[-n,0]}$. Evidently, $\delta_{[-n,0]} = M_{[-n,0]} - 1$ since, starting from $\tau_{[-n,0]}$, the server route in a $[-n,0]$-model coincides with $\Psi^{(1)}$.

**Lemma 7.** *The sequences $\tau_{[-n,0]}$, $\delta_{[-n,0]}$ do not decrease in $n$.*

Proof. Let $n \geq 0$. Introduce a $[-n-1,0]'$-model that differs from the $[-n-1,0]$-model in the instant of the arrival of the message with number $-n-1$ only. Define $T'_{-n-1}$ in such a way that the transmission of the $(-n-1)$st message would be completed before $T_{-n}$. Let

$$\begin{aligned}
\ell_0 &= \max\left\{\ell : \Psi_\ell^{(-n-1)} + \sigma_{-n-1} \equiv \Psi_\ell^{(-n)} < T_{-n}\right\},\\
\ell_1 &= \max\left\{\ell \leq \ell_0 : \text{during the } \ell\text{th cycle, the server visits the station } \mu_{-n-1}\right\},\\
j_0 &= \max\left\{j : a_j^{(-n-1)} \leq \min\left\{T_{-n-1}, \Psi_{\ell-1}^{(-n-1)}\right\}\right\}.
\end{aligned}$$

Now, it suffices to put $T'_{-n-1} = a_{j_0}^{(-n-1)}$.

Let us compare the models $[-n-1,0]'$ and $[-n-1,0]$. By Theorem 4, $\tau'_{[-n-1,0]} \leq \tau_{[-n-1,0]}$. Since the total switching time of all messages is the same for these models, the total switching time over the time interval $\left[a_{j_0}^{(-n-1)}, \tau'_{[-n-1,0]}\right]$ for the first model is less than the total switching time over the interval $\left[a_{j_0}^{(-n-1)}, \tau_{[-n-1,0]}\right]$ for the second one. Therefore, $\delta'_{[-n-1,0]} \leq \delta_{[-n-1,0]}$ as well.

Comparing the $[-n-1,0]'$- and $[-n,0]$-models, we obtain

$$\tau'_{[-n-1,0]} \equiv \tau_{[-n,0]}, \qquad \delta'_{[-n-1,0]} \equiv \delta_{[-n,0]}$$

since $\Psi^{(-n-1)} + \sigma_{-n-1} \equiv \Psi^{(-n)}$. Thus, the lemma is proved. △

Let us assume that the condition below holds.

$A_0$: The limit $\lim_{n\to\infty} \tau_{[-n,0]} \equiv \tau < \infty$ exists.

If $A_0$ holds and $a_i \to \infty$ as $i \to \infty$, then $\tau$ coincides with one of the $a_i$'s, $i \geq 1$. Moreover, the statement below, which is a direct consequence of Lemma 7 and condition $A_0$, is valid.

**Corollary 7.** 1. *There can be found $N \geq 1$ and $M \geq 1$ such that, for any $n \geq N$, $\tau_{[-n,0]} = \tau = a_M$.*

2. *The limit $\lim_{n\to\infty} \delta_{[-n,0]} \equiv \delta \leq M < \infty$ exists.*

3. *There can be found $N_0$ such that $\delta_{[-n,0]} \equiv \delta$ for all $n \geq N_0$.*

4. *Therewith, $\delta = M - 1$.*

For a $[-n,0]$-model, denote by $c_i^{(-n)}$ the actual starting instant of the $i$th switching. By definition, $c_i^{(-n)} = a_i^{(-n)}$ if $a_i^{(-n)} < T_{-n}$, and $c_i^{(-n)} = a_i$ if $i \geq M_{[-n,0]}$.

Introduce the following notation: for any queue $k$ and any switching number $i$, let $G^k_{i,[-n,0]}$ denote the number of messages to be transmitted in the $k$th queue after the instant $c_i^{(-n)}$.

Repeating the reasoning of Lemma 7 and using statement 3 of Theorem 4, we obtain the corollary below.

**Corollary 8.** *For any $i, k$, the sequence $\left\{G^k_{i,[-n,0]}\right\}$ does not decrease in $n$.*

The following statement is valid.

