Geometry over algebras

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We discuss how geometric structures arise on projective spaces from Hermitian forms on linear spaces over real algebras.

Real/complex/quaternionic hyperbolic and Fubini-Study structures are naturally constructed by projectivizing Hermitian linear spaces over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. A notable feature of this approach is that objects such as metrics, geodesics, symplectic forms, bisectors, Levi-Civita connections, curvature tensors, and so on, can be expressed in linear algebraic terms. This framework is particularly advantageous for computational applications.

We extend this discussion to include the **dual numbers** $\mathbb{D} = \mathbb{R} + \varepsilon\mathbb{R}$, with $\varepsilon^2 = 0$, the **split-complex numbers** $\mathbb{SC} = \mathbb{R} + j\mathbb{R}$, with $j^2 = 1$, and the **split-quaternions** $\mathbb{SH} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$, with $i^2 = -1$, $j^2 = 1$, $k^2 = 1$, and $ij = k = -ji$.

We present the projective geometry over these algebras alongside their natural pseudo-Riemannian structures. Geodesics and curvature tensors are described in a linear algebraic framework.

Additionally, we explore the transition of geometries between projective lines over $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{SC}$. These projective lines are naturally interpreted as configuration spaces for oriented geodesics on the 2-sphere, the Euclidean plane, and the hyperbolic plane.