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Abstracts

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Metric Space: Classification of Finite Subspaces Instead of Constraints on the Metric

The result obtained by A. D. Alexandrov [1] that internal geometry of two-dimensional convex surface can be described in terms of only metric, is generalized on the case of arbitrary metric space. Let in the metric space $\{\rho, \Omega\}$, where Ω is a set of points and ρ is the metric given on $\Omega\times\Omega$, one introduce so called world function $\sigma = \frac{1}{2}\rho^2$ and equip the metric space by a series of mappings

$$F_n : \qquad \Omega^{n+1} \to \mathbf{R}, \qquad \Omega^{n+1} \equiv \bigotimes_{i=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots, \tag{1}$$

$$F_n\left(\mathcal{P}^n\right) = \det \left|\left| \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right)\right|\right|, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, \ldots, n$$

$$\begin{aligned} \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right) &\equiv \Gamma\left(P_0, P_i, P_k\right) \\ &\equiv \sigma\left(P_0, P_i\right) + \sigma\left(P_0, P_k\right) - \sigma\left(P_i, P_k\right), \quad i, k = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

Here $\mathcal{P}^n \equiv \{P_0, P_1, \ldots, P_n\} \in \Omega^{n+1}, \quad P_i \in \Omega, \quad i = 0, 1, \ldots, n$. Mappings F_n classify sets $\mathcal{P}^n$ of $n+1$ points, assigning a real number to each set $\mathcal{P}^n$. Using this classification, one can construct analogs of all those geometrical objects which can be constructed in the Euclidean geometry. The nth order natural geometric object (tube) $\mathcal{T}(\mathcal{P}^n) \subset \Omega$ is defined as the set of points, determined by geometry and $n+1$ points $\mathcal{P}^n$ $(F_n(\mathcal{P}^n) \neq 0)$. It may be described as a set of points $R \subset \Omega$ [2]

$$\mathcal{T}(\mathcal{P}^n) = \{R \mid F_{n+1}(\mathcal{P}^n, R) = 0\},$$

$$\{\mathcal{P}^n, R\} \equiv \{P_0, P_1, \ldots, P_n, R\} \in \Omega^{n+2}, \qquad F_n(\mathcal{P}^n) \neq 0.$$

Classification (1) admits one to introduce concept of nth order multivector $\overrightarrow{\mathcal{P}^n}$, defining it as an ordered set $\{\overrightarrow{P_0, P_1, \ldots, P_n}\}$ of $n+1$ points $\mathcal{P}^n$. On can introduce scalar product of two multivectors of the same order, length of multivector, parallelness of two multivectors of the same order.

In the Euclidean space, where the metric ρ is the Euclidean distance, the quantity $V_n(\mathcal{P}^n) = (n!)^{-1}\sqrt{F_n(\mathcal{P}^n)}$ is the Euclidean volume of $(n+1)$-edr, expressed via metric (distances between its vertices $P_0, P_1, \ldots, P_n$). In this case the set $\mathcal{T}(\mathcal{P}^n) \subset \Omega$, $(F_n(\mathcal{P}^n) \neq 0)$ is n-dimensional plane, passing through $n+1$ points $\mathcal{P}^n$. The metric space $\{\rho, \Omega\}$, equipped by the mappings (1) becomes as pithy as the Euclidean space. For instance, if $\{\rho, \Omega\}$ is the n-dimensional Euclidean space, $F_k(\Omega^{k+1}) = 0$, for $k = n+1, n+2, \ldots$ and $\exists \mathcal{P}^n$, $F_n(\mathcal{P}^n) \neq 0$. The points $\mathcal{P}^n$ form a basis with n vectors $\mathbf{P}_0\mathbf{P}_i$, defined as an ordered set of two points $\{P_0, P_i\}$, $(i = 1, 2, \ldots, n)$. In this basis one can introduce coordinate system and metric tensor $g_{ik}(\mathcal{P}^n)$ for the metric space $\{\rho, \Omega\}$. One sets

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad g_{ik}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k),$$
$$P \in \Omega, \qquad P_i, P_k \in \mathcal{P}^n, \qquad i, k = 1, 2, \ldots, n,$$

where $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$, defined by the relation (2), coincides with the scalar product in the Euclidean space. $x_i(P)$ are covariant coordinates of the point P in the basis $\mathcal{P}^n$. Then metric $\rho(P, Q)$ can be expressed via coordinates of points P and Q and metric tensor g_{ik}. Constructing geometrical objects and coordinate system, one does not use restrictions on metric, imposed by the triangle axiom and the condition $\rho^2 \geq 0$. These constraints may be removed. Then instead of metric space $\{\rho, \Omega\}$, one obtains σ-space $\{\sigma, \Omega\}$, where $\sigma : \quad \Omega \times \Omega \to \mathbf{R}$ is restricted only by constraints $\sigma(P, P) = 0, \quad \sigma(P, Q) = \sigma(Q, P) \quad \forall P, Q \in \Omega$. σ-space $\{\sigma, \Omega\}$ together with the classification (1) form foundation of T-geometry which is as pithy as the Euclidean geometry and can be used for descriptions of spaces with indefinite metrics, for instance, for description of space-time. Concept of continuity is defined in T-geometry via and only via the world function σ. T-geometry is insensitive to continuity of the set Ω and may be used for both continuous and discrete sets Ω.

References

[1] . . . . , 1948.

[2] Yu. A. Rylov. Extremal property of Synge's world function and discrete geometry. *J. Math. Phys.*, **31**(12):2876–90, 1990.