О соболевских классах функций со значениями в метрических пространствах

Ю. Г. Решетняк [1]

Институт математики им. С. Л. Соболева СО РАН

[1]Институт математики им. С. Л. Соболева СО РАН, пр. Акад. Коптюга, 4, Новосибирск 630090, Россия.
E-mail: ugresh@math.nsc.ru

Потоки Риччи и геометрическая гипотеза Тёрстона

А. А. Борисенко [1]

Харьковский национальный университет им. В. Н. Каразина

[1]Харьковский национальный университет им. В. Н. Каразина, пл. Свободы, 4, Харьков 61077, Украина.
E-mail: borisenk@univer.kharkov.ua

Description of Traces for Sobolev Spaces Defined on Piecewise Smooth Surfaces

V. I. Burenkov [1]

School of Mathematics, Cardiff University

Let $Q = \{x \in \mathbb{R}^n : 0 < x_s < 1,\ s = 1, \dots, n\}$ be the unit open cube in $\mathbb{R}^n$ and $\Gamma_{i\alpha} = \{x \in \mathbb{R}^n : x_i = \alpha,\ s = 1, \dots, i-1, i+1, \dots, n\}$, where $i = 1, \dots, n$, $\alpha = 0, 1$, be its open faces. Moreover, let, for $l \in \mathbb{N}$ and $1 < p < \infty$, $\operatorname{tr}_{\partial Q} W_p^l(Q)$ be the space of all traces on ∂Q of functions in Sobolev space $W_p^l(Q)$. Given a function $f \in \operatorname{tr}_{\partial Q} W_p^l(Q)$, let $f_{i\alpha}$ denote the restriction of f to $\Gamma_{i\alpha}$.

Theorem. *Let $l \in \mathbb{N}$; $n = 2$ and $1 < p < \infty$, or $n > 2$ and $1 < p < n$. Then $f \in \operatorname{tr}_{\partial Q} W_p^l(Q)$ if and only if*

1) $f_{i\alpha} \in W_p^{l-\frac{1}{p}}(\Gamma_{i\alpha})$, $i = 1, \dots, n$, $\alpha = 0, 1$;

2) *if $l > 1$, or $l = 1$ and $p > 2$, then for all adjacent faces $\Gamma_{i\alpha}$ and $\Gamma_{j\beta}$*

$$\operatorname{tr}_{\overline{\Gamma_{i\alpha}} \cap \overline{\Gamma}_{j\beta}} f_{i\alpha} = \operatorname{tr}_{\overline{\Gamma}_{i\alpha} \cap \overline{\Gamma_{j\beta}}} f_{j\beta} \quad \text{a.e. on } \overline{\Gamma}_{i\alpha} \cap \overline{\Gamma}_{j\beta};$$

3) *if $l = 1$ and $p = 2$, then for all adjacent faces $\Gamma_{i\alpha}$ and $\Gamma_{j\beta}$*

$$\underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \left(\int_0^1 \frac{|f_{i\alpha}(|t-\alpha|, \bar{x}) - f_{j\beta}(|t-\beta|, \bar{x})|^2}{t}\, dt \right) d\bar{x} < \infty,$$

where $\bar{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

For $n = 2$ much more complicated descriptions of $\operatorname{tr}_{\partial Q} W_p^l(Q)$ were given by G. N. Yakovlev [1] and M. Yu. Vasil'chik [2]. In [1] the description involves the existence of functions $f_{i\alpha,m} \in W_p^{l-m-\frac{1}{p}}(\Gamma_{i\alpha})$, $m = 1, \dots, l-1$, satisfying certain conditions. In the description given in [2] only the functions $f_{i\alpha}$ are used, but the pasting conditions are more complicated and involve the derivatives of the functions $f_{i\alpha}$. For $n > 2$ no descriptions of $\operatorname{tr}_{\partial Q} W_p^l(Q)$ were known in terms of restrictions $f_{i\alpha} \in W_p^{l-\frac{1}{p}}(\Gamma_{i\alpha})$ and some pasting conditions.

---

[1]School of Mathematics, Cardiff University, 23 Senghennydd Road, Cardiff CF24 4YH, UK.

E-mail: burenkov@cf.ac.uk

Supported by RFBR (grant 99–01–00868).

The following simple example confirms that pasting of the derivatives of the restrictions $f_{i\alpha}$ to the faces $\Gamma_{i\alpha}$ of a function f defined of ∂Q is not required for the function f to belong to the space $\mathrm{tr}_{\partial Q} W_p^l(Q)$. Let $n = 2$ and $f = \{f_{i\alpha}\}_{i,\alpha=0,1}$, where $f_{10}(x_2) = x_2 + \ldots + x_2^{l-1}$, $f_{11}(x_2) = 1 + x_2 + \ldots + x_2^{l-1}$, $f_{20}(x_1) = x_1^l$, $f_{21}(x_1) = x_1^l + l - 1$. Then $f_{10}(0) = f_{20}(0)$, $f_{10}(1) = f_{21}(0)$, $f_{11}(0) = f_{21}(1)$, $f_{11}(1) = f_{21}(1)$, but, say, $f_{10}^{(m)}(0) \neq f_{20}^{(m)}(0)$, $m = 1, \ldots, l-1$. However, $f \in \mathrm{tr}_{\partial Q} W_p^l(Q)$ because $f = \mathrm{tr}_{\partial Q}(x_1^l + x_2 + \ldots + x_2^{l-1})$.

For $n = 2$ we give two proofs. One of them continues the proof given in paper [1]: under the assumptions of the Theorem functions $f_{i\alpha,m}$, $m = 1, \ldots, l-1$, are explicitly constructed which satisfy the conditions required in paper [1]. Another proof is a continuation of the proof given in [2]: it is proved that the pasting conditions of that paper can be essentially simplified and reduced to conditions 1) and 2) of the Theorem.

The case $n > 2$ requires much further work. First a special partition of the unity $\gamma_{i\alpha}$, $i = 1, \ldots, n$, $\alpha = 0, 1$, for the cube Q is constructed such that $\gamma_{i\alpha}|_{\Gamma_{i\alpha}} = 1$, $\gamma_{i\alpha}|_{\Gamma_{j\beta}} = 0$ if $(j, \beta) \neq (i, \alpha)$; $D^k \gamma_{i\alpha}|_{\Gamma_{j\beta}} = 0$, $0 < |k| \leqslant l - 1$, for all (j, β); and the derivatives $D^k \gamma_{i\alpha}(x)$ have the minimal possible growth as x approaches $\partial\Gamma_{i\alpha}$. Next it is assumed that there exist functions $f_{i\alpha,m} \in W_p^{l-m-\frac{1}{p}}(\Gamma_{i\alpha})$, $m = 1, \ldots, l-1$, satisfying the conditions similar to those in [1] and functions $u_{i\alpha} \in W_p^l(Q)$ are considered such that $\mathrm{tr}_{\Gamma_{i\alpha}} u_{i\alpha} = f_{i\alpha}$ and $\mathrm{tr}_{\Gamma_{i\alpha}} \frac{\partial^m u_{i\alpha}}{\partial x_i^m} = f_{i\alpha,m}$, $m = 1, \ldots, l-1$. After that it is proved that $u = \sum\limits_{i=1}^{n} \sum\limits_{\alpha=0}^{1} \gamma_{i\alpha} u_{i\alpha} \in W_p^l(Q)$ and $\mathrm{tr}_{\partial Q} u = f$. The last part of the proof is dedicated to construction of functions $f_{i\alpha,m}$ for a function f satisfying the assumptions of the Theorem, which is much more difficult than for $n = 2$.

Let, for $m = 1, \ldots, n$,

$$H^{(m)} = \{x \in \mathbb{R}^n : |x_s| < 1,\ s = 1, \ldots, m,\ 0 < x_s < 1,\ s = m+1, \ldots, n\}$$

and $Q^{(m)} = H^{(m)} \setminus \overline{Q}$. A statement similar to the Theorem holds also for $Q^{(m)}$. Finally, an analogue of the Theorem holds for bounded domains $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundaries satisfying the following condition: for of each point $x \in \partial\Omega$ there exists a neighbourhood U_x such that $U_x \cap \Omega$ can be transformed by a 'good' transformation ν_x to Q or $Q^{(m)}$ and $\overline{\nu_x(U_x \cap \partial\Omega)} = \bigcap\limits_{i=1}^{m} \overline{\Gamma}_{i0}$.

The results presented above were obtained jointly with Dr. S. Al-Mezel.

References

[1] *Yakovlev G. N.* Boundary properties of functions of the class W_p^l in regions with angular points // Dokl. Akad. Nauk SSSR. 1961. V. 140. P. 73–76.

[2] *Vasil'chik M. Yu.* On traces of functions from a Sobolev space defined on a plain domain with non-Lipschitz boundary // Dolk. Akad. Nauk SSSR. 1991. V. 319. P. 275 – 277.

Asymptotic Geometry of Metric Spaces

Sergei Buyalo [1]

Petersburg Department of
Steklov Institute of Mathematics RAS

I plan to discuss a number of key results in geometry of hyperbolic spaces and in asymptotic geometry.

The proofs of results in hyperbolic geometry are based on the hyperbolic approximation of metric spaces, a new construction of a hyperbolic space with prescribed boundary at infinity. This construction will be described in details.

Using hyperbolic approximation I'll give new proofs of

1. The extension theorem of Paulin–Bonk–Schramm that any quasi-symmetric homeomorphism between the boundaries at infinity of Gromov hyperbolic spaces is induced by a quasi-isometry of that spaces.

2. The Assuad' embedding theorem that any (compact) doubling metric space can be biLipschitz embedded in an Euclidean space after taken any power $p \in (0, 1)$ of the metric.

3. The Bonk–Schramm' embedding theorem that any Gromov hyperbolic space having a bounded growth rate at some scale can be quasi-isometrically embedded in some hyperbolic space H^n.

Next, I'll discuss some embedding and nonembedding results in asymptotic geometry when the target space is usually not hyperbolic. I plan to explain as some known constructions as well as some new ones, e.g., the quasi-isometric embedding of the hyperbolic space H^n into the n-fold metric product of metric trees.

Finally, I'll discuss 4 quasi-isometry invariants which give most of known at the moment obstacles to quasi-isometric embeddings of metric spaces. These are Gromov' hyperbolic rank, subexponential corank, t-rank and hyperbolic dimension. The definitions, properties and proofs of two key results on the invariants will be given as well as number of interesting applications.

[1] Petersburg Department of Steklov Institute of Mathematics RAS, 27, Fontanka, 191011, St.Petersburg, Russia.
E-mail: sbuyalo@pdmi.ras.ru

Minkowski-Orthogonalities and -Angles from a Projective-Geometric View Point

Gunter Weiss [1]

Dresden University of Technology

[1] Dresden University of Technology, Zellescher Weg 12–14, Dresden, D–01069, Germany. weiss@math.tu-dresden.de

О перестановочно-инвариантной оболочке пространств Бесова, Кальдерона и Соболева

М. Л. Гольдман [1]

Российский университет дружбы народов

Цель цикла лекций: познакомить слушателей с некоторыми современными концепциями и результатами теории функциональных пространств.