**Lemma 8.** *The sequence $G^k_{\delta,[-n,0]}$ is upper bounded.*

Proof. Let $n \geq \max\{N, N_0\}$, and $k'$ be such that $G^{k'}_{i,[-n,0]} > 0$, $r_0 = \min\{j \geq 0 : \ \mu_{-j} = k'\}$. By the definition of transmission policies, it is evident that, in a $[-n,0]$-model for $n \geq r_0$, the transmission of the last group of messages of the queue $k'$ after the instant $c_\delta^{(-n)}$ cannot start before $T_{-r_0}$. Therefore, the total transmission time during the last visit of the server to this queue is not greater than $\tau - T_{-r_0}$. Then, for any $k$,

$$n_0 = \min\left\{\ell \geq 0 : \sum_{j=-\ell}^{0} \sigma_j I(\mu_j = k') \geq \tau - T_{-r_0}\right\} \geq G^k_{\delta,[-n,0]}. \ \triangle$$

Due to the monotonicity and boundness, the integer sequence $G^k_{\delta,[-n,0]}$ has properties as follows.

1. For any $k$, the limit $G^k_\delta = \lim_{n\to\infty} G^k_{\delta,[-n,0]} < \infty$ exists, and there exists $N_1 < \infty$ such that for all $n \geq N_1$, $k = 1, \ldots, K$, $G^k_\delta = G^k_{\delta,[-n,0]}$.

2. By definition, only one of the $G^k_\delta$'s is nonzero, i.e., $G^{k_0}_\delta > 0$, $G^k_\delta = 0$ for $k \neq k_0$. Then, in a $[-n,0]$-model, for any $n \geq \max\{N, N_0, N_1\}$, the last servicing is made in the queue $k_0$, and during this visit, the server transmits $G^{k_0}_\delta$ messages.

3. For the same $n$'s, $c_\delta^{(-n)} \equiv c_\delta = a_{\delta+1} - \sum\limits_{j=-m_\delta}^{0} \sigma_j I(\mu_j = k_0) - w_\delta$ does not depend on $n$, where $m_\delta = \min\left\{\ell : \sum\limits_{j=-\ell}^{0} I(\mu_j = k_0) = G^{k_0}_\delta\right\}$, $-m_\delta$ is the number of the message that is sent first in the last group of $G^{k_0}_\delta$ messages.

Consider now the sequence $G^k_{\delta-1,[-n,0]}$. Let us show the validity of Lemma 8 and Properties 1–3 for this sequence. Similarly to Lemma 8, the lemma below can be proved.

**Lemma 9.** *The sequence $G^k_{\delta-1,[-n,0]}$ is upper bounded.*

Since the sequence $G^k_{\delta-1,[-n,0]}$ is integer, nondecreasing, and upper bounded, the following statements hold.

1. For any $k$, the limit $G^k_{\delta-1} = \lim_{n\to\infty} G^k_{\delta-1,[-n,0]} < \infty$ exists, and there exists $N_2 < \infty$ such that for all $n \geq N_2$, $k = 1, \ldots, K$,

$$G^k_{\delta-1} = G^k_{\delta-1,[-n,0]}.$$

2. At most one of the $G^k_{\delta-1}$'s is nonzero.

3. For $n \geq \max\{N, N_0, N_1, N_2\}$, $c_{\delta-1}^{(-n)} \equiv c_{\delta-1}$ does not depend on $n$.

Set $m_0 = \max\{m \geq 0 : \ c_{\delta-m} \geq 0\}$. Since $c_{\delta-m_0} \leq a_{\delta-m_0}$, we have $m_0 < M$. Repeat the above reasoning for $G^k_{\delta-2,[-n,0]}, \ldots, G^k_{\delta-m_0-1,[-n,0]}$. After $m_0+1$ steps, we conclude that in a $[-n,0]$-model, for any $n \geq \overline{N} = \max\{N, N_0, N_1, N_2, \ldots, N_{m_0+2}\}$ and $t \geq 0 > c_{\delta-m_0-1}$, the system state at an instant $t$ does not depend on $n$.

