Categoricity of computable infinitary theories

W. Calvert, S. S. Goncharov,
J. F. Knight, and J. Millar

Computable infinitary formulas are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets. The formulas are classified as computable Σ_α and computable Π_α, for computable ordinals α. Taken all together, the computable infinitary formulas are essentially the same as the $L_{\omega_1\omega}$ formulas contained in the least admissible set.

For a hyperarithmetical structure $\mathcal{A}$, we write $Th_\infty(\mathcal{A})$ for the computable infinitary theory of $\mathcal{A}$.

Problem. When is $Th_\infty(\mathcal{A})$ $\aleph_0$-categorical?

We get some information by considering "Scott rank".

The Scott Isomorphism Theorem says that for any countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$ sentence σ whose countable models are just the isomorphic copies of $\mathcal{A}$. The sentence is called a "Scott sentence".

Scott's proof involved assigning ordinals, first to the tuples in $\mathcal{A}$, and then to the structure itself. This is the "Scott rank". There are different definitions in use. Instead of choosing one, I will give a theorem, describing the possible Scott ranks for hyperarithmetical structures.

We write $SR(\mathcal{A})$ for the Scott rank of $\mathcal{A}$.

Theorem. If $\mathcal{A}$ is hyperarithmetical, then

1. (**Nadel**) $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$.

2. (**Folklore**) Moreover,

(a) $SR(\mathcal{A})$ is computable if for some computable ordinal α, the orbits of all tuples are definable by computable Σ_α formulas,

(b) $SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits of all tuples are defined by computable infinitary formulas, but there is no computable bound on the complexity of the formulas,

(c) $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is some tuple whose orbit is not defined by any computable infinitary formula.

Examples of computable structures of Scott ranks $< \omega_1^{CK}$, $\omega_1^{CK} + 1$

1. Computable ordinals, computable superatomic Boolean algebras, and computable reduced Abelian p-groups all have Scott rank $< \omega_1^{CK}$.

2. The *Harrison ordering*, a computable ordering of type $\omega_1^{CK}(1+\eta)$, has Scott rank $\omega_1^{CK}+1$.

So do the *Harrison Boolean algebra* (the interval algebra of the Harrison ordering), and the *Harrison Abelian p-group* (a computable group of length ω_1^{CK}, with all infinite ulm invariants, and with a divisible part of infinite dimension).

Easy cases of the problem

Suppose $\mathcal{A}$ is a hyperarithmetical structure.

1. If $SR(\mathcal{A}) < \omega_1^{CK}$, then $Th_\infty(\mathcal{A})$ is $\aleph_0$-categorical.

To see this, we use the following.

Nadel. $SR(\mathcal{A}) < \omega_1^{CK}$ iff $\mathcal{A}$ has a computable infinitary Scott sentence.

2. If $SR(\mathcal{A}) = \omega_1^{CK} + 1$, then $Th_\infty(\mathcal{A})$ is not $\aleph_0$-categorical.

The tuple witnessing that $SR(\mathcal{A}) = \omega_1^{CK} + 1$ realizes a type that is non-principal. There is a model of $Th_\infty(\mathcal{A})$ which omits this type.

The Omitting Types Theorem holds for infinitary logic, just as it does for finitary logic—the proof is a Henkin construction.

For our problem, the interesting case is where $SR(\mathcal{A}) = \omega_1^{CK}$. There are some hyperarithmetical, even computable, structures having this rank.

Scott rank ω_1^{CK}

Makkai. There is an arithmetical structure $\mathcal{A}$ of Scott rank ω_1^{CK}.

K-Millar. Makkai's example can be made computable.

Calvert-K-Millar. There is a computable tree $\mathcal{T}$ of Scott rank ω_1^{CK}.

Makkai's original example was a complicated kind of structure, obtained from a tree by putting a group on each level and then throwing away most of the tree and group structure, leaving a family of unary functions, one for each element. The tree is easier to describe.

Tree rank. Let $\mathcal{T}$ be a subtree of $\omega^{<\omega}$, and let $x \in \mathcal{T}$.

1. $tr(x) = 0$ if x has no successors,
2. $tr(x) = \beta$ if β is the least ordinal greater than $tr(y)$ for all successors y of x,
3. $tr(x) = \infty$ if x does not have ordinal tree rank.