Будут рассмотрены общие методы описания интегральных свойств функций на основе понятий банахова функционального пространства и перестановочно-инвариантного пространства, а также способы описания дифференциальных свойств с использованием обобщённых производных (пространства Соболева), разностных или аппроксимативных характеристик гладкости (пространства Бесова и Кальдерона).

Будут разобраны две недавно решённые важные проблемы теории вложений разных метрик для пространств Соболева, Бесова и Кальдерона: установление критериев вложения этих пространств в перестановочно-инвариантные пространства и нахождение их перестановочно-инвариантных оболочек (описание минимального перестановочно-инвариантного пространства, в которое вложено данное пространство Соболева, Бесова, или Кальдерона). Данные результаты вбирают в себя ряд конкретных точных теорем вложения разных метрик для этих пространств. Число публикаций по этой тематике огромно. Приведённый в конце список литературы заведомо не полон. Он затрагивает лишь несколько монографий и небольшое число работ, наиболее тесно связанных с обсуждаемыми вопросами.

1. Характеризация интегральных свойств функций.

1.1. Банаховы функциональные пространства (БФП). В литературе используются близкие понятия для общей характеризации интегральных свойств функций: понятие *идеального пространства* (*векторной решетки*), см., например, книгу С. Г. Крейна, Ю. И. Петунина и Е. М. Семёнова [2], а также несколько более узкое понятие *банахова функционального пространства* (кратко: БФП), см., например книгу К. Беннетта и Р. Шарпли [8]. В число аксиом БФП включено свойство Фату, обеспечивающее справедливость обобщённого неравенства Минковского для бесконечных сумм и интегралов. Это не сужает набор интересующих нас примеров и приложений теории БФП, но делает теорию более компактной и прозрачной. Мы

[1] Российский университет дружбы народов, ул. Миклухо-Маклая, 6, Москва 117198, Россия.

будем следовать аксиоматике теории БФП, развитой в книге К. Беннетта и Р. Шарпли.

Будут введены: понятие *банаховой функциональной нормы*, основанное на ней понятие БФП, понятие *ассоциированного пространства* к БФП (пространства линейных интегральных функционалов). Будут рассмотрены основные свойства БФП (предельный переход при монотонной сходимости, обобщённое неравенство Гёльдера, *принцип двойственности*: совпадение исходного БФП с дважды ассоциированным пространством). Для конусов неотрицательных измеримых функций будут обсуждены понятия *поглощения* и *эквивалентности конусов* и их связь с вложениями конусов в БФП.

1.2. Перестановочно-инвариантные пространства. Будут введены понятия *функции распределения* и *убывающей перестановки* для измеримой функции и обсуждены их основные свойства: равноизмеримость функции и ее перестановки, эквивалентность их интегральных свойств и экстремальные свойства убывающих перестановок (теорема Харди и Литтлвуда). Будет введено понятие *второй перестановки* (среднего значения убывающей перестановки на интервале $(0, t)$), рассмотрены её основные свойства и приведён важный результат об эквивалентности второй перестановки функции и убывающей перестановки её максимальной функции Харди — Литтлвуда. Будет введено понятие *перестановочно-инвариантного пространства* (кратко: ПИП), как такого БФП, норма в котором инвариантна относительно перестановок. Понятие ПИП несколько уже рассмотренного в книге [2] понятия *симметричного пространства*, построенного на базе идеальных пространств. Для наших целей понятие ПИП достаточно, поскольку включает основные примеры теории симметричных пространств: пространства L_p, классические пространства Лоренца, Марцинкевича, Орлича и их обобщения.

Будут приведены выражение нормы функции в ПИП через норму её убывающей перестановки (теорема представления Люксембурга) и выражение для нормы ассоциированного ПИП. Будет введено понятие *фундаментальной функции* ПИП, рассмотрены её свойства и приведены примеры её вычисления.

Будет приведено важное для дальнейшего решение задачи о построении минимального ПИП, содержащего данный конус неотрицательных убывающих функций. Оно основано на принципе двойственности для ПИП.

2. Характеризация дифференциальных свойств функций.

2.1. Характеризация с помощью обобщённых производных. Будут обсуждены определения классических пространств Соболева, теория которых изложена в книгах С. Л. Соболева [5], Р. Адамса [7], О. В. Бесова, В. П. Ильина, С. М. Никольского [1], В. И. Буренкова [9], В. Г. Мазьи [3], С. М. Никольского [4], Г. Трибеля [6] и др., а также их обобщения, связанные с использованием тех или иных ПИП в качестве базовых пространств (например, пространства Соболева — Орлича). Их теория развивалась в работах В. С. Климова (см., например, [19]), А. Чьянки и Л. Пика (см., например, [27, 28]) и многих других.

Кратко будут обсуждены обобщения, связанные с использованием дробных производных, которые привели к концепции пространств Соболева — Лиувилля (бесселевых потенциалов), а затем и более общей шкалы пространств Лизоркина — Трибеля (см., например, книгу Г. Трибеля [6], работы П. И. Лизоркина [20], Г. А. Калябина [18] и др.). Пространства Лизоркина — Трибеля, построенные на базе ПИП, и их обобщения активно исследовались в последние годы Ю. В. Нетрусовым (см., например, [22]).

2.2. Характеризация с помощью разностей и модулей непрерывности. Будут обсуждены определения классических пространств Никольского и Бесова (в терминах *разностей* и *модулей непрерывности*), построенных на базе пространств L_p, и их обобщений на базе ПИП. Будут введены пространства Бесова с обобщённой гладкостью и обсуждена более общая концепция *пространств Кальдерона* (см. [26]), и её развитие в работах К. К. Головкина, Ю. А. Брудного и В. К. Шалашова, М. Л. Гольдмана (см. [12–14]).

2.3. Аппроксимативные характеристики гладкости. Будет рассмотрено семейство подпространств в ПИП, состоящих из *целых функций экспоненциального типа* (кратко: ЦФЭТ) и введено понятие *наилучшего приближения* функций по норме ПИП с помощью ЦФЭТ. Скорость убывания наилучших приближений при расширении приближающих подпространств позволяет характеризовать свойства гладкости приближаемой функции, причём в очень широком диапазоне. Будут даны эквивалентные описания классических и обобщённых пространств Бесова в этих терминах и обсуждена более общая концепция соответствующих пространств Кальдерона и связь аппроксимативных и разностных характеристик гладкости.

2.4. Конструктивные характеристики гладкости. С аппроксимативными характеристиками тесно связаны конструктивные характеристики в терминах разложений функций из ПИП в ряды по ЦФЭТ. Будут приведены эквивалентные описания классических пространств Бесова в этих терминах и обсуждена концепция соответствующих пространств Кальдерона. Мы не сможем затронуть здесь других типов разложений (по системам всплесков, атомарных и т. д.), отсылая интересующихся к книгам Г. Трибеля, статьям Ю. В. Нетрусова, Фразье и Явертса и др.

3. О теоремах вложения разных метрик.

Будут обсуждены две следующие основные проблемы теории вложения разных метрик. Рассматривается функциональное пространство (обозначим его $\Lambda(E,F)$). Оно построено на основе ПИП E и состоит из тех его функций, которые обладают определёнными дополнительными свойствами гладкости в его норме (что отражено символом F).

Проблема 1. Для произвольного ПИП X найти точные (неулучшаемые) условия взаимосвязи свойств гладкости F с нормами в E и в X, обеспечивающие вложение $\Lambda(E,F) \subset X$ (вложение разных метрик).

Проблема 2. Для данного пространства $\Lambda(E,F)$ найти минимальное (самое узкое) ПИП X, в которое оно вложено, т. е. описать так называемую *перестановочно-инвариантную оболочку* пространства $\Lambda(E,F)$.

Для полного решения поставленных задач ключевым является нахождение эквивалентного описания конуса M убывающих перестановок функций из пространства $\Lambda(E,F)$ (во многих работах, где получены точные теоремы вложения разных метрик, подобное описание содержится в неявной форме). При наличии такого описания критерий решения первой проблемы состоит в справедливости вложения $M \subset X$. Решением второй проблемы будет минимальное ПИП X, содержащее конус M.

Мы не будем касаться здесь еще одной основной проблемы вложения разных метрик: об описании свойств гладкости в норме X функций из пространства $\Lambda(E,F)$.

4. Об оптимальном вложении обобщённых пространств Соболева.

4.1. Краткий обзор известных результатов. Будут приведены классические теоремы вложения разных метрик С. Л. Соболева [5], устанавливающие критерии вложения пространства Соболева в L_q и в пространство ограниченных непрерывных функций. В предельном случае будут приведены результаты В. И. Юдовича [24], С. И. Похожаева [23], Н. Трудингера [35] о вложении в пространство Орлича и отмечены дальнейшие усиления этих результатов в работах К. Ханссона [33], Г. Брезиса — С. Вейнгера [25], О'Нейла [34], В. Г. Мазьи [3], М. Цвикеля и Е. И. Пустыльника [30], А. Чьянки и Л. Пика [27, 28] и др.

4.2. Решение задачи об оптимальном вложении. Будет получено решение задачи об эквивалентном описании конуса

$$M_m = \{h = u^* : u \in C_0^m(\Omega),\ |\nabla^m u|^* \in E\}, \quad m \in \mathbb{N},$$

убывающих перестановок финитных функций из пространства Соболева в ограниченной области $\Omega \subset \mathbb{R}^n$ (с мерой $|\Omega|$), построенного на основе ПИП E. Показано, что $M_m \approx C_m$, $m = 1, 3, \ldots$; $M_m \approx K_m$, $m = 2, 4, \ldots$, где

$$C_m = \left\{ h(t) = \int\limits_t^{|\Omega|} g s^{m/n-1}\, ds : g \geqslant 0,\ g^* \in E \right\}, \quad t \in (0, |\Omega|];$$

$$K_m = \left\{ h(t) = t^{m/n-1} \int\limits_0^t g\, ds + \int\limits_t^{|\Omega|} g s^{m/n-1}\, ds : g \geqslant 0,\ g^* \in E \right\}.$$

На основе данного описания получены решения упомянутых выше проблем вложения разных метрик для пространств Соболева. Мы разберём ряд примеров и приложений. В основу этих построений положены модифицированные (и частично скорректированные) результаты работы Д. Эдмундса, Р. Кермана и Л. Пика [31]. Будут отмечены также результаты Ю. В. Нетрусова об оптимальных вложениях в пространств Лизоркина — Трибеля, включающих пространства дробной гладкости Соболева — Лиувилля (см. [22]).

5. Об оптимальном вложении пространств Бесова и Кальдерона.

5.1. Краткий обзор известных результатов. Будут приведены классические теоремы вложения разных метрик С. М. Никольского, О. В. Бесова (см. [10, 1, 4]) и отмечено их развитие в исследованиях по обобщённой гладкости П. Л. Ульянова и его школы (в первую очередь отметим здесь работы В. И. Коляды), а также в работах М. З. Берколайко, А. В. Бухвалова, М. Л. Гольдмана, Г. А. Калябина, Ю. В. Нетрусова и др.

