On Growth of Algebras.

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We will talk about groups, monoids, algebras: all things that grow to become infinite.

Groups. $G = \langle x_1, \dots, x_m \rangle =$

$\{ x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1} \}$

$B(n) = \{ x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1}, \ k \leq n \}$

$B(1) \subseteq B(2) \subseteq \cdots, \quad G = \bigcup_{n \geq 1} B(n)$

$g_X(n) = |B(n)|$ growth function

Unfortunately, it depends on

$X = \{x_1, \dots, x_m\}$.

$N = \{1, 2, \dots\}$.

Definition. Let $f, g : N \to [1, \infty)$ be two increasing functions. We say that $f \preceq g$ (f is asymptotically less or equal to g) if $\exists\, C \in N$:

$f(n) \leq C\, g(Cn) \quad \forall n.$

If $f \preceq g$, $g \preceq f$ then $f \sim g$, asymptotically equivalent.

If $G = \langle X \rangle = \langle Y \rangle$, $|X| < \infty$, $|Y| < \infty$

then $\qquad g_X(n) \sim g_Y(n)$.

Growth of G = class of equivalence.

<u>Motivation</u>:

1) Geometry (Shwarz, Efremovich, Milnor),

2) Comparison of infinite groups.

<u>Monoids</u>: All the same, but no inverses.

<u>Hereditary Languages</u>.

<u>Alphabet</u> $a_1, \dots, a_m$

$W = \{w_i\}_i$ "bad" prohibited words

$L(W) = \{$all words that do not contain w_i's as subwords$\} \cup \{0\} =$ monoid with zero $\langle a_1, \ldots, a_m \mid w_i = 0 \rangle$.

Stepin (1989), Grigorchuk (1995): some explicit formulas for growth.

Algebras. F a field, $A = \langle V \rangle_F$ an algebra generated by a finite dimensional subspace V.

$V^n =$ span of all products $v_1 \cdots v_k$ (with all brackets), $v_i \in V$, $k \leq n$.

$$V^1 \subseteq V^2 \subseteq \cdots, \quad A = \bigcup_{n \geq 1} V^n$$

$$g_V(n) = \dim_F V^n$$

Gelfand-Kirillov (1965):

$$\mathrm{GKdim}(A) = \inf\{\alpha \mid g_V(n) \lesssim n^\alpha\}.$$

J. Milnor (1968, Math Monthly)

Problem 1. Is it true that G has polynomially bounded growth $\iff$ $G \triangleright H$, $|G : H| < \infty$, H is nilpotent.

Problem 2. Do there exist groups with intermediate growth?

1980, M. Gromov : Yes for Problem 1.

Groups with polynomially bounded growth are virtually nilpotent.

1983, R. Grigorchuk : Yes for Problem 2.

There exists a group of intermediate growth

polynomials < growth(G) < exponentials.

What is the growth of the Grigorchuk group?

The equation

$$x^3 - x^2 - 2x - 4 = 0$$

has a root $\alpha \approx 2.46$. Let $\beta = \log_\alpha 2 \approx 0.7674$

Theorem (L. Bartholdi, 1998; A. Erschler - T. Zheng, 2018). The growth of the Grigorchuk group is $\sim e^{n^{\beta}}$.

Basic Question (for groups, algebras, monoids etc.):

which functions appear as growth functions (up to asymptotic equivalence)?

If $g(n)$ is a growth function then:

(1) $g(n)$ is an increasing function,

(2) $g(m+n) \leq g(m)\, g(n)$,

submultiplicativity.

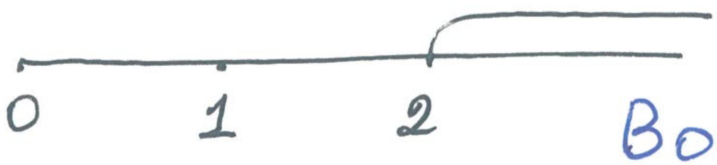
L. Bartholdi - A. Erschler:

$e^{n^\beta} \lneq f(n) \lneq e^n$ and $f(n)$ is "sufficiently regular" $\Longrightarrow$ $f(n) \sim$ growth function of a group.

Conjecture (R. Grigorchuk): there are no growth functions in the gap $]$ polynomials, $e^{\sqrt{n}}$ $[$.

True for residually-p groups (Grigorchuk).

Algebras: Gelfand-Kirillov dimension.



Bohro-Kraft

Hedlund - Morse (1938), G. Bergman (1970)

L. Bartholdi - A. Smoktunowicz (2016):

$\gamma(n)$ increasing + submultiplicative

$\Rightarrow \exists$ growth function $g(n)$

$$\gamma(n) \leq g(n) \leq n^2 \gamma(n).$$

There are "more" algebras of intermediate growth than groups.

Martha Smith (1976): L Lie algebra of subexponential growth $\Leftrightarrow$ $U(L)$ has subexponential growth.

Example. $L = \mathfrak{g}[t]$, $\dim_F \mathfrak{g} < \infty$

Growth (L) = linear

$U(L)$ has intermediate growth.

Description of growth functions of algebras, monoids, hereditary languages.

Let $F(n)$ be an increasing function,

$f(n) = F(n) - F(n-1)$ derivative.

Theorem 1. (J. Bell - E. Z., 2021)

Suppose that

(1) $n+1 \leq f(n)$,

(2) $f(m+n) \leq f(m)f(n)$,

(3) $f(n+i) \leq f(n)^2$; $i = 0, 1, 2, \ldots, n$.

Then $F(n) \sim$ growth function of a hereditary language, hence of a monoid, hence of an algebra.