Denote by $Q^k_{[-n,0]}(t)$ the queue length for the $k$th station at an instant $t$; by $W^k_{[-n,0]}(t)$, the residual message-transmission time for the $k$th queue at an instant $t$; by $j_{[-n,0]}(t)$, the number of the switching that is made at $t$ or of the preceding switching if the server is busy with transmission at $t$. Let us state the main result of this section, which follows from the above reasoning.

**Theorem 8.** *If the condition $A_0$ holds, a process $\{Q^k(t), W^k(t), j(t), t \geq 0\}$ and numbers $\overline{N}, M < \infty$ exist such that*

1. $\big\{Q^k_{[-n,0]}(t), W^k_{[-n,0]}(t), j_{[-n,0]}(t)\big\} = \{Q^k(t), W^k(t), j(t), t \geq 0\}$ *for any* $k = 1, \dots, K$, $t \geq 0$, $n \geq \overline{N}$,
2. $Q^k_{[-n,0]}(t) = W^k_{[-n,0]}(t) = Q^k(t) = W^k(t) = 0$ *for any* $k = 1, \dots, K$, $t \geq a_M$, $n \geq 0$.

Return to the stochastic model of a polling system described in Secs. 2 and 4. Since $\tau_{[-n,0]} \leq \widetilde{X}_{[-n,0]}$ a.s., under the condition $\lambda_\gamma(0) < 1$ we have

$$\tau \leq \lim \widetilde{X}_{[-n,0]}(T) < \infty \quad \text{a.s.}$$

by Theorem 6, i.e., the condition $A_0$ holds a.s. The validity of the additional assumption that $a_i \to \infty$ as $i \to \infty$ is obvious due to the stationarity of $\Psi^{(1)}$. Thus, the proof of Theorem 3 is completed.

*Remark 5.* With the help of the process $\{Q^k(t), W^k(t), j(t)\}$ introduced in Theorem 8, for the polling systems under consideration one can construct (as is done in [21] for Jackson-type queueing networks) the process $\big\{\big(\widehat{Q}^k(t), \widehat{W}^k(t), \widehat{j}(t)\big), -\infty < t < \infty\big\}$ that determines the stationary regime on the whole axis, and this regime is the minimum possible one. Stationarity is understood in the following sense: distributions of the processes $\big\{\big(Q^k(t,n), W^k(t,n), j(t,n)\big), t \geq 0\big\}$ coincide for all $n = 0, \pm 1, \pm 2, \dots$, where $Q^k(t,n) = \widehat{Q}^k(T_n + t)$, $W^k(t,n) = \widehat{W}^k(T_n + t)$, $j(t,n) = \widehat{j}(T_n + t)$. There exist natural examples where a stationary regime is not unique.

*Remark 6.* The maximum stationary regime also exists and can be constructed analogously to what is done in [21]. Schematically, this looks as follows: for any $0 < c < 1$, a polling system is introduced with the same control sequences as before, with the only change being that messages arrive at time instants $cT_i$. For $c$ close enough to 1, the traffic condition is fulfilled and, hence, the corresponding minimum stationary regimes exist. For $c \nearrow 1$, these stationary regimes monotonically decrease, and their limit provides the desired maximum stationary regime.

## 7. Possible generalizations of the model considered

**1. Batch arrivals.** All statements of Sec. 2 remain unchanged if batch message arrival in the input flow is admitted. Let $(\tau_n, \boldsymbol{\mu}_n, \boldsymbol{\sigma}_n)$ be a stationary metric-transitive sequence. Here, $\tau_n$ is the time between arrival instants of the $(n-1)$st and $n$th batch, $\boldsymbol{\mu}_n = (\mu_{n,1}, \dots, \mu_{n,K})$ is the number of messages of the $n$th batch sent to queues $1, \dots, K$ respectively, $\boldsymbol{\sigma}_n = (\boldsymbol{\sigma}_{n,1}, \dots, \boldsymbol{\sigma}_{n,K})$, $\boldsymbol{\sigma}_{n,k} = (\sigma_{n,k,1}, \dots, \sigma_{n,k,\mu_{n,k}})$ are the transmission times for the $n$th batch at the $k$th station. Let $\mathbf{E}\tau_1 = \lambda^{-1} < \infty$ and $\neq 0$, $\mathbf{E}\sum\limits_k \sum\limits_i \sigma_{1,k,i} = \sigma < \infty$, $0 < \mathbf{E}\mu_{1,k} = \mu_k < \infty$ for any $k = 1, \dots, K$. Then

$$\rho = \lambda \left[\sigma + \max_k \frac{\mu_k}{F_k C_k} W\right].$$

The proofs of Theorems 1–3 remain valid without any considerable changes (except for some obvious distinctions in terminology, such as "batch call" instead of "call," etc.)