5.2. Об оптимальном вложении пространств Кальдерона. Будут рассмотрены пространства Кальдерона $\Lambda(E, F)$, где E — ПИП. Дифференциальные свойства функций $f \in \Lambda(E, F)$ описываются в терминах принадлежности их наилучших приближений $e_t(f)_E$ по норме E с помощью ЦФЭТ (как функций параметра приближения $t > 0$) заданному БФП $F = F(0, \infty)$. Таким образом,

$$f \in \Lambda(E, F) \Longleftrightarrow f \in E(\mathbb{R}^n),\ e_t(f)_E \in F(0, \infty).$$

Они включают классические пространства Бесова и различные их обобщения. Для них будет приведено решение задачи об эквивалентном описании конуса убывающих перестановок и решения двух основных проблем вложения разных метрик, упомянутых выше. В частности, будет получено описание их перестановочно-инвариантной оболочки. Важную роль здесь играют модифицированные вторые перестановки измеримых функций, определяемые фундаментальной функцией ПИП E (они совпадают с классическими вторыми перестановками, если $E = L_1$).

5.3. Приложения для обобщённых пространств Бесова. Большое внимание будет уделено конкретизации этих построений в случае обобщённых пространств Бесова (в терминах наилучших приближений), позволяющей получить описание перестановочно-инвариантной оболочки в явном виде. Для этого используется ряд недавних результатов об оценках норм интегральных операторов типа Харди на конусах монотонных функций, полученных в работах А. Гогатишвили и Л. Пика [32], М. Л. Гольдмана [15], М. Карро и Х. Сориа [29] и др. Отметим, что для разностных вариантов обобщённых пространств Бесова оптимальные вложения были получены Ю. В. Нетрусовым [21].

Результаты, приведённые в пп. 5.2, 5.3, получены в работах [16] (изотропный случай) и [17] (анизотропный случай).

План цикла лекций.

1–2. Характеризация интегральных свойств функций.

Банаховы функциональные пространства, их общие свойства. Убывающие перестановки функций и их свойства. Перестановочно-инвариантные пространства, их основные характеристики. Примеры перестановочно-инвариантных пространств: пространства Лебега, Лоренца, Орлича.

3–4. Характеризация дифференциальных свойств функций.

Обобщённые производные и классические пространства Соболева. Их обобщения с использованием перестановочно-инвариантных пространств. Дробные производные и пространства Соболева — Лиувилля (бесселевых потенциалов). Шкала пространств Лизоркина — Трибеля.

Разности и модули непрерывности. Пространства Бесова и некоторые их обобщения. Концепция пространств Кальдерона.

Наилучшие приближения с помощью целых функций экспоненциального типа (аппроксимативные характеристики гладкости).

Поведение разложений в ряды по целым функциям экспоненциального типа (конструктивные характеристики гладкости)

5. Об оптимальном вложении обобщённых пространств Соболева.

Описание конуса убывающих перестановок функций из пространства Соболева. Критерий вложения пространства Соболева в перестановочно-инвариантное пространство. О перестановочно-инвариантной оболочке пространства Соболева.

6. Об оптимальном вложении пространств Бесова и Кальдерона.

Описание конуса убывающих перестановок функций из пространства Кальдерона. Критерий вложения пространства Кальдерона в перестановочно-инвариантное пространство. О перестановочно-инвариантной оболочке пространства Кальдерона. Приложения для обобщённых пространств Бесова.

Список литературы

Монографии

[1] *Бесов О. В., Ильин В. П., Никольский С. М.* Интегральные представления функций и теоремы вложения. М.: Наука, 1996.

[2] *Крейн С. Г., Петунин Ю. И., Семёнов Е. М.* Интерполяция линейных операторов. М.: Наука, 1978.

[3] *Мазья В. Г.* Пространства Соболева. Л.: Изд-во ЛГУ, 1985.

[4] *Никольский С. М.* Приближение функций многих переменных и теоремы вложения. М.: Наука, 1977.

[5] *Соболев С. Л.* Некоторые применения функционального анализа в математической физике. Л.: Изд-во ЛГУ, 1950.

[6] *Трибель Г.* Теория функциональных пространств. М.: Мир, 1986.

[7] *Adams R. A.* Sobolev spaces. New York–London: Acad. Press, 1975.

[8] *Bennett C., Sharpley R.* Interpolation of Operators. New York: Academic. 1988 (Pure Appl. Math. V., 129).

[9] *Burenkov V. I.* Sobolev spaces on domains. Stuttgart–Leipzig: Teubner, 1998.

Статьи

[10] *Бесов О. В.* Исследование семейства функциональных пространств в связи с теоремами вложения // Тр. Мат. ин-та им. В. А. Стеклова. 1961. Т. 60. С. 42–81.

[11] *Берколайко М. З., Овчинников В. И.* Неравенства для целых функций экспоненциального типа в симметричных пространствах // Тр. Мат. ин-та им. В. А. Стеклова. 1983. Т. 161. С. 3–17.

[12] *Брудный Ю. А., Шалашов В. К.* Липшицевы пространства функций // Метрические вопросы теории функций и отображений. Киев: Наукова Думка, 1973. С. 3–60.

[13] *Гольдман М. Л.* О вложении конструктивных и структурных липшицевых пространств в симметричные // Тр. Мат. ин-та им. В. А. Стеклова. 1986. Т. 173. С. 90–112.

[14] *Гольдман М. Л.* О вложении разных метрик для пространств типа Кальдерона // Тр. Мат. ин-та им. В. А. Стеклова. 1988. Т. 181. С. 70–94.

[15] *Гольдман М. Л.* Точные оценки норм операторов типа Харди на конусах квазимонотонных функций // Тр. Мат. ин-та им. В. А. Стеклова. 2001. Т. 232. С. 115–143.

[16] *Гольдман М. Л., Керман Р. А.* Об оптимальном вложении пространств Кальдерона и обобщённых пространств Бесова // Тр. Мат. ин-та им. В. А. Стеклова. 2003. Т. 243. С. 161–193.

[17] *Гольдман М. Л., Энрикес Ф. Э.* Об оптимальном вложении анизотропного пространства Кальдерона. 2003. Деп. ВИНИТИ.

[18] *Калябин Г. А.* Характеризация пространств типа Бесова — Лизоркина — Трибеля с помощью обобщенных разностей // Тр. Мат. ин-та им. В. А. Стеклова. 1988. Т. 181. С. 95–116.

[19] *Климов В. С.* Теоремы вложения и геометрические неравенства // Изв. АН СССР. Сер. мат. 1976. Т. 40, № 3. С. 645–671.

[20] *Лизоркин П. И.* Свойства функций из пространств $\Lambda^r_{p\theta}$ // Тр. Мат. ин-та им. В. А. Стеклова. 1974. Т. 131. С. 158–181.

[21] *Нетрусов Ю. В.* Теоремы вложения пространств Бесова в идеальные пространства // Зап. науч. семинаров ЛОМИ/Мат. ин-т им. В. А. Стеклова. Ленингр. отд-ние. 1987. Т. 159. С. 69–82.

[22] *Нетрусов Ю. В.* Теоремы вложения для пространств Лизоркина — Трибеля // Зап. науч. семинаров ЛОМИ/Мат. ин-т им. В. А. Стеклова. Ленингр. отд-ние. Т. 159. С. 103–112.

[23] *Похожаев С. И.* О собственных функциях уравнения $\Delta u + \lambda f(u) = 0$ // Докл. АН СССР. 1965. Т. 165. С. 36–39.

[24] *Юдович В. И.* Некоторые оценки связанные с интегральными операторами и с решениями эллиптических уравнений // Докл. АН СССР. 1961. Т. 2. С. 746–749.

[25] *Brezis H., Weinger S.* A note on limiting case of Sobolev embedding and convolution inequalities // Comm. Partial Diff. Eq. 1980. V. 5. P. 773–789.

[26] *Calderon A. P.* Intermediate Spaces and Interpolation, the Complex Method // Stud. Math. 1964. V. 24, № 2. P. 113–190.

[27] *Cianchi A.* A sharp embedding theorem for Orlich–Sobolev spaces // Indiana Univ. Math. J. 1996. V. 45. P. 39–65.

[28] *Cianchi A., Pick L.* Sobolev embeddings into BMO, VMO, and l_∞ // Ark. Mat. 1998. V. 36. P. 317–340.

[29] *Carro M., Soria J.* Weighted Lorentz Spaces and the Hardy Operator // J. Func. Anal. 1993. V. 112. P. 480–494.

[30] *Cwikel M., Pustylnik E.* Sobolev type embedding in the limiting case // J. Fourier Anal. Appl. 1998. V. 4. P. 433–446.

[31] *Edmunds D., Kerman R., Pick L.* Optimal Sobolev imbeddings involving rearrangement-invariant quasi-norms // J. Func. Anal. 2000. V. 170. P. 307–355.

[32] *Gogatishvili A., Pick L.* Discretization and anti-discretization of rearrangement invariant norms. Prague, 2002. (Preprint/Charles Univ.; № MATH–KMA–2001/70).

[33] *Hansson K.* Imbedding theorems of Sobolev type in potential theory // Math. Scand. 1979. V. 45. P. 77–102.

[34] *O'Neil R.* Convolution operators and $L(p,q)$ spaces // Duke Math. J. 1963. V. 30. P. 129–142.

[35] *Trudinger N. S.* On imbeddings into Orlicz spaces and some applications // J. Math. Mech. 1967. V. 17. P. 473–483.

Null Lagrangians

Tadeusz Iwaniec [1]

Syracuse University

The term null Lagrangian pertains to a nonlinear differential expression whose integral mean over any domain depends only on its boundary values, like integrals of exact differential forms. Because of that it may very well be right to call null Lagrangians exact nonlinear differential forms. An important special case is furnished by the Jacobian determinant $J(x, f) = \det[Df(x)]$ of a Sobolev mapping f. One rather surprising discovery, that brought null Lagrangians to the theory of nonlinear PDEs, is the higher integrability phenomenon. In this category of important results we include the L^1-estimates of $J(x, f)$ under weakest possible regularity hypotheses on the mapping f. Recently, these estimates became critical in formulating a theory of mappings with finite distortion. Yet, within somewhat wider context vast progress has been made in understanding the role of null Lagrangians in the study of so-called very weak solutions of nonlinear elliptic PDEs. The present lectures will certainly fresh light on these issues. Every effort will be made to reduce to a minimum the technical aspects of the subject in the interest of mathematical insights.

It is a special pleasure and honor for me to address these lectures at the Sobolev Institute of Mathematics in Novosibirsk, the town of Yurii Grigor'evich Reshetnyak.

[1]Department of Mathematics, Syracuse University, 215 Carnegie Bldg., Syracuse, NY 13244, USA.
E-mail: tiwaniec@mailbox.syr.edu

Composition of Functions in Besov Spaces with Critical Exponent and Spaces of Functions of Bounded p-Variation

(Based on joint work with
Gérard Bourdaud (Paris VII) and Winfried Sickel (Jena).)

