Integro-local limit theorem for a supercritical branching process in a random environment

Overview

A branching process in a random environment (BPRE) is a natural and important generalization of the Galton–Watson process, where the reproduction law varies according to a random environment indexed by time.
The random environment is represented by a sequence $\xi = (\xi_0, \xi_1, \dots)$ of i.i.d. random variables taking values in an abstract space Ξ.
The random variable ξ_n represents the random environment at time n; to each realization of ξ_n corresponds a probability law $\{p_i(\xi_n) : i \in \mathbb{N}\}$ on $\mathbb{N} = \{0, 1, 2, \dots\}$.
Define the process $(Z_n)_{n>0}$ by the relations

$$Z_0 = 1, Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i}, \quad \text{for } n \geqslant 0$$

where $N_{n,i}$ is the number of children of the ith individual of the generation n. Conditionally on the environment ξ , the r.v.'s $N_{n,i}$ $(i = 1, 2, \dots)$ are independent of each other with common probability distribution, and also independent of Z_n.
Set $m_n = m_n(\xi) = \sum_{i=0}^{\infty} i p_i(\xi_n)$ - the average number of children of an individual of generation n when the environment ξ is given.
An important tool in the study of a BPRE is the **associated random walk**

$$S_n = \log \mathbb{E}_\xi Z_n = \sum_{i=1}^{n} X_i, \quad n \geqslant 1$$

where the r.v.'s $X_i = \log m_{i-1} (i \geqslant 1)$ are i.i.d. depending only on the environment ξ. The behavior of the process (Z_n) is mainly determined by the associated random walk which is seen from the decomposition :

$$\log Z_n = S_n + \log W_n,$$

where $W_n = Z_n / \mathbb{E}_\xi Z_n$. The sequence $(W_n)_{n>0}$ is well known to be a positive martingale with respect to the natural filtration $\mathcal{F}_n = \sigma\left(\xi, N_{k,i}, 0 \leqslant k \leqslant n-1, i = 1, 2 \dots\right)$.
Then the limit $W = \lim W_n$ exists $\mathbb{P}$-a.s. and $\mathbb{E}W \leqslant 1$.