Fact: $tr(x) = \infty$ iff x extends to a path.

Rank-homogeneous trees. Let $\mathcal{T}_n$ denote the set of elements at level n (i.e., of length n). Then $\mathcal{T}$ is *rank-homogeneous* if for all $x \in \mathcal{T}_n$,

1. if $tr(x) = \alpha$, and there exists $y \in \mathcal{T}_{n+1}$ s.t. $tr(y) = \beta$, where $\beta < \alpha$, then x has infinitely many successors of rank β,

2. if $tr(x) = \infty$, then x has infinitely many successors y of rank ∞.

Calvert-K-Millar. Let $\mathcal{T}$ be a hyperarithmetical rank-homogeneous tree.

1. $SR(\mathcal{T}) < \omega_1^{CK}$ if there is a computable bound on the ordinal tree ranks.

2. $SR(\mathcal{T}) = \omega_1^{CK}$ if for each n, there is a computable bound on the ordinal tree ranks of elements of $\mathcal{T}_n$, but there is no computable bound over all.

3. $SR(\mathcal{T}) = \omega_1^{CK} + 1$ if for some n, there is no computable bound on the ordinal tree ranks of elments of $\mathcal{T}_n$.

For Makkai's example and the computable variant, the computable infinitary theory is $\aleph_0$-categorical. The same is true for the tree. We have the following general result, inspired by the tree.

Theorem. Let $\mathcal{A}$ be a hyperarithmetical structure. Let $(A_n)_{n \in \omega}$ be an increasing sequence of subsets, defined by a computable sequence of computable Σ_α formulas (for fixed α). Suppose

1. $|\mathcal{A}| = \cup_n A_n$,

2. for each n, there is a computable ordinal α_n s.t. the orbits of tuples in A_n are all defined by computable Σ_{α_n} formulas.

Then $Th_\infty(\mathcal{A})$ is $\aleph_0$-categorical.

Proof: There is a computable infinitary sentence saying $(\forall x) \bigvee\!\!\!\bigvee_n x \in A_n$. For each n, for each $\overline{a}$ in A_n, let $\psi_{\overline{a}}(\overline{x})$ be the conjunction of the computable Σ_{α_n} formulas true of $\overline{a}$. There is a computable infinitary sentence saying that each tuple from A_n satisfies one of the formulas $\psi_{\overline{a}}(\overline{x})$. For any formula $\gamma(\overline{x})$ true of $\overline{a}$, there is a computable infinitary sentence saying that $(\forall \overline{x})\,[\psi_{\overline{a}}(\overline{x}) \rightarrow \gamma(\overline{x})]$.

These sentences are true in $\mathcal{A}$, so they are is true in any model of $Th_\infty(\mathcal{A})$.

Corollary. For the computable tree $\mathcal{T}$ of Scott rank ω_1^{CK} (from Calvert-K-Millar), $Th_\infty(\mathcal{T})$ is $\aleph_0$-categorical.

Proof: The tree $\mathcal{T}$ has the feature that for each n, there is a computable bound on the Scott ranks of tuples included in the first n levels of $\mathcal{T}$. Let A_n consist of the elements at levels $\leq n$.

There are further examples of Scott rank ω_1^{CK}.

Calvert-Goncharov-K. There is a computable structure of Scott rank ω_1^{CK} in each of the following classes:

1. undirected graphs,

2. fields of any desired characteristic,

3. linear orderings.

Proof: Starting with the tree, we apply some transformations—from trees to undirected graphs, from graphs to fields, and from graphs to linear orderings. We show that each of these transformations "preserves rank", in the sense that for a computable input structure, either the output structure has the same Scott rank, or else both structures have computable Scott rank. (More generally, for an arbitrary input structure $\mathcal{A}$, either the output structure has the same Scott rank, or else both have Scott rank computable in $\mathcal{A}$.)

For these further structures of Scott rank ω_1^{CK}, the computable infinitary theory is again $\aleph_0$-categorical.

To prove this, we consider the same transformations. We show that, in addition to preserving rank, they preserve categoricity; i.e., assuming that the input structure is hyperarithmetical, if the computable infinitary theory of the input structure is $\aleph_0$-categorical, then the same is true for the output structure.

Recall the transformation from trees to undirected graphs.

