On the asymptotics for the minimal distance between extreme vertices in a generalised Barak—Erdos graph

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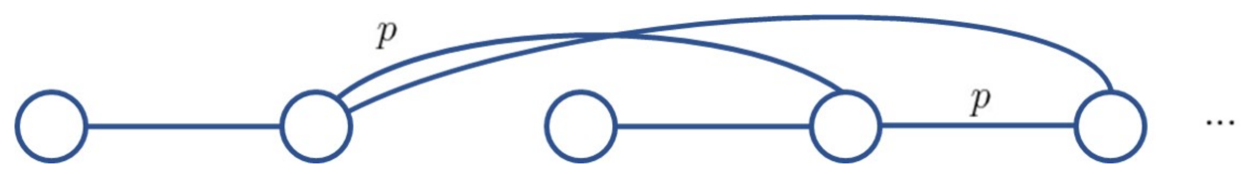
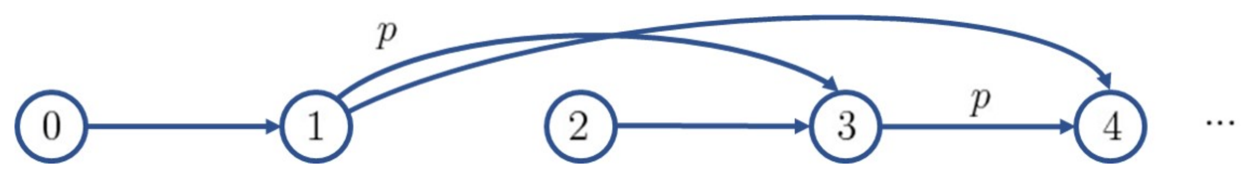
Introduction

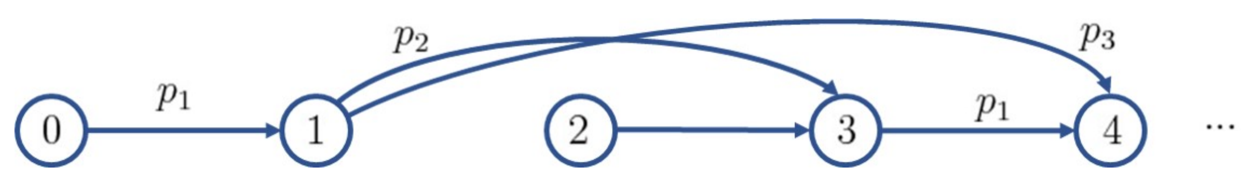
The classical *Erdös—Rényi graph* is an undirected graph with a fixed non-random set of vertices and a random set of edges. Each edge exists independently of the others with a given probability $p \in [0, 1]$.



By numbering the vertices of the Erdös—Rényi graph and directing all the edges from the smaller vertices to the larger ones, we obtain the *Barack—Erdös random graph*.



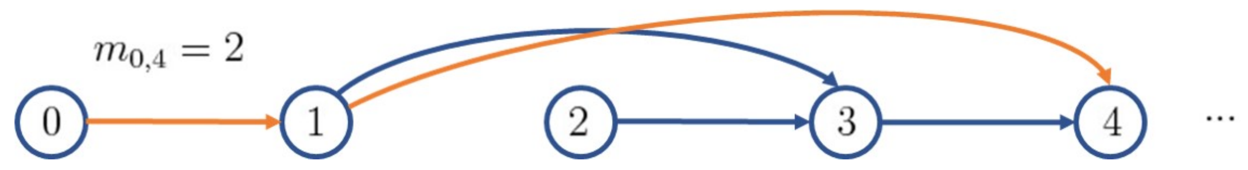
We deal with a *generalization of the Barack—Erdös graph*. We assume that an edge between two arbitrary vertices i and j $(i < j)$ exists with a probability p_{j-i}, depending only on the difference $j - i$.



If an edge exists, its length is assumed to be equal to one.

The path $\pi = \pi_{i,j}$ from vertex i to vertex j is an ordered set of edges $(i \to i_1, i_1 \to i_2, \ldots, i_{k-1} \to j)$. The path π exists if all edges in it exist. The length of the path is the number k, that is, the sum of the lengths of its edges.

Let $m_{i,j}$ be the minimum path length between the vertices i and j. If such paths do not exist, then by definition $m_{i,j} = \infty$.



Our *main goal* is to explore the asymptotic behavior for the distribution of the random variable $m_{0,n}$ when $n \to \infty$.

Background

The generalized Barak–Erdös graph was first considered in [1]. The authors studied the asymptotic behavior of the maximum length of all paths from 1 to n as $n \to \infty$ and proved the Strong Law of Large Numbers and the Central Limit Theorem for this object.

For classical Barak—Erdös graphs, similar results were proved by Foss and Konstantopoulos in [2]. In [3] Mallein and Ramassamy obtained an analytical expression for the growth rate of the maximum path length, using the natural correspondence between such graphs and the Infinite Bin Model introduced in [2].

There are other options for generalizing the Barak—Erdös model. For example, we can assume that each edge in graph has a random length. In [4], the authors investigated the asymptotic behavior of the maximum path length among all paths from 0 to n in such graph when $n \to \infty$. In this setting, the studied problem is the Last-passage percolation problem.

Main results

First, we give the results related to the asymptotic behavior of the distribution of the sequence $m_{0,n}$.

Theorem 1. *Let $p_n \to p$ as $n \to \infty$.*

1. *If $p > 0$, then $m_{0,n} \xrightarrow{d} \xi$, where*
$$\mathbb{P}(\xi = 1) = 1 - \mathbb{P}(\xi = 2) = p;$$
2. *If $p = 0$ and $p_n\sqrt{n} \to \infty$ as $n \to \infty$, then $m_{0,n} \xrightarrow{\mathbb{P}} 2$;*
3. *If $p = 0$ and $\limsup_{n\to\infty} n^\alpha p_n < \infty$ for some $\alpha \in (0, 1)$, then*
$$\mathbb{P}\left(m_{0,n} < \frac{1}{1-\alpha}\right) \to 0.$$

Theorem 1 implies

Corollary 1. *If $\limsup_{n\to\infty} \frac{\ln p_n}{\ln n} \le -1$, then $m_{0,n} \xrightarrow{\mathbb{P}} \infty$.*

Having imposed slightly stronger conditions on the sequence p_n, we obtain

Proposition 1. *If the series $\sum_{n>0} p_n$ converges, then $m_{0n} \to \infty$ with probability one.*

The final set of results is devoted to the study of the connectedness of the vertices 0 and n in the graph as $n \to \infty$. The following theorem holds.

Theorem 2. *Suppose that:*

1. $\sum_{n>0} np_n < \infty$;
2. *$p_n \neq 1$ for all $n \in \mathbb{N}$.*

Then $\mathbb{P}(m_{0,n} = \infty) \to 1$ as $n \to \infty$, i.e the set of all paths from 0 to n is empty with high probabilty.

Finally, we provide sufficient conditions for the connectedness of the vertices 0 and n in the graph as $n \to \infty$.

Theorem 3. *Let $p_1 > 0$ and at least one of the following:*

1. $\liminf_{n\to\infty} np_n > 1$;
2. *There exists a number k such that $p_k = 1$.*

Then $\mathbb{P}(m_{0,n} = \infty) \to 0$ as $n \to \infty$.

Acknowledgements

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References

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