International Conference on Analysis and Geometry

August 30 – September 3, 1999, Novosibirsk, Russia

Abstracts

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Approximation of Jacobians in Sobolev Spaces

My talk is based on a recent joint paper with Jan Malý [1]. The main motivation for our research was the following elementary problem.

Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leq p < \infty$. Here $W^{1,p}$ stands for the usual Sobolev space of function with the L^p integrable gradient. Assume that in addition the Jacobi determinant $\det Du \in L^q(\Omega)$ for some $q \geq 1$. Now we can ask:

Does there exists a sequence of "more regular" mappings $u_k : \Omega \to \mathbb{R}^n$ such that

$$u_k \to u \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \qquad \textit{and} \qquad \det Du_k \to \det Du \text{ in } L^q(\Omega) \text{ ?}$$

It was not clarified here what we shall mean by "more regular" mappings. Observe that if $q \leq p/n$, then we can get desired approximation using the approximation by convolution and the resulting sequence u_k will be then C^∞ smooth, so this is not very interesting case and in what follows we assume $q > p/n$.

In general, in the case $q > p/n$, it is not possible to find an approximating sequence consisting of Lipschitz mappings. Indeed, the radial projection mapping $u_0 : B^n(0,1) \to S^{n-1}(0,1)$, $u_0(x) = x/|x|$ satisfies $u_0 \in W^{1,p}$ for any $p < n$ and $\det Du_0 = 0 \in L^q$ for any $q \geq 1$. However if $n-1 \leq p < n$ and $1 \leq q < \infty$, u_0 cannot be approximated by Lipschitz mappings in the above sense.

This sudden lack of the approximation is caused by the high nonlinearity of the determinant — trying to approximate u by convolution or any other related method we immediately loose information about the integrability of the Jacobian above the exponent p/n.

The obstacle to the existence of the approximating sequence is of a topological nature. The mapping u_0 when restricted to spheres centred at 0 has degree 1, while continuous mappings (from the approximating sequence) must have degree 0 on all such spheres. No such obstacle occurs in the case $p = n$, and indeed in this case we can prove an approximation result in the positive direction.

Theorem. *Let* $u \in W^{1,n}(\Omega, \mathbb{R}^n)$, *where* $\Omega \subset \mathbb{R}^n$ *is a bounded domain. Assume that in addition* $\det Du \in L^q(\Omega)$ *for some* $1 < q < \infty$. *Then there is a sequence of continuous and differentiable a.e. mappings* $u_k \in W^{1,n}(\Omega, \mathbb{R}^n)$ *with the Lusin property and such that*

$$u_k \to u \quad \text{in} \quad W^{1,n}(\Omega, \mathbb{R}^n) \qquad \textit{and} \qquad \det Du_k \to \det Du \quad \text{in} \quad L^q(\Omega)\,.$$

The Lusin property means that u maps sets of the Lebesgue measure zero onto sets of the Lebesgue measure zero. This is a very important property as it is crucial for the validity of the change of variables formula.

In the case $n - 1 < p < n$ we also have some results in the positive direction. Roughly speaking, in this case we can approximate u by mappings which are differentiable a.e. and which are continuous outside a set of the Hausdorff dimension $n - p$. As the example of $x/|x|$ shows, in this case it is not possible to avoid discontinuity.

We develop a new method of approximation. Actually our method is very general and it applies to approximation of much more general nonlinear expressions involving derivative than just a Jacobian. Our method can be used to find "more regular" mappings u_k such that $u_k \to u$ in $W^{1,p}(\Omega, \mathbb{R}^n)$, $n - 1 < p \leq n$ and $f(x, u_k, Du_k) \to f(x, u, Du)$ in $L^q(\Omega)$ for a large class of nonlinear function f including polyconvex functions. The method gives also approximation results for mappings in the John Ball class $A_{p,q}$ [2].

References

[1] P. Hajłasz, J. Malý. *Approximation of nonlinear expressions involving gradient in Sobolev spaces.* Preprint, 1999.

[2] V. Šverak. Regularity properties of deformations with finite energy. *Arch. Rat. Mech. Anal.*, **100**:105–127, 1988.