Integrable Hamiltonian systems.

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The Euler–Lagrange equations

Consider a manifold M, $dim\ M = n$ with coordinates $x = (x^1, \dots, x^n)$ as a configurational space of a certain mechanical system.

The Lagrangian $L(x, \dot{x}, t)$ can be defined as follows $L = T - V$.

The principle of least action:

$$S = \int_a^b L(x, \dot{x}, t) dt \to min.$$

The Euler–Lagrange equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0.$$

The Legendre transformation and Hamiltonian systems

A Legendre transform is used in classical mechanics to derive the Hamiltonian formulation from the Lagrangian formulation, and conversely.

Introduce **the momenta** $p = \frac{\partial L}{\partial \dot{x}}$ and define **the Hamiltonian**

$$H(x,p,t) = \dot{x}p - L(x,\dot{x},t).$$

Then the Euler–Lagrange equations go to the Hamiltonian equations:

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x}.$$

The first integral of this Hamiltonian system is any function $F(x,p)$ such that

$$\dot{F} = \{F, H\} = \left(\frac{\partial F}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial x}\right) \equiv 0.$$

Liouville integrability

Theorem (Liouville–Arnold)
Let (M, ω, H) be a symplectic manifold ($dim M = 2m$), and let the Hamiltonian system with the Hamiltonian H admit m first integrals $F_1, ..., F_m$ which are pairwise in involution: $\{F_j, F_k\} = 0$. Consider the common level surface of these integrals

$$M_f = \{z : F_j(z) = f_j = const, \quad j = 1, ..., n\}.$$

Let all the functions F_j be independent on M_f. Then
• *M_f is a smooth invariant manifold.*
• *Each compact component M_f is diffeomorphic to m-dimensional torus $\mathbb{T}^m$.*
• *In certain coordinates $(\varphi_1, ..., \varphi_m)$ $mod\ 2\pi$ on $\mathbb{T}^m$ the Hamiltonian equations have the form $\dot{\varphi} = v$ where $v = v(f) \in \mathbb{R}^m$ is a constant vector.*

Quasi-linear systems of PDEs

Quasi-linear systems of the form

$$A(U)U_x + B(U)U_y = 0,$$

$$U_t = A(U)U_x, \qquad U = (u_1, \dots, u_n)^T$$

appears in such areas like

- gas-dynamics
- non-linear elasticity
- integrable geodesic flows on 2-surfaces

and many others.

Certain methods

1) **The Cauchy–Kovalevskaya theorem** is the main local existence and uniqueness theorem for analytic PDEs associated with Cauchy initial value problems.

2) **The classical hodograph method** - interchange the independent and dependent variables. Given a quasi-linear system of PDEs, this method yields a linear one.

3) **The generalized hodograph method (S.P. Tsarev, 1985)** is applicable for semi-Hamiltonian systems of PDEs and allows to construct implicit solutions.

4) **The method of separation of variables.**

5) Other methods...

N-body problem

In physics, **the N-body problem** is the problem of predicting the individual motions of a group of celestial objects interacting with each other gravitationally.

- $N = 2$ — complete integrability;
- $N > 2$ — non-integrable in general case.

Certain Hamiltonian systems

We will consider few Hamiltonian systems which correspond to

- geodesic flows,
- natural mechanical systems,
- magnetic geodesic flows.

Integrable geodesic flows on a 2-surface

Let

$$ds^2 = g_{ij}(x)dx^i dx^j, \qquad i, j = 1, 2$$

be a Riemannian metric on a 2-surface $\mathbb{M}^2$. The geodesic flow is called *integrable* if the Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad H = \frac{1}{2} g^{ij} p_i p_j$$

admits an additional first integral $F : T^*\mathbb{M}^2 \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right) \equiv 0$$

and F is functionally independent of H almost everywhere.

Polynomial integrals of geodesic flows

Choose the conformal coordinates (x, y), such that

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2), \qquad H = \frac{p_1^2 + p_2^2}{2\Lambda}.$$

Theorem (V.V. Kozlov, 1995)
For any $n \geq 1, n \in \mathbb{N}$ there exists an analytic function $\Lambda(x, y)$ such that the corresponding geodesic flow admits an irreducible polynomial integral of degree n with analytic (in a small neighborhood of a point $x = y = 0$) coefficients.