Massimo Lanza de Cristoforis [1]

Universita di Padova (Italy)

We introduce the space $BV_p^1(\mathbb{R})$ of the real-valued Lipschitz continuous functions g on the real line $\mathbb{R}$ whose first order derivative equals almost everywhere a function with bounded p-variation (in the sense of Wiener), and we characterize the set of functions f of $\mathbb{R}$ to itself such that the (nonlinear) superposition operator T_f which takes g to the composite function $T_f[g] \equiv f \circ g$ maps $BV_p^1(\mathbb{R})$ to itself. Then by using Peetre's Imbedding Theorem, we deduce that if $f \in BV_p^1(\mathbb{R})$ and $f(0) = 0$, then T_f maps the Besov space $B_{p,1}^{1+(1/p)}(\mathbb{R})$ to $B_{p,\infty}^{1+(1/p)}(\mathbb{R})$. Finally, we prove that $BV_p^1(\mathbb{R})$ is contained in a class $U_p^1(\mathbb{R})$ introduced by Bourdaud and Kateb, and that if $f \in U_p^1(\mathbb{R})$ and $f(0) = 0$, then T_f maps $B_{p,1}^{1+(1/p)}(\mathbb{R})$ to $B_{p,\infty}^{1+(1/p)}(\mathbb{R})$. Corresponding results are also obtained for functions g defined in $\mathbb{R}^n$.

[1]Dipartimento di Matematica Pura ed Applicata, Universita di Padova, Via Belzoni 7, Padova 35131, Italy.
E-mail: mldc@math.unipd.it

Квазиконформно плоские поверхности в римановых многообразиях

В. М. Миклюков [1]

Волгоградский государственный университет

В терминах изопериметрии и основной частоты сечений $\mathcal{X} \setminus \Pi$ геодезическими сферами $S_{\mathcal{X}}(a,t)$ описываются некоторые глобальные свойства локально квазиконформно плоских поверхностей в римановых многообразиях общего вида.

1. Постановка задачи.
2. Дифференциальные формы. Связь с уравнениями.
3. Лемма Лебега — Куранта на многообразии.
4. Неравенство Гарнака на многообразии.
5. D-свойство квазиконформных отображений $f\colon \mathcal{X} \to \mathbb{R}^n$.
6. Изопериметрический профиль многообразия.
7. Основная частота и N-средние геодезической сферы $S_{\mathcal{X}}(a,t)$.
8. Оценки постоянной в неравенстве Пуанкаре для дифференциальных форм на многообразии.
9. Оценки интеграла энергии.
10. Квазиконформные гиперплоскости.
11. Квазиплоскости коразмерности > 1.

[1]Волгоградский государственный университет, ул. 2-ая Продольная, 30, Волгоград 400062, Россия.
E-mail: miklyuk@vlink.ru

On the Sasaki Distance Between Directions in a Metric Space and Solution of a Problem by A. D. Aleksandrov on Synthetic Description of Riemannian Manifolds

I. G. Nikolaev [1]

University of Illinois

1. Field and Metric Definition of Riemannian Manifold.

It is customary to define Riemannian space as a pair $\langle \Re, g \rangle$ of a C^3-smooth connected differentiable manifold $\Re$ of dimension greater than one and a C^2-smooth metric tensor g on it. This definition that came to us from Riemann is called the *field definition* of Riemannian space; a priori existing metric tensor field g completely determines all geometric characteristics of Riemannian geometry: *length*, *angle*, *parallel translation*, *curvature* etc. In contrast, the metric description of Riemannian geometry assumes that all basic features traditionally postulated in the field definition (differentiable structure, existence and smoothness of the metric tensor, etc.) can be derived from purely metric axioms combined with simplest topological assumptions. From metric point of view, a Riemannian space is a metric space which, in some sense, admits a continuous, metrically defined, sectional curvature.

2. Two-Dimensional Results.

A. Wald [10] obtained a metric characterization of two-dimensional Riemannian spaces. K. Menger described Wald's theorem as follows: "I venture to predict that the theorem just stated (*Wald's Theorem*) will become a cornerstone in the geometry of the future... This result should make geometers realize that (contrary to the traditional view) the fundamental notion of curvature does not depend on coordinates, equations, parametrizations, or differentiability assumptions..." [4].

A. D. Aleksandrov [1] used a very different approach, based on his studies of non-regular convex surfaces, to obtain an independent characterization of two-dimensional Riemannian spaces. Aleksandrov's theorem combined with classical results by Yu. G. Reshetnyak [9] gives a complete solution of the metric description of 2-dimensional Riemannian spaces.

We emphasize that the 2-dimensional case is very special because the curvature is defined on the space itself rather then on its Grassman manifold $G(2, n)$.

[1] University of Illinois at U-C, Department of Mathematics, 273 Altgeld Hall, MC–382, 1409 W. Green Street, Urbana, IL 61801, USA.
E-mail: inik@math.uiuc.edu

3. Aleksandrov's Problem.

A. D. Aleksandrov conjectured that it is possible to find a purely metric curvature-like conditions on the distance function of a metric space which ensure that such metric space is isometric to a C^2-smooth Riemannian manifold, that is, a Riemannian manifold with continuous curvature tensor. In the multidimensional case, first progress towards solving Aleksandrov's conjecture was done by V. N. Berestovskii [3] who proved that so called Aleksandrov spaces of two-sided bounded curvature, are in fact C^1-differentiable manifolds whose metric is given by a continuous metric tensor. Later, the author established that such spaces are almost Riemannian: they are C^3-differentiable manifolds and their metric is given by a metric tensor of class $W^2_p \cap C^{1,\alpha}$, for every $p \geqslant 1$ and $\alpha \in (0,1)$, see [5–7]. These results give a partial solution of Aleksandrov's conjecture: there is a metric description of "almost Riemannian" spaces. In 1982 A. D. Aleksandrov included the conjecture on metric characterization of classical multidimensional C^2-Riemannian spaces into the list of unsolved problems in synthetic geometry. Solution of Aleksandrov's conjecture requires a new notion of the generalized tangent "bundle" of a metric space with the Sasaki distance on it.

4. Sasaki Distance and $T^k(\mathcal{M})$

The Sasaki distance between a pair of bound vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in a Euclidean space is

$$\sqrt{(\mathrm{dist}(A,C))^2 + |\overrightarrow{AB} - \overrightarrow{AD'}|^2},$$

where $\overrightarrow{AD'}$ is the result of the parallel translation of the vector $\overrightarrow{CD}$ to the point A. Hence, the tangent bundle of a Euclidean space can be viewed as the space of bound vectors furnished with the Sasaki distance. The construction of the Sasaki distance in a Riemannian space uses the Levi–Civita parallelism. Let $\langle \mathcal{M}, g \rangle$ be a C^∞-Riemannian space. The length of a smooth path

$$t \to \Xi(t) = [c(t), \xi(t)] \in T_{c(t)}(M), \quad a \leqslant t \leqslant b,$$

in the tangent bundle $T(\mathcal{M})$, in the Sasaki metric, is given by

$$\ell_S(\Xi) = \int\limits_a^b \sqrt{|\dot c(t)|^2 + |\nabla_{\dot c(t)} \xi(t)|^2}\, dt.$$

Below we introduce the *quadrilateral cosine* that plays a key role in our work. It enables us to construct a metric substitute for $|\nabla_{\dot c(t)} \xi(t)|$ and eventually introduce generalized tangent "bundles" $T^k(\mathcal{M})$ of an arbitrary metric space $\mathcal{M}$. If $A, B \in \mathcal{M}$, then $\overrightarrow{AB}$ denotes the ordered pair (A, B). The *quadrilateral cosine* is defined by

$$\operatorname{cosq}(\overrightarrow{AB}, \overrightarrow{CD}) = \frac{\rho^2(A,D) + \rho^2(B,C) - \rho^2(A,C) - \rho^2(B,D)}{2\rho(A,B)\rho(C,D)}.$$

As one expects, $\mathcal{M}$ can be isometrically imbedded into $T(\mathcal{M})$, and for a Riemannian space with at least twice continuously differentiable metric tensor our construction produces the standard tangent bundle and the standard Sasaki distance on it. Our construction makes sense for any metric space, in particular, for non-manifolds. As an example, we consider the generalized tangent bundle of a graph.

5. Solution of Aleksandrov's Problem.

For metric spaces, the concepts of *geodesic segment*, a curve of minimal length joining a given pair of points, *angle* between geodesic segments, "tangent" *direction* and *triangle* made up of geodesic segments are defined. Let $\Omega_P(\mathcal{M})$ denote the space of directions at the point $P \in \mathcal{M}$ and $\mathcal{M}_P$ denote the space of tangent elements.

Now let $(\mathcal{M}, \rho)$ be a metric space. Let $\Omega_P(\mathcal{M}) \neq \varnothing$. Consider two pairs of directions (ξ, ζ) and (ξ', ζ') at the point P. We define the distance between these pairs by $\psi((\xi,\zeta),(\xi',\zeta')) = \max\{\measuredangle(\xi,\xi'), \measuredangle(\zeta,\zeta')\}$. A pair of directions $\sigma = (\xi, \zeta)$ is said to be a *section* at P if $0 < \measuredangle(\xi,\zeta) < \pi$; $\Omega^2_P(\mathcal{M})$ denotes the set of all sections at P.

Let $\{\mathcal{T}_k = PB_kC_k\}_{k=1,2,\dots}$ be a sequence of non-degenerate triangles with a common vertex P and $\sigma = (\xi,\zeta) \in \Omega^2_P(\mathcal{M})$. Let ξ_k and ζ_k denote the directions of geodesic segments $\mathcal{PB}_k$ and $\mathcal{PC}_k$, respectivey. We let $\sigma(\mathcal{T}_k)$ be the pair (ξ_k, ζ_k), $k = 1, 2, \dots$. The sequence $\{\mathcal{T}_k\}_{k=1,2,\dots}$ is said to be *σ-convergent* to the point P (notation: $\mathcal{T}_k \xrightarrow{\sigma} P$) if $B_k, C_k \to P$ and $\sigma(\mathcal{T}_k) \to \sigma$ as $k \to \infty$.

A metric space $(\mathcal{M}, \rho)$ *admits sectional curvature* K_σ in the direction of the section $\sigma \in \Omega^2_P(\mathcal{M})$ if

(i) There is a sequence $\{\mathcal{T}_k = PB_kC_k\}_{k=1,2,\dots}$ of non-degenerate triangles in $\mathcal{M}$, which is σ-convergent to the point P;

(ii) For every σ-convergent sequence $\{\mathcal{T}'_k = PB'_kC'_k\}_{k=1,2,\dots}$ of non-degenerate triangles in $\mathcal{M}$ the limit $\lim\limits_{\mathcal{T}'_k \xrightarrow{\sigma} P} \frac{\delta(\mathcal{T}'_k)}{\mathfrak{s}(\mathcal{T}'_k)}$ exists and equals $K_\sigma = K_\sigma(P; \mathcal{M})$.