**2. Random policies.** Theorems 1–3 also hold for natural extensions of the classes $A$, $\widehat{A}$, $M$, which include random functions. Assume that transmission policies depend on a random variable, namely, $f_k(x, D^j_k)$ messages are transmitted during the $j$th visit of the server to the $k$th station, where for any $k$, random variables $D^j_k$, $j = 1, 2, \dots$, are i.i.d. It is necessarily required therewith (say, in the extended class $M$) that for any $k, j$, $f_k(x, D^j_k) \leq f_k(x+1, D^j_k) \leq f_k(x, D^j_k) + 1$ a.s. In this case,

$$F_k = \mathbf{E} \lim_{x \to \infty} f_k\big(x, D^j_k\big) = \mathbf{E} F_k\big(x, D^j_k\big) \leq \infty.$$

The validity of the statements of Secs. 3, 4, and 6 is evident; we just explain the changes in Sec. 5.

Let $F_k < \infty$. Then $F_k(D_k^j) < \infty$, and random variables $x_k^j = \min\{x: \ f_k(x, D_k^j) = F_k(D_k^j)\}$ are i.i.d. and a.s. finite. Also,

$$\frac{\sum_{j=1}^{n} F_k(D_k^j)}{n} \to F_k \quad \text{a.s.} \tag{15}$$

Let us compute the limit $Z_N/N$ in the sense of convergence in probability. Let $N_k$ be a (random) length of the $k$th queue at $t = 0$,

$$m_k = \max\left\{n: \ N_k \geq x_k^1, \ N_k - F_k(D_k^1) \geq x_k^2, \ \ldots, \ N_k - \sum_{j=1}^{n-1} F_k(D_k^j) \geq x_k^n\right\}.$$

Since $m_k \to \infty$ a.s., we also have $n_k = \max\{j: \ L_k(j) \leq m_k\} \to \infty$ a.s. Then

$$1 \geq \frac{\sum_{j=1}^{m_k} F_k(D_k^j)}{N_k} \geq 1 - \frac{x_{m_k+1}}{N_k} \xrightarrow{\text{P}} 1.$$

Using (15), we obtain

$$\begin{aligned} \frac{N_k}{m_k} &= \frac{N_k}{\sum_{j=1}^{m_k} F_k(D_k^j)} \frac{\sum_{j=1}^{m_k} F_k(D_k^j)}{m_k} \xrightarrow{\text{P}} F_k, \\ \frac{n_k}{N} &= \frac{n_k}{m_k}\frac{m_k}{N_k}\frac{N_k}{N} \xrightarrow{\text{P}} \frac{1}{C_k} F_k p_k. \end{aligned}$$

Further computation of the limit desired is made in the same way as in Sec. 5. In the case $F_k = \infty$, the changes are analogous to the above ones.

**3. Exhaustive policies.** Theorems 1–3 are also valid in a more general case, where some transmission policies can have an "exhaustive" character, i.e., not only the queue but also new messages that arrive while the server works at a station can be transmitted before switching of the server. Such a policy can be defined by a sequence of functions $\{f_k^j\}_{j=1}^{\infty}$, where for all $j$, $f_k^j$'s belong to one of the classes introduced in Sec. 1, and $f_k^1(1) = 1$. If at the instant of a server arrival, there are $x_1$ messages in the queue, $f_k^1(x_1)$ of them are to be transmitted. Then, if at the instant of the transmission end, there are $x_2$ messages in the queue, $f_k^2(x_2)$ of them are to be transmitted, and so on. The server switches to the next station only if $f_k^j(x_j) = 0$ for some $j$ (for an analogous way of describing such policies, see, e.g., [1]).