Sketch of the proof

Let

$$F = a_n(x,y)p_1^n + a_{n-1}(x,y)p_1^{n-1}p_2 + \ldots + a_1(x,y)p_1p_2^{n-1} + a_0(x,y)p_2^n$$

be the first integral of this geodesic flow. The following relations hold:

$$\frac{\partial a_n}{\partial x}\Lambda + \frac{n}{2}a_n\frac{\partial \Lambda}{\partial x} + \frac{a_{n-1}}{2}\frac{\partial \Lambda}{\partial y} = 0,$$

$$\frac{\partial a_n}{\partial y}\Lambda + \frac{\partial a_{n-1}}{\partial x}\Lambda + \frac{n-1}{2}a_{n-1}\frac{\partial \Lambda}{\partial x} + a_{n-2}\frac{\partial \Lambda}{\partial y} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{a_1}{2}\frac{\partial \Lambda}{\partial x} + \frac{\partial a_0}{\partial y}\Lambda + \frac{n}{2}a_0\frac{\partial \Lambda}{\partial y} = 0.$$

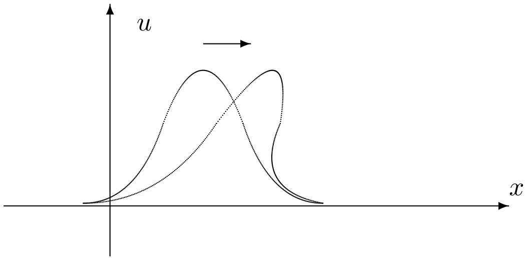
Let $a_1 \neq 0$. Then this system can be solved with respect to $\frac{\partial \Lambda}{\partial x}, \frac{\partial a_0}{\partial x}, \ldots, \frac{\partial a_n}{\partial x}$. So we can consider a Cauchy problem on the line $x = 0$ and apply the Cauchy–Kovalevskaya theorem to prove the existence and uniqueness of an analytic solution.

Hopf equation (inviscid Burgers' equation)

Consider the following equation $u_t + uu_x = 0$. The solution to the Cauchy problem $u|_{t=0} = g(x)$ is given by the implicit formula

$$u(x,t) = g(x - ut).$$

It follows from this formula that the higher any point is placed, the faster it is.



Topological obstacles to the complete integrability

Theorem (V.V. Kozlov, 1979)
If a genus of a surface $\mathbb{M}^2$ *is different from 0 or 1 (that is* $\mathbb{M}^2$ *is homeomorphic neither to a sphere* $\mathbb{S}^2$ *nor to a torus* $\mathbb{T}^2$*), then the geodesic flow of any analytical Riemannian metric on this surface has no first integral which is analytical on* $T^*\mathbb{M}^2$ *and independent of the Hamiltonian.*

Polynomial integrals of low degrees

G.D. Birkhoff (1927): Local linear and quadratic integrals (on a fixed energy level).

$$ds^2 = \Lambda(x)(dx^2 + dy^2), \qquad F_1 = p_2,$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

V.N. Kolokoltsov (1982): Linear and quadratic integrals on the 2-sphere and the 2-torus.

The 2-sphere
There exist integrable cases with polynomial first integrals of degrees **1, 2, 3, 4**.

Conjecture on possible degrees (V.V. Kozlov, 1993).
The maximal possible degree of any *irreducible* polynomial in momenta first integral of the geodesic flow on a surface of a genus g seems to be not larger than $4 - 2g$.

Natural mechanical systems and the Maupertuis principle

Let $\mathbb{M}^n$ be a smooth manifold with the Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Consider a Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad H = \frac{1}{2}g^{ij}(x)p_i p_j + V(x), \quad i, j = 1, \ldots, n,$$

where $V(x)$ is a smooth potential. Define

$$Q^{2n-1} = \{H(x,p) = h, h > maxV(x)\}.$$

Construct a new Hamiltonian

$$\widetilde{H} = \frac{1}{2}\frac{g^{ij}(x)p_i p_j}{h - V(x)}$$

such that $\widetilde{H} = 1$ on Q^{2n-1}. $\widetilde{H}$ corresponds to the new metric

$$\widetilde{g_{ij}} = (h - V(x))g_{ij}.$$

Natural mechanical systems and the Maupertuis principle

So we have

$$Q^{2n-1} = \{H(x,p) = h\} = \{\widetilde{H}(x,p) = 1\}.$$

It follows from here that trajectories of these two Hamiltonian systems coincide (up to a parametrization).