A metric space $(\mathcal{M}, \rho)$ *possesses sectional curvature* at the point P if $\Omega^2_P(\mathcal{M}) \neq \varnothing$ and it admits K_σ for every $\sigma \in \Omega^2_P(\mathcal{M})$. Finally, the *curvature of a metric* *space* $(\mathcal{M}, \rho)$ *exists at a point* $P \in \mathcal{M}$ if $(\mathcal{M}, \rho)$ possesses sectional curvature at the point P and the lower and upper curvatures at P satisfy the following inequalities: $\underline{K}_{\mathcal{M}}(P) > -\infty$ and $\overline{K}_{\mathcal{M}}(P) < +\infty$. The construction of the Sasaki distance on $T(\mathcal{M})$ enables us to introduce the condition of continuity and Hölder continuity of the curvature of a metric space. Solution of Aleksandrov's problem is given by the following theorem (see, [8]).

Theorem 1. *A locally compact metric space with intrinsic metric and Hölder-continuous, with exponent* $\alpha \in (0, 1)$*, curvature which admits local geodesic extandability is isometric to a* C^2*-Riemannian manifold.*

Further we have:

Theorem 2. *Any locally compact metric space* $\mathcal{M}$ *with intrinsic metric, which is not linear at one of its points, in which geodesics are locally extendable and such that, for some* $m = 0, 1, 2, \ldots$*, the curvature of the generalized tangent bundle* $T^m(\mathcal{M})$ *is Hölder-continuous with exponent* $\alpha \in (0, 1)$*, is isometric to a* C^{m+2}*-smooth Riemannian manifold.*

As a corollary, we derive the following metric description of C^∞-Riemannian manifolds.

Corollary 3. *Let* $(\mathcal{M}, \rho)$ *be a locally compact metric space with intrinsic metric and which admits local geodesic extandability. Suppose that the curvature of* $T^m(\mathcal{M})$ *is Hölder-continuous for arbitrarily large* m *and that* $\mathcal{M}$ *is not linear at one of its points. Then* $(\mathcal{M}, \rho)$ *is isometric to a* C^∞*-Riemannian manifold.*

6. Curvature and Quadrilateral Cosine.

In conclusion we mention two results representing an interplay between the curvature condition and the notion of the quadrilateral cosine [2].

Theorem 4. *A Riemannian space* $\langle \Re, g \rangle$ *is of non-positive sectional curvature if and only if each point* $P \in \mathcal{M}$ *has a neighborhood such that, for each quadruple* $\{A, B, C, D\}$ *of distinct points in this neighborhood the absolute value of their quadrilateral cosine is bounded by* 1.

Theorem 5. *Let* $(\mathcal{M}, \rho)$ *be a metric space such that every pair of points can be joined by a geodesic segment. For every quadruple* $\{A, B, C, D\}$ *of distinct points, let the absolute value of their quadrilateral cosine is bounded by* 1*. If for some quadruple* $\{P, Q, R, S\}$ *of distinct points their quadrilateral cosine is equal to* 1*, then the geodesic convex hull of the set* $\{P, Q, R, S\}$ *is either isometric to a quadrilateral in a Euclidean plane or a segment of straight line.*

References

[1] *Aleksandrov A. D.* Intrinsic Geometry of Convex Surfaces. Moscow–Leningrad: OGIZ, 1948. (Russian). German transl.: Die innere Geometrie der konvexen Flächen. Berlin: Akademie-Verlag, 1955.

[2] *Berg I. D., Nikolaev I. G.* On a distance between directions in an Aleksandrov space of curvature $\leqslant K$ // Michigan Math. J. 1998. V. 45. P. 257-289.

[3] *Berestovskii V. N.* Introduction of a Riemannian structure in certain metric spaces // Sibirsk. Mat. Zh. 1975. V. 16, № 4. P. 651–662. (Russian). Engl. transl. in Sib. Math. J. 1976. V. 16. P. 499–507.

[4] *Menger K.* The formative years of Abraham Wald and his work in geometry // Ann. Math. Statistics. 1952. V. 23. P. 14–20.

[5] *Nikolaev I. G.* Parallel translation and smoothness of the metric of spaces with bounded curvature // Dokl. Akad. Nauk SSSR. 1980. V. 250, № 5. P. 1056–1058. (Russian). Engl. transl. in Sov. Math. Dokl. 1980. V. 21. P. 263–265.

[6] *Nikolaev I. G.* Parallel translation of vectors in spaces with curvature that is bilaterally bounded in the sense of A. D. Aleksandrov // Sibirsk. Mat. Zh. 1983. V. 24, № 1. P. 130–145. (Russian). Engl. transl. in Sib. Math. J. 1983. V. 24. P. 106–119.

[7] *Nikolaev I. G.* Smoothness of the metric of spaces with bilaterally bounded curvature in the sense of A. D. Aleksandrov // Sibirsk. Mat. Zh. 1983. V. 24, № 2. P. 114–132. (Russian). Engl. transl. in Sib. Math. J. 1983. V. 24. P. 247–263.

[8] *Nikolaev I. G.* Approximation of a convex hypersurface with bounded positive specific scalar curvature // Siberian Adv. Math. 2003. V. 13, № 4. P. 1–16.

[9] *Reshetnyak Yu. G.* Isothermal coordinates on manifolds of bounded curvature. I, II // Sibirsk. Mat. Zh. 1960. V. 1. P. 88–116, 248–276. (Russian).

[10] *Wald A.* Begründung einer koordinatenlosen Differentialgeometrie der Flächen // Ergeb. eines Math. Kolloq. 1935. V. 7. P. 24-46.

Nonlinear Capacities and Blow-Up Solutions to Nonlinear PDE's

Stanislav I. Pohozaev [1]

Steklov Institute of Mathematics

The plan of the lectures:

1. Nonlinear Capacity and Blow-Up for Nonlinear PDE's Equations and Inequalities. (Introduction.)
2. Nonlinear Elliptic Capacity Blow-Up for Nonlinear Elliptic Problems.
3. Nonlinear Parabolic Capacity and Blow-Up for Nonlinear Parabolic Problems.
4. Nonlinear Hyperbolic Capacity and Blow-Up for Nonlinear Hyperbolic Problems.

(and if there will be time)

5. Integral Capacity and Blow-Up for Nonlinear Nonlocal Problems.

[1]Steklov Institute of Mathematics, Gubkina str. 8, Moscow 119991, Russia.
E-mail: pohozaev@mi.ras.ru

Ivory's Theorem in Hyperbolic Spaces

Hellmuth Stachel[1]

Vienna University of Technology

[1]Vienna University of Technology, Wiedner Hauptstr. 8–10/104, A 1040 Wien, Austria.
E-mail: stachel@geometrie.tuwien.ac.at

A New Pointwise Selection Principle for Mappings of One Real Variable

V. V. Chistyakov [1]

State University Higher School of Economics, *and*
Institute of Mathematics, Technical University of Łódź

The well known *Helly selection principle* [6] asserts that *any uniformly bounded sequence of monotone real functions on the closed interval $T = [a, b]$ of the real line $\mathbb{R}$ contains a pointwise convergent subsequence.* This result implies several pointwise selection principles for functions and mappings of one real variable of various types of uniformly bounded and bounded generalized variations including numerous applications [7, 8, 3, 1, 4].

The aim of this work is to present a general selection principle with no condition of uniform boundedness of variations of any kind [5].

Let X be a metric space with metric $d(\cdot,\cdot)$ and X^T be the set of all mappings from the interval T into X. Given a positive integer n and $f \in X^T$, we set $\nu(n, f) = \sup \sum_{i=1}^{n} d(f(b_i), f(a_i))$, where the supremum is taken over all numbers a_i, b_i $(i = 1, \ldots, n)$ from T such that $a \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant b$. The sequence $\{\nu(n, f)\}_{n=1}^{\infty}$, called the *modulus of variation of f*, was introduced by Chanturiya in [2]. He has shown that a mapping $f \in X^T$ (with complete X) has left and right limits at all points of T if and only if $\nu(n, f) = o(n)$ (that is, $\lim_{n\to\infty} \nu(n, f)/n = 0$).

Our main result is the following *pointwise selection principle*:

Theorem. *Suppose a sequence of mappings $\{f_j\}_{j=1}^{\infty} \subset X^T$ is such that* (i) $\limsup_{j\to\infty} \nu(n, f_j) = o(n)$, *and* (ii) *for each $t \in T$ the closure in X of the set $\{f_j(t)\}_{j=1}^{\infty}$ is compact. Then there exists a subsequence of $\{f_j\}_{j=1}^{\infty}$, which converges in X pointwise on T to a mapping $f \in X^T$ satisfying $\nu(n, f) = o(n)$.*

We also prove that: 1) conditions (i) and (ii) are essential; 2) condition (i) is not necessary, but it is necessary for the uniform convergence; 3) all point-

[1]Department of Mathematics, State University Higher School of Economics, Bol'shaya Pechërskaya Street 25, Nizhny Novgorod 603600, Russia.

Current address: Institute of Mathematics, Technical University of Łódź, ul. Wólczańska 215, 93-005 Łódź, Poland.

E-mail: czeslaw@mail.ru

This work is supported by the Russian Foundation for Basic Research (Grant 03–01–00473) and J. Mianowski Fund / Foundation of Polish Science (2004).

wise selection principles referred to above are consequences of our Theorem; 4) variants of the Theorem hold for the almost everywhere convergence and weak pointwise convergence if X is a reflexive separable Banach space (cf. [5]).

References

[1] *Belov S. A., Chistyakov V. V.* A selection principle for mappings of bounded variation // J. Math. Anal. Appl. 2000. V. 249. P. 351–366.

[2] *Chanturiya Z. A.* The modulus of variation of a function and its application in the theory of Fourier series // Dokl. Akad. Nauk SSSR. 1974. V. 214. P. 63–66. (Russian). Engl. transl. in Soviet Math. Dokl. 1974. V. 15. P. 67–71.

[3] *Chistyakov V. V.* On the theory of multivalued mappings of bounded variation of one real variable // Mat. Sb. 1998. V. 189. P. 153–176. (Russian). Engl. transl. in Sbornik Math. 1998. V. 189. P. 797–819.

[4] *Chistyakov V. V.* Selections of bounded variation // J. Appl. Anal. 2004. V. 10. P. 1–82.

[5] *Chistyakov V. V.* The optimal form of selection principles for functions of a real variable // J. Math. Anal. Appl. 2004. (submitted).

[6] *Helly E.* Über lineare Funktionaloperationen // Sitzungsber. Naturwiss. Kl. Kaiserlichen Akad. Wiss. Wien. 1912. V. 121. P. 265–297.

[7] *Musielak J., Orlicz W.* On generalized variations (I) // Studia Math. 1959. V. 18. P. 11–41.

[8] *Waterman D.* On Λ-bounded variation // Studia Math. 1976. V. 57. P. 33–45.

Capacity and its Applications

Mikhail Shubin [1]

Northeastern University

Abstract. Capacity appears in electrostatics as a characteristic of a body made of conducting material: it is the maximal charge that can be put on this body so that the electric potential of the field created by this charge is bounded by 1. In 1924 Norbert Wiener introduced capacity in mathematics as a positive function of compact sets. It is not additive as a measure but subadditive. Starting with Wiener's famous work, the theory was fast developed and applied in many problems of analysis, partial differential equations, mathematical physics and geometry.

