Some Index Sets in Computable Algebra

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Some Survey

In classical paper by Frohlich, Shepherdson some partial enumeration of commutative rings was proposed to answer some natural problems in negative way. One of this problems was related to existence of general splitting algorithm for the class of computable fields (of zero characteristic). Only after negative answer to this problem they had constructed concrete computable field without any splitting algorithm.

Index sets are considered as precise mesuare of complexity of given class of constructive objects or some given problem on such objects in modern computability theory.

Example.

The notion of identically truth coincide with one of provability.

Usually index sets are m-complete in appropriate class Γ of arithmetical, Ershov's or hyperarithmetical hierarchy. For short, it is convenient to call corresponding set Γ-complete.

List of theorems

Theorem 1 *(Folklore) Index sets of fields and radical rings are Π^0_2-complete.*

Theorem 2 *(Kudinov?) Index set of the class of (commutative) semisimple rings is Π^0_3-complete.*

Theorem 3 *(CHKM) Index set of the class of fields with splitting algorithm is Σ^0_3-complete.*

Theorem 4 *(Kudinov)Index set of the class of (commutative) Noetherian rings is Π^1_1-complete.*

Theorem 5 *(Calvert) Index set of the class of commutative Artinian rings is Σ^0_5-set.*

As for the theorem 2, there exist more general metatheorem, that is useful tool in similar situations.

Proposition 6 *(Kudinov) Let K be some nontrivial class of computable rings with the following properties.*

1. *K is closed under direct sums*
2. *The complement of K in the class of rings is closed under direct sums*
3. *There exists the ring R such that it is not in K, but all finitely generated subrings of R do lie in K.*

Then index set of the class K is Π^0_3-hard.

Idea for Theorem 2

Take $R_\infty = \mathbf{Z}[\frac{1}{2k+1} \mid k \in \omega]$.
This ring is not semisimple since $2 \in J(R)$.
But rings like $R_m = \mathbf{Z}[\frac{1}{2k+1} \mid k < m]$ are semisimple for finite m.
So, one can apply Proposition.

Idea for Theorem 3

For any c.e. set $A \subseteq \omega$ one can construct the computable field F_A by the following natural rule

$$F_A \;=\; \mathbf{Q}(\,\sqrt{p_k}\,|\,k \in A).$$

It is easy to observe that polinomial $x^2 - p_k$ is reducible over F_A iff $k \in A$. So, for noncomputable A field F_A does not possess a splitting algorithm. More careful but simple calculation shows that for computable A this field is computable in its algebraic closure $\bar{\mathbf{Q}}$, hence, has a splitting algorithm.

Idea for Theorem 4

Given any computable tree $T \subseteq \omega^{<\omega}$, one can construct some integral domain R_T by the following way.

To begin with, we assign each vertex v of given tree T with some variable x_i in some effective bijective way.

For any path f in T we can consider the following prime ideals in the polinomial ring $\mathbf{Q}[x_i \,|\, i \in \omega]$

$$I_f \ = (x_i \,|\, x_i \in f)_R.$$

And for multiplicative system $S = \ R - \bigcup_f \ I_f$ we consider standard ring of fractions $R_S = R_T$.

Fact R_T is Noetherian iff T has only finite pahts.

References

[1] W. Calvert, V. Harizanov, J. Knight, S. Miller, Index sets of computable structures, *Algebra and Logic*, V. 45, N 5, September 2006 , pp. 306-325.