Suppose that the initial natural mechanical system (with H as a Hamiltonian) admits a first integral $f(x,p)$ on a fixed energy level Q^{2n-1}. Then the geodesic flow (with $\widetilde{H}$ as a Hamiltonian) admits a first integral $\widetilde{f}(x,p) = f(x, \frac{p}{|p|})$ on the whole $T^*\mathbb{M}^n$ (except maybe a zero energy level) and $f\,|_{Q^{2n-1}} = \widetilde{f}\,|_{Q^{2n-1}}$.

Natural mechanical systems on the 2-torus

Consider a Hamiltonian system with the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + V(x_1, x_2),$$

where V is assumed to be periodic function on the plane $\mathbb{R}^2$ with a period lattice $\Lambda \subset \mathbb{R}^2$.

1) If

$$V(x_1, x_2) = V(\alpha x_1 + \beta x_2),$$

where $\alpha, \beta \in \mathbb{R}$, then there exists a polynomial integral $F_1 = \alpha p_2 - \beta p_1$.

2) If

$$V(x_1, x_2) = V_1(\alpha_1 x_1 + \beta_1 x_2) + V_2(\alpha_2 x_1 + \beta_2 x_2),$$

where $\alpha_i, \beta_i \in \mathbb{R}$ are constants compatible with the period lattice Λ, then there exists a polynomial integral
$F_2 = (d_1 + d_2)p_1^2 + 4p_1 p_2 - (d_1 + d_2)p_2^2 + 2(d_1 - d_2)(V_1 - V_2)$, $d_i = \alpha_i/\beta_i$.

Polynomial integrals of natural mechanical systems

- **3 degree**
  M. Bialy
  N.V. Denisova, V.V. Kozlov
- **4 degree**
  N.V. Denisova, V.V. Kozlov, D.V. Treschev
- **5 degree**
  A.E. Mironov
- **Higher degrees**
  Open problem

The cubic first integral on the 2-torus

Choose the conformal coordinates (x, y), such that $ds^2 = \Lambda(x,y)(dx^2 + dy^2)$.

$$H = \frac{p_1^2 + p_2^2}{2\Lambda}, \quad F = a_0(x,y)p_1^3 + a_1(x,y)p_1^2 p_2 + a_2(x,y)p_1 p_2^2 + a_3(x,y)p_2^3.$$

The problem reduces to searching for solutions to the following quasi-linear system of PDEs

$$\begin{pmatrix} 3a_0 & 2\Lambda & 0 \\ 1+a_0 & 0 & 0 \\ a_1 & 0 & \Lambda \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_x + \begin{pmatrix} a_1 & 0 & 0 \\ 3a_1 & 0 & 2\Lambda \\ 1+a_0 & \Lambda & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_y = 0.$$

Integrable geodesic flows on the 2-torus

Theorem (N.V. Denisova, V.V. Kozlov, 1993)
Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral F_n which is independent on the Hamiltonian. Suppose that
1) either F_n is even on p_1, p_2
2) or F_n is even on $p_1(p_2)$ and odd on $p_2(p_1)$,
then there exists an additional polynomial in momenta first integral of degree ≤ 2.

Theorem (N.V. Denisova, V.V. Kozlov, 1994)
Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral F_n which is independent on the Hamiltonian. The metric $\Lambda(x, y)$ is assumed to be a trigonometric polynomial. Then there exists an additional polynomial in momenta first integral of degree ≤ 2.