Any nonlinear growth function of an algebra satisfies (1), (2), (3).

Be'eri Greenfeld: an increasing ~~function~~ submultiplicative function $F(n)$ such that its derivative $F'(n)$ $\sim$ a submultiplicative function $\Rightarrow$ $F(n) \sim$ growth function.

Remarks.

1) the conditions (1), (2), (3) are not asymptotically invariant.

2) many words in these languages are not extendable to infinite words. What if we count only words that are extendable? Then it would imply $F''(n) \geq 0$.

All subsequent unassigned results are joint with <u>Be'eri Greenfeld</u>.

<u>Growth of Modules.</u>

Let A be a finitely generated algebra, $A = \langle V \rangle$, $\dim V < \infty$; let M be a finitely generated A-module, $W \subset M$, $\dim W < \infty$, $M = AW$.

$$g_{V,W}(n) = \dim\left(\sum_{i=0}^{n} V^i W\right)$$

is the growth function of the module M.

Theorem 2. Every increasing function $f(n)$ such that $f(n) \leq e^n$, is asymptotically equivalent to a growth function of a module over an associative algebra.

The following theorem answers a question from [Krause - Lenagan].

Theorem 3. There exist finitely generated modules M_1, M_2 and a short exact sequence $(0) \to M_1 \to M \to M_2 \to (0)$ such that $GKdim(M_1), GKdim(M_2) < \infty$, but $GKdim(M) = \infty$.

Nil Algebras.

An associative algebra A is nil if $\forall a \in A \ \exists\, n(a) \geq 1 : a^{n(a)} = 0$.

The Kurosh Problem (1941):

A is finitely generated & nil $\overset{?}{\Rightarrow}$ $\dim_F A < \infty$.

An analog of The Burnside Problem.

Counterexamples (E. Golod, 1964)

Question (Latyshev, Small)

$\exists$? counterexamples of polynomially bounded growth, i.e. of finite GKdim?

In groups NO (Gromov's Theorem).

T. Lenagan, A. Smoktunowicz (2007):

Yes, they do exist over countable fields.

T. Lenagan, A. Smoktunowicz, A. Young (2012): $2 \leq \mathrm{GKdim}(A) \leq 3$.

J. Bell, A. Young (2011): $\forall$ field, $\forall$ superpolynomial function $\omega(n)$ there exists a finitely generated nil algebra: $g_A \leq \omega(n)$.

What are growth functions of nil algebras?

Conjecture. All functions of Theorem 1, $g(n) \geq n^2$, can be realized on nil algebras.

Theorem 4.

(1) Countable fields. Let $p(n)$ be the growth function of the Lenagan-Smoktunowicz-Young algebra, $n^2 \leq p(n) \leq n^3$. Let $f(n)$ be an arbitrary increasing submultiplicative function. Then $p(n)^2 f(n) \sim$ growth of a finitely generated nil algebra.

(2) Arbitrary fields. Let $\delta(n)$ be an arbitrary increasing function, $\delta(n) \to \infty$.

Let $\omega(n)$ be an arbitrary increasing superpolynomial function.

Let $f(n)$ be an arbitrary increasing submultiplicative function.

Then there exists a finitely generated nil algebra A:

$$f\left(\frac{n}{\delta(n)}\right) \leq g_A(n) \leq \omega(n) f(n).$$

Oscillating Growth.

M. Kassabov, I. Pak (2013): groups with growth functions oscillating between a given intermediate function (like $e^{n^{4/5}}$) and an arbitarity rapid subexponential function.

V. Petrogradsky (2021): analogs for nil restricted Lie algebras of positive characteristics, oscillating between functions very close to linear and functions very close to exponentials.

We constucted associative nil algebras (of arbitrary characteristics!) with oscillating growth functions.

Theorem 5

(1) Countable fields. Let F be a countable field. Let $f(n)$ be a subexponential function. There exists a finitely generated nil F-algebra A such that:

- $g_A(n) \leq n^{6+\varepsilon}$ infinitely often $\forall \varepsilon > 0$,
- $g_A(n) \geq f(n)$ infinitely often.

(2) Arbitrary fields. Let $f(n)$ be a subexponential function. Let $\omega(n)$ be an increasing super-polynomial function. There exists a finitely generated nil F-algebra A such that:

- $g_A(n) \leq \omega(n)$ infinitely often,
- $g_A(n) \geq f(n)$ infinitely often.

Domains.

<u>Theorem 6.</u> Let $f(n)$ be a subexponential function. There exists a finitely generated domain A such that:

- $g_A(n) \leq e^{n^{3/4+\varepsilon}}$ infinitely often,

$\forall \varepsilon > 0$,

- $g_A(n) \geq f(n)$ infinitely often.

Tensor Products.

$$g_{A\otimes B}(n) \leq g_A(n)\, g_B(n)$$

Therefore:

$$\mathrm{GKdim}(A \otimes_F B) \leq \mathrm{GKdim}(A) + \mathrm{GKdim}(B).$$

Warfield (1984): LHS may be smaller than RHS.

Question (Warfield): let $\mathrm{GKdim}(A) = \alpha$, $\mathrm{GKdim}(B) = \beta$; $2 \leq \alpha \leq \beta$. Is it true that any value in $[2+\beta, \alpha+\beta]$ is attainable for $\mathrm{GKdim}(A \otimes B)$?

Krempa-Okninski (1987): Yes.

But the counterexamples A, B had large radicals.

Question (Krempa-Okninski (1987), Krauze-Lenagan (2000)): what about semiprime A, B?

Theorem 7. There exist semiprime examples.