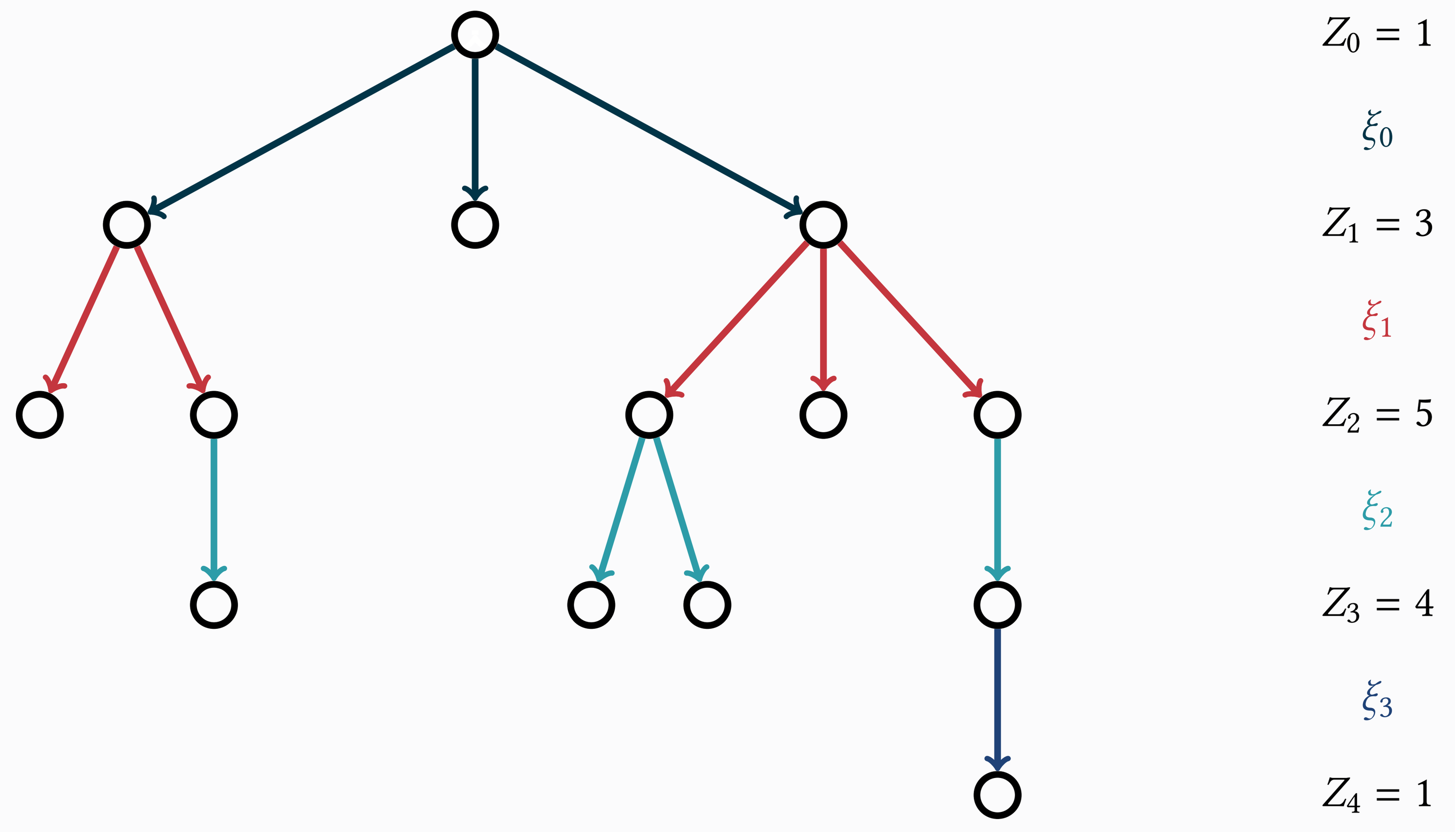
Assumptions

We assume that each individual has at least one child, which means that

$$p_0 = 0 \quad \mathbb{P}\text{-a.s.}$$

This implies that the associated random walk has non negative increments increments, $Z_n \to \infty$ as $n \to \infty$ and $W > 0$ $\mathbb{P}$-a.s.
Let $X = \log m_0$, $\mu = \mathbb{E}X$ and $\sigma^2 = \mathbb{E}(X - \mu)^2$.
We shall also assume that the BPRE is **supercritical** with $\mu \in (0, \infty)$, i.e the population size tends to infinity with positive probability.

Visual representation of the model



Rate function

A function $\Lambda : \mathbb{R} \to \mathbb{R}^+$, defined for all $x \in \mathbb{R}$ as

$$\Lambda(\alpha) = \sup_\lambda (\alpha\lambda - \ln \psi(\lambda)).$$

is called a **rate function** (where $\psi(\lambda) := \mathbb{E}e^{\lambda X}$ is the Laplace transform of r.v. X at λ).
Under the following assumption:
$[\mathbf{C}_0]$ There is a number $\lambda_+ > 0$, such that $\mathbb{E}e^{\lambda_+ X} < \infty$
there exist two constants α_- and α_+, such that $\mu \in (\alpha_-, \alpha_+)$, $\Lambda(\alpha)$ is analytical for all $\alpha \in (\alpha_-, \alpha_+)$ and satisfies the following properties (see [2] for details):

- $\min_\alpha \Lambda(\alpha) = \Lambda(\mu) = 0$
- $\Lambda(\mu) = \Lambda'(\mu) = 0, \quad \Lambda''(\mu) = \frac{1}{\sigma}$
- $\Lambda(t)$ can be associated with a Cramér series defined by

$$\mathscr{L}(t) := \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \cdots$$

which converges for $|t|$ small enough, where

$$\gamma_k = \frac{d^k \psi}{d\lambda^k}(0)$$

are the cumulants of order k of the random variable X (see [6]).

Denote $\Delta[x) := [x, x + \Delta)$ ($\Delta_n[x)$ has the same meaning but Δ_n is a function of n).

Our main goal

Theorem (Integro-local limit theorem, ILT)

Assume $[\mathbf{C}_0]$ *and the associated random walk* $(S_n)_{n\geq 0}$ *of BPRE* $(Z_n)_{n\geq 0}$ *to be non-trivial and non-lattice. Let* $\alpha \in (\alpha_-, \alpha_+)$.
Then for $\alpha = x/n$ *we have as* $\Delta_n \to 0$ *sufficiently slow as* $n \to \infty$,

$$\mathbb{P}\left(\log Z_n \in \Delta_n[x)\right) = \frac{\Delta_n C(\alpha)}{\sqrt{2\pi n}} e^{-n\Lambda(\alpha)}(1 + o(1)),$$

where $C(\alpha)$ *is a continuous function and*
$o(1)$ *uniformly goes to* 0 *for* x/n *in any compact from* (α_-, α_+).

Corollary (Large deviations)

Let $\alpha_0 \in (\mu, \alpha_+)$. *Then for any* $\{x_n\}$ *such as* $\alpha := x_n/n \to \alpha_0$ *as* $n \to \infty$ *holds*

$$\mathbb{P}\left(\log Z_n \geq x_n\right) = \frac{C(\alpha_0)}{\Lambda'(\alpha_0)\sqrt{n}} e^{-n\Lambda(\alpha)}(1 + o(1)).$$

Similarly, for $\alpha_0 \in (\alpha_-, \mu)$, *we have*

$$\mathbb{P}\left(\log Z_n \leq x_n\right) = \frac{C(\alpha_0)}{\Lambda'(\alpha_0)\sqrt{n}} e^{-n\Lambda(\alpha)}(1 + o(1)).$$

Here are some known results, related to the topic.

Theorem (Large deviation principle, LDP [3], [4], [5])

Under some additional assumptions, a sequence $\left(\log Z_n / n\right)_{n\geq 1}$ *of r.v.'s from* $\mathbb{R}^d$ *satisfies the* ***large deviation principle*** *with a rate function* $\Lambda(\alpha)$, *i.e for any Borel set* $B \subset \mathbb{R}^d$ *two inequalities hold:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \leq -\Lambda([B]),$$

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \geq -\Lambda((B)),$$

where $\Lambda(B) := \inf_{\alpha\in B} \Lambda(\alpha)$, $\Lambda(\emptyset) := \infty$ *and* $[B]$, (B) *are the closure and the interior of the set* B *respectively.*

The following results can be obtained from **ILT** and **LDP**.

Theorem (Moderate-large deviations [1])

For any $y = y(n)$ *such that* $0 < y = o(\sqrt{n})$, *as* $n \to \infty$,

$$\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geqslant y\right) = \overline{\Phi}(y) \exp\left\{\frac{y^3}{\sqrt{n}} \mathscr{L}\left(\frac{y}{\sqrt{n}}\right)\right\}(1 + o(1)),$$

where

$$\overline{\Phi}(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-\frac{u^2}{2}} \mathrm{d}u$$

is the tail of standard normal distribution.

Corollary (Normal deviation)

For any $y = y(n) > 0$, $y = o(n^{1/6})$ *holds*

$$\mathbb{P}\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq y\right) = \overline{\Phi}(y)(1 + o(1))$$

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