This paper presents a short summary of my lectures given in Summer 2004 for graduate students in Northeastern University (Boston) and in Conference-School on Analysis and Geometry (Novosibirsk). It starts with a definition and simplest properties of the Wiener capacity. Then we describe some classical applications of capacity in partial differential equations: removable singularities of bounded harmonic functions and regularity of boundary points for the Dirichlet boundary value problem (Wiener). In the last section we formulate a recent result by V. Maz'ya and M. Shubin on two-sided estimates for the bottom of the spectrum of the Laplacian with the Dirichlet boundary conditions in open subsets of $\mathbb{R}^n$. We comment on corollaries of this result and formulate unsolved problems.

1. Introduction to Capacity.

In this section we describe definition and simplest properties of the Wiener capacity. The detailed proofs and more details can be found e.g. in [2].

Capacity is a function on sets

$$\mathrm{cap}\colon \{\text{ Borel subsets of } (\mathbb{R}^n)\} \to [0, +\infty]$$

where for simplicity we only consider the case $n \geqslant 3$.

We will only need compact sets $F \subset \mathbb{R}^n$. In this case $0 \leqslant \mathrm{cap}(F) < +\infty$. The notion of capacity comes from electrostatics, where a unit of capacity is called **Farad**, in memory of a great English scientist Michael Faraday (1791–1867). Faraday first was a bookbinder. Though being self-trained and having no

[1]Department of Mathematics, Northeastern University, 360 Huntington Ave., Boston, MA 02115, USA.
E-mail: shubin@neu.edu

grasp of mathematics, he became interested in electricity and eventually became a great physicist. He discovered electromagnetic induction and introduced the notion of field. His picture is printed on 20 British Pounds banknotes.

Capacity characterizes an electric device which is called *capacitor* and used to store electric energy by accumulating imbalance of electric charge. 1 coulomb of charge causes a 1 volt difference of potentials across 1 farad capacitor. We can formulate this as the relation

$$V = \frac{Q}{C}$$

where V is the voltage drop, Q is the charge of the body and C is the capacitance or capacity of the capacitor. V can also be considered as the work which is needed to drag a unit charge through the capacitor. Since farad is a very big unit (SI), generally we use smaller units: microfarad ΩF, nanofarad nF and picofarad pF which are equal to $10^{-6}F$, $10^{-9}F$, $10^{-12}F$ respectively.

It is known from electrostatics that a charge q at $x_0 \in \mathbb{R}^3$ creates electric field $\mathbb{E} = -\nabla V$ at any point $x \in \mathbb{R}^3$. Here

$$V(x) = \frac{q}{4\pi|x - x_0|},$$

hence by an easy calculation

$$\mathbb{E} = \frac{q}{4\pi} \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^3},$$

which corresponds to the inverse squares law of interaction discovered in electrostatics by Coulomb (and earlier in gravitation by Newton). Since the force acting on a charge e at x is $e\mathbb{E}(x)$, then the work done over a test charge e to drag it along a curve $\gamma\colon [0,1] \to \mathbb{R}^3$ against the field is

$$\int_\gamma (-e\mathbb{E}) \cdot d\vec{x} = \int_\gamma e\, dV = e(V(\gamma(1)) - V(\gamma(0))),$$

In particular, we see that this work is independent of the curve, provided that the initial and terminal points of the curve are fixed.

In n-dimensional case ($n \geqslant 3$) a natural generalization of the potential above is obtained if we take $V(x) = q\mathcal{E}(x)$ where

$$\mathcal{E}(x) = \frac{1}{(n-2)\omega_n |x - x_0|^{n-2}}, \quad x \in \mathbb{R}^n,$$

which is the fundamental solution for the operator $(-\Delta)$ in $\mathbb{R}^n$, $n \geqslant 3$. In this formula ω_n is the $(n-1)$-dimensional volume of the $(n-1)$-dimensional unit sphere.

If we have several charges $q_1, q_2, \dots, q_N$ at the points $y_1, y_2, \dots, y_N$ respectively, then we can use so called *superposition principle* which is equivalent to the linearity of the electrostatics equations (or, more generally, Maxwell equations, which describe electrodynamics): if charges are added, then the forces and potentials are added too. So we get

$$V(x) = \frac{1}{4\pi} \sum_{i=1}^{N} \frac{q_i}{|x - y_i|}, \quad x \in \mathbb{R}^3,$$

and more generally

$$V(x) = \sum_{i=1}^{N} q_i \mathcal{E}(x - y_i), \quad x \in \mathbb{R}^n, \ n \geqslant 3.$$

In case when the charge distribution is not discrete and given by a density function $\rho(x)$ (say, continuous and compactly supported), then we need to take the potential in the form

$$V(x) = \int_{\mathbb{R}^n} \mathcal{E}(x - y) \rho(y) \, dy,$$

or more generally if distribution is given by a compactly supported measure (possibly signed) μ:

$$V(x) = \int_{\mathbb{R}^n} \mathcal{E}(x - y) \, d\mu(y).$$

Note that $\Delta V(x) = 0$ outside the support of the measure μ. If $d\mu(x) = \rho(x)\, dx$, where $\rho \in C^1$, then $\Delta V(x) = -\rho(x)$.

Now let us turn to a precise definition of capacity which is due to Norbert Wiener (1894–1964). He contributed to many areas of mathematics and applied mathematics. (In particular, he is known as the father of cybernetics.) He once said: "One of the chief duties of the mathematician in acting as an advisor to scientists is to discourage them from expecting too much from mathematics". Wiener gave the following definition of capacity:

Definition 1.1. *Capacity* of a compact set $F \subset \mathbb{R}^n$, $n \geqslant 3$, is

$$\operatorname{cap}(F) = \sup_{\mu} \{\mu(F) | \int_F \mathcal{E}(x - y) \, d\mu(y) \leqslant 1 \text{ for all } x \in \mathbb{R}^n \setminus F\}, \tag{1.1}$$

where μ is a measure on F (possibly signed), supremum is taken over all such measures.

In fact, maximum or maximizing measure on F exists and it is unique. It is positive and supported on the boundary of F, which is $\partial F = F \setminus \mathrm{Int}(F)$, where $\mathrm{Int}(F)$ is the set of all interior points of F. The maximizing measure is called *equilibrium distribution of charges*. When the total measure $\mu(F)$ is fixed, then the equilibrium distribution of charges minimizes the energy of the system of charges. The corresponding potential (the integral in (1.1)) is called the *equilibrium potential*.

It was Faraday who demonstrated that in equilibrium the charges only reside on the exterior boundary of a charged conductor, and an exterior charge had no influence on anything enclosed within a conductor (this shielding effect is used in what is now known as the Faraday cage). This means that $V(x)$ is constant on every connected component of $\mathrm{Int}(F)$ (so $V = \mathrm{const}$ on $\mathrm{Int}(F)$ if $\mathrm{Int}(F)$ is connected). This property is equivalent to saying that $\mathbb{E}(x) = 0$ on $\mathrm{Int}(F)$.

Let us also provide an alternative definition of the Wiener capacity:

Definition 1.2. *Capacity* of a compact set $F \subset \mathbb{R}^n$, $n \geqslant 3$, is

$$\mathrm{cap}(F) = \inf_u \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \;\middle|\; u \equiv 1 \text{ on } F,\ u(x) \to 0 \text{ as } |x| \to \infty \right\}. \tag{1.2}$$

Here the infimum is taken over all $u \in C^\infty(\mathbb{R}^n)$ satisfying the conditions in (1.2). It is easy to see that instead we can take infimum over functions from $u \in C_0^\infty(\mathbb{R}^n)$, such that $u = 1$ in a neighborhood of F (with the neighborhood depending upon u). In yet another convenient version instead of requiring $u \in C^\infty$ (or C_0^∞) we can take Lipschitz functions i.e. functions u satisfying

$$|u(x) - u(y)| \leqslant C|x - y|, \quad x, y \in \mathbb{R}^n,$$

with C depending upon u. Such functions are known to be differentiable almost everywhere with the derivatives coinciding with the corresponding distributional derivatives (so that we can integrate by parts).

Definition 1.2 is equivalent to Definition 1.1 but we will not prove this now. Let us only mention that the minimizing function in (1.2) in fact coincides with the equilibrium potential for any set F as in Definition 1.1, provided F is sufficiently regular (e.g. if every connected component of F is the closure of an open set with a smooth boundary).

Proposition 1.3. *The capacity as a function on compact sets with values in* $[0, +\infty)$ *has the following properties*:

1. Monotonicity: $F_1 \subset F_2$ *implies that* $\operatorname{cap}(F_1) \leqslant \operatorname{cap}(F_2)$.

2. Continuity: *for every compact* F *and every* $\epsilon > 0$*, there exists an open set* $U \supset F$*, such that for every compact* F' *with* $U \supset F' \supset F$*, we have*

$$\operatorname{cap}(F') \leqslant \operatorname{cap}(F) + \epsilon.$$

3. Choquet inequality: *for any compact sets* $F_1, F_2 \subset \mathbb{R}^n$

$$\operatorname{cap}(F_1 \cup F_2) + \operatorname{cap}(F_1 \cap F_2) \leqslant \operatorname{cap}(F_1) + \operatorname{cap}(F_2).$$

   In particular, the capacity is subadditive i.e.

$$\operatorname{cap}(F_1 \cup F_2) \leqslant \operatorname{cap}(F_1) + \operatorname{cap}(F_2)$$

   for any compact sets $F_1, F_2 \subset \mathbb{R}^n$.

Proof. Monotonicity follows immediately from Definition 1.2. Continuity easily follows from the same definition of $\operatorname{cap}(F)$ if we use the test functions from $C_0^\infty(\mathbb{R}^n)$ which are equal to 1 near F.

To prove the Choquet inequality we can use test functions u, $v \in C_0^\infty(\mathbb{R}^n)$ which have compact support and are almost minimizing for F_1, F_2 in (1.2), and then take $\varphi = \max\{u, v\}$, $\psi = \min\{u, v\}$ (which are Lipschitz functions with compact support). Then φ, ψ can be test functions for $F_1 \cup F_2$, $F_1 \cap F_2$, and

$$\begin{aligned} \operatorname{cap}(F_1 \cup F_2) + \operatorname{cap}(F_1 \cap F_2) &\leqslant \int |\nabla \varphi|^2 \, dx + \int |\nabla \psi|^2 \, dx \\ &= \int |\nabla u|^2 \, dx + \int |\nabla v|^2 \, dx \leqslant \operatorname{cap}(F_1) + \operatorname{cap}(F_2) + \epsilon, \end{aligned} \tag{1.3}$$

where $\epsilon > 0$ can be made arbitrarily small.

In this calculation we used the fact (which follows from the implicit function theorem) that for any function $f \in C^\infty(\mathbb{R}^n)$ the set

$$\{x : f(x) = 0, \ \nabla f(x) \neq 0\}$$

has measure zero (we should apply this to $f = u - v$). The resulting inequality in (1.3) holds for every $\epsilon > 0$, hence for $\epsilon = 0$, which ends the proof. □

It can be shown that the Choquet inequality allows to extend capacity to all Borel sets, like a measure. We can start with the following two functions of sets which are related to capacity and defined for any set in $\mathbb{R}^n$ (unlike capacity):

Definition 1.4.