Transformation from trees to undirected graphs (due to Marker, Nies, and others)

We have a "universal" computable graph $\mathcal{G}$ with a computable sequence $(b_k)_{k\in\omega}$. For each k, b_k is connected to one vertex of a triangle. For each pair (j,k), there are elements $c_{j,k}$, $d_{j,k}$, connected directly to b_j and joined to b_k by a 2-chain, s.t. $c_{j,k}, d_{j,k}$ is one vertex of a square, pentagon. The triangles, squares, pentagons, and connecting elements, are disjoint from each other and the special points b_k, and there are no other elements or connections.

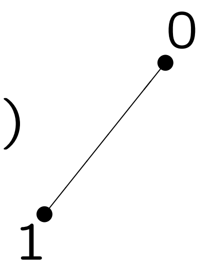
For an input tree $\mathcal{T}$, we have output $\mathcal{G}(\mathcal{T}) \subseteq \mathcal{G}$ consisting of the following:

1. b_k, with the triangle, for $k \in \mathcal{G}$,

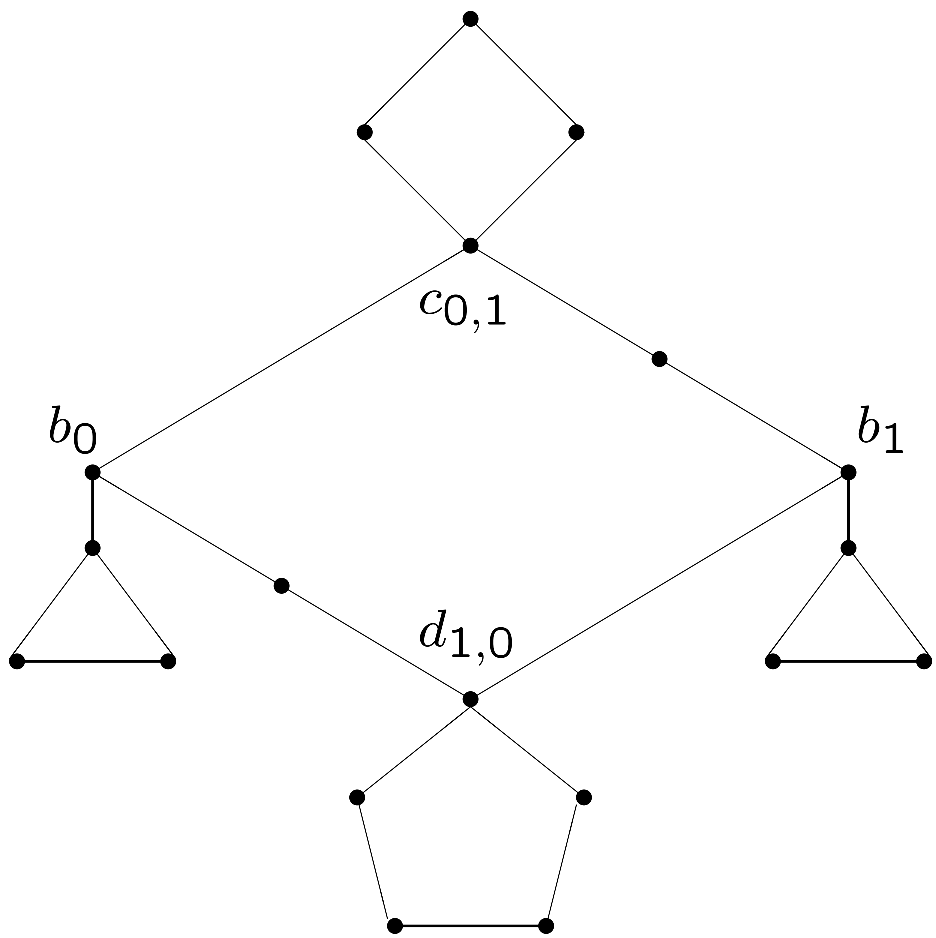
2. $c_{j,k}$, with the square, if k is a successor of j in $\mathcal{G}$, or $d_{j,k}$, with the pentagon, if j,k are in $\mathcal{G}$ but k is not a successor of j.

Here is a simple example. The tree has just two nodes.

$\mathcal{T}$ (1 is successor of 0)



$\mathcal{G}(\mathcal{T})$



A copy of the input tree is definable in the output graph, and the orbits of tuples in the graph are simply definable over the tree. We have the following general result for this setting.