Under these conditions, Theorems 1–3 are also valid. The quantity $F_k$ in the ergodicity condition for an exhaustive policy is determined as $F_k = \sum_j \lim_x f_k^j(x)$. If, for example, $f_k^j(x) \equiv x$ for all $k$ and $j$, then $F_k = \infty$ for all $k$ and $\rho = \lambda\sigma$.

The proofs of Theorems 1–3 remain unchanged; certain changes should be made in the proof of Theorem 4 only.

**4. Systems with infinite number of servers.** Under certain conditions, analogous results hold for this case as well. A paper on this subject is being prepared for publication.

## Appendix

Let $\{\xi_i\}$, $i \in \mathbb{N}$, be a stationary sequence of stationary random variables. Define a stationary sequence $\{\phi_i\}$, $i \in \mathbb{N}$, of random variables with nonsingular distributions as $\phi_i = \xi_i - \xi_{i+1}$. The following statement holds.

**Lemma 10.** *If* $\mathbf{E}(\phi_1^+) < \infty$, *then* $\mathbf{E}|\phi_1| < \infty$, $\mathbf{E}\phi_1 = 0$.

PROOF. The sequence $\{\phi_i\}$ obeys the LLN:

$$\frac{\phi_1 + \ldots + \phi_m}{m} = \frac{\xi_1}{m} - \frac{\xi_{m+1}}{m} \xrightarrow{\mathrm{P}} 0 \tag{16}$$

since the corresponding random variables have nonsingular distributions.

Let us show that $\mathbf{E}|\phi_1| < \infty$. Let $N > 0$, $\widetilde{\phi}_i = \max\{\phi_i, -N\} \geq -N$. Then, a random variable $\widetilde{\phi}$ exists such that $\mathbf{E}|\widetilde{\phi}| < \infty$, $\mathbf{E}\widetilde{\phi} = \mathbf{E}\widetilde{\phi}_1$, and

$$(\widetilde{\phi}_1 + \ldots + \widetilde{\phi}_m)/m \to \widetilde{\phi} \quad \text{a.s.}$$

Assume that $N$ exists such that $\mathbf{E}\widetilde{\phi} = \mathbf{E}\widetilde{\phi}_1 < 0$. Then, for some $\varepsilon > 0$, $\mathbf{P}(\widetilde{\phi} < -\varepsilon) = \Delta > 0$, i.e.,

$$\mathbf{P}\Big(\lim_m (\widetilde{\phi}_1 + \ldots + \widetilde{\phi}_m)/m < -\varepsilon\Big) = \Delta > 0.$$

Then, $m_0$ exists such that for all $m \geq m_0$,

$$\mathbf{P}\big((\widetilde{\phi}_1 + \ldots + \widetilde{\phi}_m)/m \leq -\varepsilon/2\big) \geq \Delta/2 > 0.$$

But, since $\phi_1 \leq \widetilde{\phi}_i$,

$$\mathbf{P}\big((\phi_1 + \ldots + \phi_m)/m \leq -\varepsilon/2\big) \geq \Delta/2 > 0,$$

which contradicts (16); thus, $\mathbf{E}\widetilde{\phi}_1 \geq 0$ for any $N$, and $\mathbf{E}\phi_1 \geq 0$. Then, obviously, $\mathbf{E}\phi_1^- \leq \mathbf{E}\phi_1^+ < \infty$, whence the desired statement follows.

Now, let us show that $\mathbf{E}\phi_1 = 0$. Let, on the contrary, $\mathbf{E}\phi_1 > 0$. Then a random variable $\phi$ exists such that $\mathbf{E}\phi = \mathbf{E}\phi_1 > 0$ and $(\phi_1 + \ldots + \phi_m)/m \to \phi$ a.s. Therefore, for some $\varepsilon > 0$, $\mathbf{P}(\widetilde{\phi} > \varepsilon) = \Delta > 0$. Then $m_0$ exists such that for all $m \geq m_0$,

$$\mathbf{P}\big((\widetilde{\phi}_1 + \ldots + \widetilde{\phi}_m)/m > \varepsilon/2\big) \geq \Delta/2 > 0,$$

which contradicts (16). $\triangle$

**LLN for random subadditive stochastic sequences.** The results presented below are generalizations of the corresponding properties of subadditive sequences [18, 19] for the case of shift transformations with random index. They are used in Sec. 4.

Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ be an arbitrary metric space. Consider two sequences of random variables, $\mathcal{X}$-valued $\{\xi_n\}_{-\infty}^{\infty}$ and real-valued $\{\psi_n\}_{-\infty}^{\infty}$. We assume that

(a) $\{\xi_n\}$ is a stationary ergodic sequence,

(b) $\{\psi_n\}$ is a sequence of i.i.d. nonnegative random variables, and $\mathbf{P}(\psi_1 > 0) > 0$,

(c) these sequences are mutually independent.

Set $\mathcal{F}_\xi = \sigma\{\xi_n;\, -\infty < n < \infty\}$, $\mathcal{F}_\psi = \sigma\{\psi_n;\, -\infty < n < \infty\}$, $\mathcal{F}_{\xi,\psi} = \sigma\{(\xi_n, \psi_n);\, -\infty < n < \infty\}$. Let $\theta_\xi$ and $\theta_\psi$ be two measure-preserving shift transformations of $\mathcal{F}_{\xi,\psi}$-measurable random variables, namely, if

$$S = h(\ldots, \xi_{n-1}, \xi_n, \xi_{n+1}, \ldots; \ldots, \psi_{m-1}, \psi_m, \psi_{m+1} \ldots),$$

then

$$\theta_\xi \circ S = h(\ldots, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots; \ldots, \psi_{m-1}, \psi_m, \psi_{m+1} \ldots),$$

and

$$\theta_\psi \circ S = h(\ldots, \xi_{n-1}, \xi_n, \xi_{n+1}, \ldots; \ldots, \psi_m, \psi_{m+1}, \psi_{m+2} \ldots).$$

For $n \geq 1$, let $\theta_\xi^n$ and $\theta_\psi^n$ denote the iterations of $\theta_\xi$ and $\theta_\phi$ respectively, and $\theta_\xi^{-n}$ and $\theta_\psi^{-n}$ be the shift transformations inverse to $\theta_\xi^n$ and $\theta_\psi^n$.

For each $k \geq 1$, let $Z_{[1,k]}$ be a nonnegative random variable and $\eta_{[1,k]} = \min\{n \geq 1 :\ \psi_1 + \ldots + \psi_n \geq Z_{[1,k]}\}$. Put $\eta_{[1,0]} \equiv 0$.

We assume that

(d) For any $\ell \geq 1$ and any Borel set $B$, the event $\{Z_{[1,k]} \in B;\ \eta_{[1,k]} \leq \ell\}$ belongs to the $\sigma$-algebra $\sigma\{\xi_1, \ldots, \xi_k; \psi_1, \ldots, \psi_\ell\}$.

For $m \leq n$, introduce the pair of random variables

$$\big(Z_{[m,n]}, \eta_{[m,n]}\big) = \theta_\xi^{m-1} \circ \theta_\psi^{m-1}\big(Z_{[1,n-m+1]}, \eta_{[1,n-m+1]}\big),$$

and for $r \leq m \leq n$, the pair of random variables

$$\Big(Z^r_{[m,n]}, \eta^r_{[m,n]}\Big) = \theta_\xi^{m-r} \circ \theta_\psi^{\eta_{[r,n-1]}}\big(Z_{[r,n-m+1]}, \eta_{[r,n-m+1]}\big).$$

The assumptions (a)–(d) imply that the sequences $\big\{\big(Z^r_{[m,m+k]}, \eta^r_{[m,m+k]}\big),\ k \leq 0\big\}$ and $\big\{\big(Z_{[1,1+k]}, \eta_{[1,1+k]}\big),\ k \leq 0\big\}$ are identically distributed. Assume that $Z_1 = Z_{[1,1]}$, $\eta_1 = \eta_{[1,1]}$, $Z_2 = Z^1_{[2,2]}$, $\eta_2 = \eta^1_{[2,2]}$, and for $k \geq 3$, $(Z_k, \eta_k) = \theta_\xi \circ \theta_\psi^{\eta_1} \circ (Z_{k-1}, \eta_{k-1})$.

The assumptions (a)–(d) imply the proposition below.