- *Internal capacity* of any set $E \subset \mathbb{R}^n$ is defined as

$$\underline{\operatorname{cap}}(E) = \sup_{K \subset E,\ K \text{ compact}} \operatorname{cap}(K)$$

- *External capacity* of any set $E \subset \mathbb{R}^n$ is defined as

$$\overline{\operatorname{cap}}(E) = \inf_{G \supset E,\ G \text{ open}} \underline{\operatorname{cap}}(G).$$

As in measure theory the best sets are the ones where the above two definitions coincide. There is enough of them due to the following Choquet theorem:

Theorem 1.5. $\underline{\operatorname{cap}}(E) = \overline{\operatorname{cap}}(E)$ *for any Borel set* E (*and even any analytic set*).

Let us recall that the Borel sets are sets from a minimal σ-algebra which contains all open (or closed) sets. Analytic sets form a bigger σ-algebra which we will not define here.

Any set E satisfying the above condition in Theorem 1.5 is called *capacitable.* In particular, all open and closed sets are capacitable, as well as all sets which are obtained from them by arbitrarily many countable unions and intersections.

The following theorem establishes a relation between Lebesgue measure and capacity:

Theorem 1.6. *For any Borel set* $F \subset \mathbb{R}^n$, $n \geqslant 3$,

$$\operatorname{mes}(F) \leqslant c_n (\operatorname{cap}(F))^{\frac{n}{n-2}},$$

with equality for any closed ball.

The constant c_n can be found from the explicit values of the measure and capacity of the unit ball.

Corollary 1.7. *If* $F \subset \mathbb{R}^n$ *and* $\operatorname{cap}(F) = 0$, *then* $\operatorname{mes}(F) = 0$.

Note that the converse is not always true. For example, for any open ball B_r with the radius r we have

$$\operatorname{cap}(\overline{B}_r) = \operatorname{cap}(\partial B_r) = (n-2)\omega_n r^{n-2},$$

but $\operatorname{mes}(\partial B_r) = 0$. Here $\overline{B}_r$ means the closure of B_r, i.e. the corresponding closed ball.

2. Applications to Partial Differential Equations.

We will describe without proofs two important applications of capacity in partial differential equations. In these applications the language of capacity is clearly relevant, in particular because they give necessary and sufficient conditions of some important properties to hold. For the proof we refer the reader to [2].

2.1. Removable Singularity Property.

Definition 2.1. Let E be a compact subset in $\mathbb{R}^n$. Suppose that for any open set $\Omega \subset \mathbb{R}^n$, such that $\Omega \supset E$, and any $u \in C^\infty(\Omega \setminus E)$ such that $\Delta u = 0$ in $\Omega \setminus E$ and u is bounded on $\Omega \setminus E$, there exists $U \in C^\infty(\Omega)$, such that $\Delta U = 0$ in Ω and $U = u$ on $\Omega \setminus E$ (i.e., any bounded harmonic function in $\Omega \setminus E$ can be extended to a harmonic function in Ω). Then E is said to have *removable singularities property.*

It is proved in PDE textbooks that a set consisting of one point satisfies the removable singularity property.

The following theorem completely characterizes all compact sets in $\mathbb{R}^n$ which have the removable singularity property.

Theorem 2.2. *A compact set E has the removable singularity property if and only if* $\operatorname{cap}(E) = 0$.

Note that if E is a submanifold then

$$\operatorname{cap}(E) = 0 \iff \operatorname{codim} E \geqslant 2.$$

2.2. Solvability of the Dirichlet Problem.

Consider the Dirichlet problem stated in the classical form as follows:

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n, \qquad u|_{\partial\Omega} = \varphi \in C(\partial\Omega), \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\partial\Omega$ its boundary, φ is a given continuous function on $\partial\Omega$ and we are looking for a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. (Then u is called a *classical solution.*) By the maximum principle such solution is unique. If $\partial\Omega$ is smooth or piecewise smooth, then the classical solution exists for all φ. But for general Ω the solution u may not exist for some φ.

N. Wiener discovered a necessary and sufficient condition on Ω such that the classical solution exists for all φ. We will now describe this result which is called Wiener Criterion.

Let us start by presenting Ω in the form of a union of domains with smooth boundaries:

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k, \quad \Omega_k \subset \Omega_{k+1},$$

and assume that $\varphi = \Phi\,|_{\partial\Omega}$ where $\Phi \in C(\overline{\Omega})$, i.e. Φ is a continuous extension of φ to $\overline{\Omega}$. Let us construct a harmonic function u_k in Ω_k such that

$$u_k\,|_{\partial\Omega_k} = \Phi\,|_{\partial\Omega_k}$$

Clearly,

$$|u_k| \leqslant M = \max_{\overline{\Omega}} \Phi.$$

It follows that the set of all u_k's is precompact in $C(K)$ for any compact $K \subset \Omega$. Therefore, passing to a subsequence if necessary, we can assume that $u_k \to U \in C^\infty(\Omega)$ uniformly on any compact set $K \subset \Omega$.

It is easy to see that U does not depend on Φ and if (2.1) is solvable, then $u = U$. However, in general, it is not necessarily true for all points $x \in \partial\Omega$ that

$$\lim_{x\in\Omega, x\to\overline{x}} U(x) = \varphi(\bar{x}).$$

The points $\bar{x}$ where this is true for all φ are called *regular*. (We will also call a point *irregular* if it is not regular.) The Dirichlet problem (2.1) is solvable for all φ if and only if all points $x \in \partial\Omega$ are regular.

The following theorem gives a regularity criterion:

Theorem 2.3 (Wiener). *A point $\bar{x} \in \partial\Omega$ is regular if and only if*

$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}((\overline{B}_{2^{-k}} \setminus B_{2^{-k-1}}) \cap (\mathbb{R}^n \setminus \Omega)) = +\infty,$$

where $B_r = B_r(\bar{x})$ (the open ball of radius r centered at $\bar{x}$), $\overline{B}_r$ is the closure of this ball.

The following theorem asserts that there are sufficiently many regular points:

Theorem 2.4 (Kellogg). *For any bounded open set $\Omega \subset \mathbb{R}^n$*

$$\operatorname{cap}\{\text{irregular points on } \partial\Omega\} = 0.$$

It can be shown that then $(n-1)$-dimensional Hausdorff measure of the set of irregular points is 0. In particular, the set of regular points is dense in $\partial\Omega$.

3. An Application of Capacity in Spectral Theory

In this section we describe a recent application of capacity to spectral theory, based on ideas of [1].

Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth (C^∞) boundary $\partial\Omega$. Consider the operator $-\Delta$ on the domain

$$D(-\Delta) = \{u \in C^2(\overline{\Omega}),\ u|_{\partial\Omega} = 0\}.$$

Now let λ be an eigenvalue of the operator $(-\Delta)$ with a corresponding eigenfunction u, i.e. $u \in D(-\Delta)$, $u \not\equiv 0$, and $(-\Delta)u = \lambda u$ in Ω. It is convenient to consider $-\Delta$ as an (unbounded) linear operator in the Hilbert space $L^2(\Omega)$ with the scalar product $(u, v) = \int\limits_\Omega u\bar{v}\, dx$. Then for an eigenfunction u with the eigenvalue λ we obtain, integrating by parts (or using the Green formula):

$$\lambda(u,u) = (-\Delta u, u) = \int\limits_\Omega |\nabla u|^2\, dx = \int\limits_\Omega \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}\right|^2 dx,$$

which implies

$$\lambda = \frac{\int\limits_\Omega |\nabla u|^2\, dx}{\int\limits_\Omega |u|^2\, dx} \geqslant 0.$$

It can be proved that there exists a complete orthonormal system $\psi_1, \psi_2, \ldots$ of eigenfunctions of $-\Delta$ in $D(-\Delta)$, with the eigenvalues $\lambda_1, \lambda_2, \ldots$, so that $-\Delta\psi_j = \lambda_j\psi_j$. The eigenvalues form a discrete set with the only accumulation point at $+\infty$, so we can enumerate them in the increasing order:

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots.$$

Here each eigenvalue is listed the number of times which is equal to its multiplicity. (In fact, it can be proved that the lowest eigenvalue is simple, so that $\lambda_1 < \lambda_2$.) Then

$$\min_j \lambda_j = \lambda_1 = \min_u \frac{\int\limits_\Omega |\nabla u|^2\, dx}{\int\limits_\Omega |u|^2\, dx}, \tag{3.1}$$

where the minimum is taken over all $u \in D(-\Delta)$. To show this note first that λ_1 is obviously greater or equal than the right hand side of (3.1) because we can take $u = \psi_1$. For the inverse inequality we should prove that the ratio in the right hand side of (3.1) is always greater or equal than λ_1 for any $u \in D(-\Delta)$. To this end let us expand u over the system $\{\psi_j\}$: $u = \sum\limits_j c_j\psi_j$, where c_j are the corresponding Fourier coefficients. Then $(u,u)=\sum_j |c_j|^2$ and

$$(-\Delta u, u) = \sum_j \lambda_j |c_j|^2 \geqslant (\min_j \lambda_j) \sum_j |c_j|^2 = \lambda_1(u,u),$$

which proves the desired inequality.

It can be shown that instead of taking minimum over $u \in D(-\Delta)$ in (3.1) we can take infimum over all functions $u \in C_0^\infty(\Omega)$. (To prove this we need to approximate any function $u \in D(-\Delta)$ by functions from $C_0^\infty(\Omega)$, multiplying u by appropriate cut-off functions which vanish near the boundary, and then smoothing them to put them into $C_0^\infty(\Omega)$.) But then the corresponding infimum is well defined for any open set $\Omega \subset \mathbb{R}^n$ (possibly unbounded and having non-smooth boundary). So for any such Ω we can define its invariant

$$\lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega |u|^2\, dx}. \tag{3.2}$$

In fact, $\lambda(\Omega)$ is also the bottom of the spectrum of a self-adjoint operator which can be obtained by the Friedrichs construction from the quadratic form which is the Dirichlet integral (the numerator in (3.2)). We will not describe the Friedrichs construction. The corresponding operator does not necessarily have discrete spectrum (or even eigenvalues) in $L^2(\Omega)$. It may have continuous spectrum like in the case of $\Omega = \mathbb{R}^n$. In general the spectrum may have a complicated nature instead, and investigating it is an important problem of spectral theory.

For general unbounded Ω it is important to know whether the spectrum is separated from 0, i.e. whether $\lambda(\Omega)$ is strictly positive. Clearly, positivity of $\lambda(\Omega)$ is equivalent to the inequality

$$\int_\Omega |\nabla u|^2\, dx \geqslant \lambda \int_\Omega |u|^2\, dx, \quad u \in C_0^\infty(\Omega), \tag{3.3}$$

to hold with $\lambda > 0$ (independent of u), or in other words,

$$\int_\Omega |u|^2\, dx \leqslant C \int_\Omega |\nabla u|^2\, dx, \quad u \in C_0^\infty(\Omega), \tag{3.4}$$

where $C = C(\Omega)$ is independent of u (the best possible C is $\lambda(\Omega)^{-1}$).