Transfer Theorem I. Let $\mathcal{B}$ be a hyperarithmetical structure. Suppose $\mathcal{A}$ is definable in $\mathcal{B}$ by computable infinitary formulas of bounded complexity, and the following conditions are satisfied:

1. all automorphisms of $\mathcal{A}$ extend to automorphisms of $\mathcal{B}$,

2. for some computable ordinal α, the orbit of each tuple in $\mathcal{B}$ over $\mathcal{A}$ is defined by a computable Σ_α formula $\varphi(\overline{a}, \overline{x})$.

If $Th_\infty(\mathcal{A})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{B})$.

Corollary. Suppose $\mathcal{T}$ is a hyperarithmetical tree, and let $\mathcal{G}(\mathcal{T})$ be the undirected graph obtained by Marker's transformation. If $Th_\infty(\mathcal{T})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{G}(\mathcal{T}))$.

Proof: Let $\mathcal{B} = \mathcal{G}(\mathcal{T})$. Let $\mathcal{A}$ be the copy of $\mathcal{T}$ with universe equal to the set of special elements attached to a triangle, and with successor relation consisting of the ordered pairs of special elements attached to a square.

Transformation from undirected graphs to fields (due essentially to Friedman-Stanley)

We start with a universal computable algebraically closed field $\mathcal{F}$, of the desired characteristic, with a computable algebraically independent sequence $(b_k)_{k\in\omega}$. For a graph $\mathcal{G}$, let $\mathcal{F}(\mathcal{G})$ be the subfield of $\mathcal{F}$ generated by the following:

1. elements of $acl(b_k)$, for $k \in \mathcal{G}$,

2. $\sqrt{d + d'}$, where for some i and j joined by an edge in $\mathcal{G}$, d, d' are inter-algebraic with b_i, b_j, respectively.*

*For characteristic 2, we use $(d + d')^{1/3}$ instead of $\sqrt{d + d'}$.

A copy of the input graph is interpretable in the output field as a definable quotient, and the orbits of tuples in the field are all defined relatively simply over the graph.

We have the following general result.

Transfer Theorem II. Let $\mathcal{B}$ be a hyperarithmetical structure. Suppose $\mathcal{A} = \mathcal{A}^*/_\equiv$ is a quotient structure, where $\mathcal{A}^* = (D, R_i)$, and D, R_i, and $\equiv$ are definable in $\mathcal{B}$ by computable infinitary formulas of bounded complexity. For any choice function c on $D/_\equiv$ (choosing one element from each equivalence class), let $\mathcal{A}_c$ be the copy of $\mathcal{A}$ which is isomorphic under c. Suppose the following conditions are satisfied:

1. for any choice functions c and c', any isomorphism f from $\mathcal{A}_c$ onto $\mathcal{A}_{c'}$ extends to an automorphism of $\mathcal{B}$,

2. for any tuple $\overline{b}$ in $\mathcal{B}$ and any choice function c, the orbit of $\overline{b}$ under automorphisms of $\mathcal{B}$ that fix $\mathcal{A}_c$ pointwise is defined by a computable Σ_α formula $\varphi(\overline{a}, \overline{x})$.

If $Th_\infty(\mathcal{A})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{B})$.

Corollary. Suppose $\mathcal{G}$ is a hyperarithmetical undirected graph and $\mathcal{F}(\mathcal{G})$ is the field of characteristic 0, or p, obtained by the variant of the Friedman-Stanley transformation. If $Th_\infty(\mathcal{G})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{F}(\mathcal{G}))$.

Transformation from undirected graphs to linear orderings (due to Friedman-Stanley)

We start with the lexicographic ordering on $Q^{<\omega}$ as a universal ordering. Let $(Q_k)_{k\in\omega}$ be an effective partition of Q into dense subsets. Let $(t_n)_{n\in\omega}$ be a list of the quantifier-free types for tuples in undirected graphs. For a graph $\mathcal{G}$, let $\mathcal{L}(\mathcal{G})$ be the restriction of the ordering on $Q^{<\omega}$ to elements of the form $q_0 r_0 \ldots q_n r_n k$, where for some finite sequence $a_0, \ldots, a_n$ in $\mathcal{G}$, satisfying type t_m, $q_i \in Q_{a_i}$, $r_0, \ldots, r_{n-1} \in Q_0$, $r_n \in Q_1$, and $k \leq m$.