**Proposition 1.** *The sequence $\{(Z_k, \eta_k),\ k \geq 1\}$ is stationary and metric-transitive.*

Let the conditions below hold,

(e) $d \equiv \mathbf{E}Z_1 < \infty$,

(f) "random" subadditivity: For any $r \leq m < n$,

$$Z_{[r,n]} \leq Z_{[r,m]} + Z^r_{[m+1,n]} \quad \text{a.s.}$$

In this inequality, in contrast to the subadditive processes in [18], the variable $Z^r_{[m+1,n]}$ does not equal $Z_{[m+1,n]}$; it only equals the latter in distribution, being its random shift in $\psi$. The "range" of this shift depends on the first summand on the right-hand side of (f).

Let us state simple corollaries of the assumptions (a)–(f).

**Corollary 9.** *For any $n \geq 1$, $Z_{[1,n]} \leq \sum\limits_{i=1}^{n} Z_i \equiv S_n$.*

**Corollary 10.** *For all $n \geq 1$, $m \geq 1$,*

$$\mathbf{E}Z_{[1,n]} \equiv g_n < \infty \quad \textit{and} \quad g_{n+m} \leq g_n + g + m.$$

**Corollary 11.** *Let $\gamma = \inf\limits_{n \geq 1} g_n/n$. Then $\gamma < \infty$ and $\gamma = \lim\limits_n g_n/n$.*

For the proof of the last statement, see, e.g., [18].

**Corollary 12.** *Let $\Phi = \limsup Z_{[1,n]}/n$. Then $\Phi \leq d$ a.s.*

Proof. Since $S_n/n \to d$ a.s., $\limsup Z_{[1,n]}/n \leq \lim S_n/n = d$ a.s. $\triangle$

**Corollary 13.** *The random variable $\Psi$ is $\mathcal{F}_\xi$-measurable.*

Proof. Since $Z_{[1,n]} \leq Z_{[1,m]} + Z^1_{[m+1,n]}$ a.s., we have for any fixed $m \geq 1$ and $m \to \infty$

$$\Phi = \limsup Z_{[1,n]}/n \leq \limsup Z^1_{[m+1,n]}/n = \limsup Z^1_{[m+1,n]}/(n-m+1) \equiv \Phi^1_{[m+1]}.$$

Since $\Phi \overset{\mathrm{D}}{=} \Phi^1_{[m+1]}$, we have $\Phi = \Phi^1_{[m+1]}$ a.s. For any $\delta > 0$, $k \geq 1$, and $A \in \sigma\{\phi_1, \ldots, \phi_k\}$, choose $m \gg 1$ such that $\mathbf{P}(\eta_{[1,m]} \geq k) \geq 1 - \delta$. Then, for any Borel set $B$,

$$\begin{aligned} \mathbf{P}(\Phi \in B; A) &\leq \sum_{i=k}^{\infty} \Big(\Phi^1_{[m+1]} \in B; A; \eta_{[1,m]} = i\Big) + \delta = \sum_{i=k}^{\infty} \Big(\Phi^1_{[m+1]} \in B\Big) \times \mathbf{P}\Big(A; \eta_{[1,m]} = i\Big) + \delta \\ &\leq \mathbf{P}(\Phi \in B) \times \mathbf{P}(A) + \delta. \end{aligned}$$

Analogously, $\mathbf{P}(\Phi \in B; A) \geq \ldots \geq \mathbf{P}(\Phi \in B) \times \mathbf{P}(A) - \delta$. $\triangle$

**Corollary 14.** $\Phi = \gamma$ *a.s.*

Proof. Let us show that $\Phi = \text{const}$ a.s. Indeed, $\Phi = \Phi^1_{[2]} = \theta^1_\xi \times \theta_\psi^{\eta_{[1,1]}} \times \Phi^1_{[1]} = \theta^1_\xi \times \Phi$ a.s. Therefore, $\Phi$ is an $\mathcal{F}_\xi$-invariant random variable, and $\Phi = \text{const}$ a.s.

*Remark 7.* The sequence $S_n/n$ is uniformly integrable since $S_n/n \to d$ a.s. and $\mathbf{E}S_n/n = d$ for any $n$.