Example 3.1. In 1 dimensional case, let $\Omega = \mathbb{R}$, and take $u = u_N \in C^\infty(\mathbb{R})$ such that $u_N = 1$ on $[-N, N]$, $u_N = 0$ on $[-\infty, -N-1) \cup (N+1, +\infty]$, $|u_N| \leqslant 1$, and $|u'_N| \leqslant C$ for some $C > 0$ (independent of N). Then

$$\frac{\int |\nabla u_N|^2 \, dx}{\int |u_N|^2 \, dx} \leqslant \frac{C^2}{N} \to 0 \quad \text{as } N \to \infty,$$

so the estimate (3.4) is impossible (hence $\lambda(\mathbb{R}) = 0$).

Example 3.2. Now let us consider the case of a finite interval $\Omega = (0, \ell)$, where $\ell > 0$. Let $u \in C^1([0, \ell])$, $u(0) = 0$. Then we can write

$$u(x) = \int_0^x u'(t) \, dt,$$

so by the Cauchy–Schwarz inequality

$$|u(x)|^2 \leqslant x \int_0^x |u'(t)|^2 \, dt \leqslant \ell \int_0^\ell |u'(t)|^2 \, dt,$$

hence integrating it over $(0, \ell)$, we obtain

$$\int_0^\ell |u|^2 \, dx \leqslant \ell^2 \int_0^\ell |u'(t)|^2 \, dt.$$

This means that we can take $C = \ell^2$ in (3.4) and $\lambda = \ell^{-2}$ in (3.3), so we should have

$$\lambda((0, \ell)) \geqslant \ell^{-2}.$$

Note that in this case the eigenfunctions can be explicitly found: they are $\sin \frac{\pi k x}{\ell}$, $k = 1, 2, \ldots$, and the eigenvalues are $\frac{\pi^2 k^2}{\ell^2}$. So the smallest eigenvalue for $(0, \ell)$ is in fact $\lambda((0, \ell)) = \pi^2 / \ell^2$.

In this example we were able to obtain the estimate of the form (3.4) for $\Omega = (0, \ell)$ with the only condition $u(0) = 0$.

Now we will try to characterize compact sets $F \subset \overline{B}_r$ (here B_r means a fixed open ball with an arbitrary center in $\mathbb{R}^n$, $\overline{B}_r$ its closure), such that the following estimate holds

$$\int_{B_r} |u|^2 \, dx \leqslant C \int_{B_r} |\nabla u|^2 \, dx, \quad u \in C^\infty(\overline{B}_r), \ u|_F = 0,$$

where C is independent of u. The following example shows that for the dimension $n \geqslant 3$ it is not enough to take F consisting of a single point.

Example 3.3. Let $B_r = B_r(0)$, $F = \{0\}$ in $\mathbb{R}^n$, $n \geqslant 3$. Let us show that the estimate

$$\int_{B_r} |u|^2 \, dx \leqslant C \int_{B_r} |\nabla u|^2 \, dx, \quad u \in C^\infty(\overline{B}_r), \; u(0) = 0,$$

does not hold. To this end take $u = u_\epsilon = u_\epsilon(|x|)$, where $u_\epsilon = 1$ if $|x| \geqslant \epsilon$, $u_\epsilon(0) = 0$, $|\nabla u| \leqslant C_1 \epsilon^{-1}$. Then $\int_{B_r} |u_\epsilon|^2 \, dx \geqslant C_2 > 0$, but

$$\int_{B_r} |\nabla u|^2 \, dx \leqslant C_3 \epsilon^{-2} \epsilon^n = C_3 \epsilon^{n-2} \to 0, \qquad \text{as } \epsilon \searrow 0,$$

which contradicts the estimate above.

In fact, it can be shown that the estimate

$$\int_{B_r} |u|^2 \, dx \leqslant C \int_{B_r} |\nabla u|^2 \, dx, \quad u \in C^\infty(\overline{B}_r), \; u|_F = 0,$$

holds if and only if $\operatorname{cap}(F) > 0$. More precisely,

$$\int_{B_r} |u|^2 \, dx \leqslant \frac{C_n r^n}{\operatorname{cap}(F)} \int_{B_r} |\nabla u|^2 \, dx, \quad u \in C^\infty(\overline{B}_r), \; u|_F = 0,$$

where C_n depends on the dimension n only.

Now let us try to estimate $\lambda(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^n$ in geometric terms. To this end we will first notice that $\lambda(\Omega)$ has the following *monotonicity property*:

$$\text{If } \Omega' \subset \Omega, \text{ then } \lambda(\Omega) \subset \lambda(\Omega').$$

This immediately follows from (3.2) because the supply of functions for taking the infimum is larger and therefore the infimum is smaller for Ω compared with Ω'.

Now let us compare Ω with a ball $B_r \subset \Omega$. By a scaling (a similarity transformation of variables reducing the ball B_r to a unit ball B_1), we easily obtain that

$$\lambda(B_r) = c_n r^{-2},$$

where $c_n = \lambda(B_1)$. By monotonicity we get then:

$$\lambda(\Omega) \leqslant c_n r^{-2}.$$

This estimate becomes stronger if r increases, so we should take a ball of the maximal radius to get the best estimate. Since the biggest ball $B_r \subset \Omega$ may not exist, we can define the *interior radius* of Ω by

$$r_\Omega = \sup\{r \mid \exists B_r \subset \Omega\}. \tag{3.5}$$

Clearly, $0 < r_\Omega \leqslant +\infty$ for any non-empty Ω (and it can be indeed $+\infty$ for unbounded Ω). From the previous arguments we easily conclude that

$$\lambda(\Omega) \leqslant C_n r_\Omega^{-2}. \tag{3.6}$$

This gives the best result which we can get from monotonicity. But it is still far from being precise. In particular, the opposite estimate is not true. This can be seen from the following example. Let

$$\Omega = \mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} \overline{B}_{r_z}(z),$$

where $r_z \to 0$ sufficiently fast as $|z| \to \infty$ (e.g. $r_z = 2^{-|z|}$ is sufficient). Obviously, $r_\Omega < \infty$, but it can be proved (e.g. from a more precise estimate given below) that $\lambda(\Omega) = 0$.

However by modifying the definition of r_Ω in (3.5), we can improve (3.6) and get a two-sided estimate for $\lambda(\Omega)$. This modification consists of ignoring sets of "small" capacity, which we will refer to as *negligible* sets. In fact the definition of negligibility includes a parameter γ, $0 < \gamma < 1$. We will call a compact set $F \subset \overline{B}_r$ *negligible* in $\overline{B}_r$, or, more precisely, γ-*negligible* in $\overline{B}_r$, if

$$\operatorname{cap}(F) \leqslant \gamma \operatorname{cap}(\overline{B}_r).$$

Now we can modify the definition of r_Ω by introducing a new quantity

$$r_{\Omega,\gamma} = \sup\{r \mid \exists B_r \subset \mathbb{R}^n,\ \overline{B}_r \setminus \Omega \text{ is } \gamma\text{-negligible in } B_r\}.$$

So here we allow not only balls $B_r \subset \Omega$ but balls which are in Ω up to a set of "small" capacity. (Note, however, that here "small" may mean set which is allowed to take 99% os capacity of $\overline{B}_r$, if $\gamma = 0.99$.)

The quantity $r_{\Omega,\gamma}$ is called *interior capacitary radius* of Ω.

We will now formulate a result which is essentially contained in [1] and gives the desired two-sided estimate for $\lambda(\Omega)$.

Theorem 3.4. *Let us fix $\gamma \in (0,1)$. Then there exist $c = c(\gamma, n) > 0$ and $C = C(\gamma, n) > 0$ such that*

$$cr_{\Omega,\gamma}^{-2} \leqslant \lambda(\Omega) \leqslant Cr_{\Omega,\gamma}^{-2}. \tag{3.7}$$

Let us formulate some interesting corollaries of Theorem 3.4.

Corollary 3.5. $\lambda(\Omega) > 0$ *if and only if* $r_{\Omega,\gamma} < \infty$.

This corollary gives necessary and sufficient condition of strict positivity of the operator $-\Delta$ (with the Dirichlet boundary conditions) in Ω. (Here the operator should be understood as the Friedrichs extension from $C_0^\infty(\Omega)$.)

Since the condition $\lambda(\Omega) > 0$ does not contain γ, we immediately obtain

Corollary 3.6. *Conditions $r_{\Omega,\gamma} < \infty$, taken for different γ's, are equivalent.*

Denoting $F = \mathbb{R}^n \setminus \Omega$ (which can be an arbitrary closed subset in $\mathbb{R}^n$), we obtain from the previous Corollary (comparing $\gamma = 0.01$ and $\gamma = 0.99$):

Corollary 3.7. *Let F be a closed subset in $\mathbb{R}^n$, which has the following property: there exists $r > 0$ such that*

$$\operatorname{cap}(B_r \setminus \Omega) \geqslant 0.01 \operatorname{cap}(B_r)$$

for all B_r. Then there exists $r_1 > 0$ such that

$$\operatorname{cap}(B_{r_1} \setminus \Omega) \geqslant 0.99 \operatorname{cap}(B_{r_1})$$

for all B_{r_1}.

This is a new property of capacity which is proved by spectral theory arguments.

Let us formulate two open problems related with the topics discussed in this section.

1. Find precise dependence of $c = c(\gamma, D)$ and $C = C(\gamma, n)$ from (3.7) upon γ and n.

2. Extend the results formulated in this section to the Laplacian on Riemannian manifolds.

Once upon a time Marc Kac formulated a fundamental and fascinating question: **"Can you hear the shape of a drum?"** The precise meaning of this question is as follows: is it possible to reconstruct the drum (a bounded domain in $\mathbb{R}^2$) up to an isometry by the spectrum of its Dirichlet Laplacian (i.e. Laplacian with the Dirichlet boundary conditions)? Now, decades and hundreds (if not thousands) papers after this formulation first appeared, this question and its generalizations are still in the focus of attention for many researchers in spectral geometry.

Theorem 3.4 suggest formulation of a question, which is roughly inverse to the question of Marc Kac: **"Can you see the fundamental frequency of a drum?"** More precisely, can you find a simple visual image related to a domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$), such that it allows to recover the lowest eigenvalue of the Dirichlet Laplacian in this domain? Assuming that our eye can filter out the sets of small capacity, a partial answer to this question is given by Theorem 3.4.

I will finish with a quote which I borrowed from a preface by Michel Hazewinkel to one of the books which he edited (and which is published by Kluwer Publishers): **"Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question."** (From "The Hermit Clad in Crane Feathers" in R. van Gulik's *"The Chinese Maze Murders".*)

Acknowledgments.

I am very grateful to Mustafa Kesir who helped me to edit and typeset preliminary notes of this text.

References

[1] *Maz'ya V., Shubin M.* Discreteness of spectrum and positivity criteria for Schrödinger operators. 2003. (Preprint/arXiv:math.SP/0305278; `http://xxx.lanl.gov`, 19 May 2003).

[2] *Wermer J.* Potential theory. Berlin–New York: Springer, 1974. (Lecture Notes in Mathematics; 408).