The input structure is not definable, or interpretable in the output structure, but the orbits in the input structure are represented by elements in the output structure, and the orbits in the output structure are all defined relatively simply in terms of a tuple of the special elements. We have the following general result for this setting.

Transfer Theorem III. Let $\mathcal{A}$ and $\mathcal{B}$ be hyperarithmetical structures. Let X be a definable subset of $\mathcal{B}$, and let g be a hyperarithmetical function from X onto the set of tuples in $\mathcal{A}$ s.t. the following conditions are satisfied:

1. for $d, d' \in X$, d and d' are in the same orbit in $\mathcal{B}$ iff $g(d)$ and $g(d')$ are in the same orbit in $\mathcal{A}$,

2. for each $\overline{b}$ in $\mathcal{B}$, the orbit of $\overline{b}$ over X is defined by a computable Σ_α formula $\varphi(\overline{d}, \overline{x})$,

3. for each $\overline{d} = (d_1, \ldots, d_n)$ in X, there is a computable Σ_α formula $\gamma(\overline{u})$ s.t. $\overline{d}'$ (also in X) is in the orbit of $\overline{d}$ iff it satisfies $\gamma(\overline{u})$ and for each i, $g(d_i)$ and $g(d'_i)$ are in the same orbit in $\mathcal{A}$.

If $Th_\infty(\mathcal{A})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{B})$.

Partial proof

Let $\mathcal{B}'$ be a model of the computable infinitary theory of $\mathcal{B}$, and let X' be the subset of $\mathcal{B}'$ defined by the formula that defines X. We show that the type of each $d' \in X'$ is realized by some $d \in X$.

For each computable ordinal α, there is a tuple $\overline{a}_\alpha$ in $\mathcal{A}$ s.t. if $\psi(\overline{u})$ is the conjunction of the computable Σ_α formulas true of $\overline{a}_\alpha$, and $d_\psi(x)$ defines the set of elements of X s.t. $g(x)$ satisfies $\psi(\overline{u})$, then d' satisfies $d_\psi(x)$. Let $\Gamma(\overline{u})$ be the limit of the types realized by the tuples $\overline{a}_\alpha$; i.e., for a computable Σ_α formula $\varphi(\overline{u})$, $\varphi(\overline{u}) \in \Gamma(\overline{u})$ iff $\varphi(\overline{u})$ is true of $\overline{a}_\beta$ for all $\beta \geq \alpha$.

We can show that the type $\Gamma(\overline{u})$ is realized in some model of $Th_\infty(\mathcal{A})$ (using essentially the Henkin construction). Since the theory is $\aleph_0$-categorical, $\Gamma(\overline{u})$ is realized by some tuple $\overline{a}$ in $\mathcal{A}$. We have $d \in X$ s.t. $g(d) = \overline{a}$, so d is the desired element.

Corollary. Suppose $\mathcal{G}$ is a hyperarithmetical undirected graph and $\mathcal{L}(\mathcal{G})$ is obtained from $\mathcal{G}$ by the Friedman-Stanley transformation. If $Th_\infty(\mathcal{G})$ is $\aleph_0$-categorical, then so is $Th_\infty(\mathcal{L}(\mathcal{G}))$.

There are further rank-preserving transformations.

Morozov (with Calvert, Harizanov, and K) has shown that the Mal'cev transformation, taking fields to their Heisenberg groups, preserves Scott rank.

Calvert, Goncharov, and K have a transformation from undirected graphs to Boolean algebras preserving Scott rank.

It seems likely that these transformations also preserve categoricity.

Problem. Is there a computable, or hyperarithmetical, structure $\mathcal{A}$ s.t. $SR(\mathcal{A}) = \omega_1^{CK}$ and $Th_\infty(\mathcal{A})$ is not $\aleph_0$-categorical?

Millar-Sacks. There is a structure $\mathcal{A}$ s.t. $SR(\mathcal{A}) = \omega_1^{CK}$ and $\omega_1^{\mathcal{A}} = \omega_1^{CK}$, so $\mathcal{A}$ lives in a fattening of the least admissible set, and the theory of $\mathcal{A}$ in the corresponding admissible fragment is not $\aleph_0$-categorical.