Title: Categoricity of computable infinitary theories

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Abstract: We consider conditions under which the computable infinitary theory of a hyperarithmetical structure $\mathcal{A}$ is $\aleph_0$-categorical. The interesting case is where $\mathcal{A}$ has Scott rank ω_1^{CK}. We consider known computable structures of Scott rank ω_1^{CK}, and we show that in each case, the computable infinitary theory is $\aleph_0$-categorical. We begin with a tree. If A_n is the set of elements at levels $m \leq n$ in the tree, then for each n, the orbits of tuples in A_n are defined by computable infinitary formulas of bounded complexity, and this explains why the computable infinitary theory is $\aleph_0$-categorical. We then consider further computable structures of Scott rank ω_1^{CK}, an undirected graph, fields of any desired characteristic, and a linear ordering. These are obtained from the tree by applying "rank-preserving" transformation from trees to undirected graphs, and from undirected graphs to fields and to linear orderings.

The transformation from trees to undirected graphs was described by Marker, Nies, and others. This transformation has the feature that a copy of the input tree is definable in the output graph. We give a general theorem on transfer of categoricity for this setting. The transformation from undirected graphs to fields is essentially due to Friedman and Stanley. This transformation has the feature that a copy of the input graph is interpretable in the output field as a definable quotient structure. We give a second theorem on transfer of categoricity for this setting. The transformation from undirected graphs to linear orderings is due to Friedman and Stanley. For this transformation, there is no copy of the input structure defined, or obtained as a definable quotient, in the output structure. However, the output structure has a definable set of elements representing tuples in the input structure. We give a third theorem on transfer of categoricity for this setting.

There are further transformations which yield further computable structures of Scott rank ω_1^{CK}. In particular, there is the Mal'cev transformation from fields to groups. Morozov (in unfinished work with Calvert, Harizanov, and Knight) has shown that this transformation is rank-preserving. There is a rank-preserving transformation from undirected graphs to Boolean algebras (in unfinished work of Calvert, Goncharov, and Knight). It seems likely that these transformations also preserve categoricity.

Open Question. Is there a computable (or hyperarithmetical) structure of Scott rank ω_1^{CK} for which the computable infinitary theory is *not* $\aleph_0$-categorical?

Millar and Sacks have an example of a structure $\mathcal{A}$ of Scott rank ω_1^{CK}, such that $\omega_1^{\mathcal{A}} = \omega_1$ (so $\mathcal{A}$ lives in a fattening of the least admissible set), and the theory of $\mathcal{A}$ in the corresponding infinitary fragment is not $\aleph_0$-categorical.