Next, let us show that $\gamma \le \Phi$ a.s. The sequence $\{Z_{[1,n]}/n\}$ is uniformly integrable. Hence, $\gamma = \limsup \mathbf{E}Z_{[1,n]}/n \le \mathbf{E}\limsup Z_{[1,n]}/n = \Phi$ a.s.

Finally, let us show that $\Phi \le g_k/k$ for any $k$. Fix $k$ and define the sequence $\{Z_n^{(k)}\}$,

$$Z_1^{(k)} = Z_{[1,k]}, \qquad Z_2^{(k)} = Z^1_{[k+1,2k]},$$

and for $\ell \ge 3$, $Z_\ell^{(k)} = \theta_\xi \times \theta^{\eta_{[1,k]}} \times Z_{\ell-1}^{(k)}$. If $km < n \le k(m+1)$, then $Z_{[1,n]} \le \sum\limits_{\ell=1}^{m} Z_\ell^{(k)}$ a.s., and

$$\limsup Z_{[1,n]}/n \le \limsup \left[\sum_{\ell=1}^{m} Z_\ell^{(k)}\right] \Big/ k(m-1) = g_k/k \quad \text{a.s.} \ \triangle$$

**Corollary 15.** *As $n \to \infty$, the sequence $Z_{[1,n]}/n$ tends to $\Phi$ in probability.*

Proof. Let $\Phi_{[1,n]} = \sup\limits_{m\ge n} Z_{[1,m]}/m$. The sequence $\{\Phi_{[1,n]}\}_{n\ge1}$ is nonincreasing, and $\Phi_{[1,n]} \to \Phi$ a.s. as $n \to \infty$. Therefore,

$$\Phi_{[1,n]} \times I\{\Phi_{[1,n]} \le \Phi + \varepsilon\} \to \Phi$$

a.s., and

$$\mathbf{P}\big(Z_{[1,n]}/n \ge \Phi + \varepsilon\big) \le \mathbf{P}\big(\Phi_{[1,n]} \ge \Phi + \varepsilon\big) \to 0.$$

Assume that there exists $0 < \delta < \Phi$ such that $\limsup \mathbf{P}(Z_{[1,n]}/n \le \Phi - \delta) > 0$. Then a subsequence $\{n_k\}$ can be selected, for which the limit

$$\lim_k \mathbf{P}\big(Z_{[1,n_k]}/n_k \le \Phi - \delta\big) \equiv p > 0$$

exists. Choose $q < \delta p$ and $x_0 \gg 1$ such that

$$\sup_n \mathbf{E}\big\{Z_{[1,n]}/n \times I\big(Z_{[1,n]}/n > x_0\big)\big\} \le q$$

and $\varepsilon > 0$ such that $\varepsilon(1-p) + q < \delta p$.

For $n \ge 1$, define the sequence of random variables

$$\begin{aligned}\beta_n &= (\Phi-\delta) \times I\big(Z_{[1,n]}/n \le \Phi - \delta\big) + (\Phi+\varepsilon) \times I\big(\Phi - \delta < Z_{[1,n]}/n \le \Phi + \varepsilon\big)\\ &+ x_0 \times I\big(\Phi + \varepsilon < Z_{[1,n]}/n \le x_0\big) + Z_{[1,n]}/n \times I\big(Z_{[1,n]} > x_0\big).\end{aligned}$$

It is evident that $Z_{[1,n]}/n \le \beta_n$ a.s. On the other hand,

$$\begin{aligned}\mathbf{E}Z_{[1,n_k]}/n_k \le \mathbf{E}\beta_{n_k} &\le (\Phi-\delta)\times p + (\Phi+\varepsilon)\times(1-p) + x_0 \times \mathbf{P}\big(Z_{[1,n_k]}/n_k \ge \Phi+\varepsilon\big) + q\\ &\to (\Phi-\delta)\times p + (\Phi+\varepsilon)\times(1-p) + q\\ &= \Phi - \delta\times p + \varepsilon\times(1-p) + q \ < \ \Phi\end{aligned}$$

as $k \to \infty$. This provides a contradiction, i.e., $Z_{[1,n]}/n \to \Phi$ in probability. $\triangle$

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