Integrable geodesic flows on the 2-torus

Theorem (M. Bialy, A.E. Mironov, 2011)
If the Hamiltonian system has an integral F which is a homogeneous polynomial of degree n, then on the covering plane $\mathbb{R}^2$ there exist the global semi-geodesic coordinates (t, x) such that

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$$

and F can be written in the form:

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

Here the last two coefficients can be normalized by the following way:

$$a_{n-1} = g, \; a_n = 1.$$

Integrable geodesic flows on the 2-torus

The condition $\{F, H\} = 0$ *is equivalent to the quasi-linear PDEs*

$$U_t + A(U)U_x = 0,$$

where $U^T = (a_0, \dots, a_{n-1}),\ a_{n-1} = g,$

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

Semi-Hamiltonian systems

Theorem (M. Bialy, A.E. Mironov, 2011)
The previous system is semi-Hamiltonian. Namely, there is a regular change of variables

$$U \mapsto (G_1(U), \dots, G_n(U))$$

such that for some $F_1(U), \dots, F_n(U)$ *the following conservation laws hold:*

$$(G_i(U))_x + (F_i(U))_y = 0, \qquad i = 1, \dots, n.$$

Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \dots, \lambda_n$ *of A(U) are real and pairwise distinct, there exists a change of variables*

$$U \mapsto (r_1(U), \dots, r_n(U))$$

such that the system can be written in Riemannian invariants:

$$(r_i)_x + \lambda_i(r)(r_i)_y = 0, \qquad i = 1, \dots, n.$$

The classical hodograph method (n=2)

Consider a quasi-linear system of PDEs of the form

$$\begin{pmatrix} f \\ g \end{pmatrix}_y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}_x, \qquad a_{ij} = a_{ij}(f,g)$$

on the unknown functions $f(x,y), g(x,y)$. The following relations hold:

$$\frac{\partial f}{\partial x} = \triangle \frac{\partial y}{\partial g},\ \frac{\partial f}{\partial y} = -\triangle \frac{\partial x}{\partial g},\ \frac{\partial g}{\partial x} = -\triangle \frac{\partial y}{\partial f},\ \frac{\partial g}{\partial y} = \triangle \frac{\partial x}{\partial f},$$

where $\triangle = \left(\frac{\partial x}{\partial f}\frac{\partial y}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial y}{\partial f} \right)^{-1}$. We obtain the following system of linear PDEs:

$$-\frac{\partial x}{\partial g} = a_{11}(f,g)\frac{\partial y}{\partial g} - a_{12}(f,g)\frac{\partial y}{\partial f},$$

$$\frac{\partial x}{\partial f} = a_{21}(f,g)\frac{\partial y}{\partial g} - a_{22}(f,g)\frac{\partial y}{\partial f}.$$

The extended hodograph method (n=3)

Consider a quasi-linear system of PDEs of the form

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_y = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}_x , \qquad a_{ij} = a_{ij}(f,g,h)$$

on the unknown functions $f(x,y), g(x,y), h(x,y)$. To apply the hodograph method, we need an additional flow which commutes with the previous one:

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_t = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}_x , \qquad b_{ij} = b_{ij}(f,g,h).$$

Denote $\triangle = (t_f(x_h y_g - x_g y_h) - t_g(x_h y_f - x_f y_h) + t_h(x_g y_f - x_f y_g))^{-1}$. We have

$$\frac{\partial f}{\partial x} = \triangle \left(\frac{\partial y}{\partial h}\frac{\partial t}{\partial g} - \frac{\partial y}{\partial g}\frac{\partial t}{\partial h} \right), \ \frac{\partial f}{\partial y} = -\triangle \left(\frac{\partial x}{\partial h}\frac{\partial t}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial t}{\partial h} \right), \ \frac{\partial f}{\partial t} = \triangle \left(\frac{\partial x}{\partial h}\frac{\partial y}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial y}{\partial h} \right), \ldots$$

Semi-Hamiltonian systems, the generalized hodograph method

S.P. Tsarev (1985).

A quasi-linear system of PDEs written in the diagonal form

$$r^i_t = v_i(r) r^i_x, \qquad i = 1, \ldots, n, \qquad v_i \neq v_j$$

is called *semi-Hamiltonian* if

$$\partial_{r_j} \frac{\partial_{r_i} v_k}{v_i - v_k} = \partial_{r_i} \frac{\partial_{r_j} v_k}{v_j - v_k}, \qquad i \neq j \neq k \neq i.$$

Here r_j are Riemann invariants. Semi-Hamiltonian systems possess infinitely many symmetries, i.e. commuting flows of the form $r^i_\tau = \omega_i(r) r^i_x,\ i = 1, \ldots, n,$ wherein the following relations on v_i, ω_i hold:

$$\frac{\partial_{r_k} v_i}{v_k - v_i} = \frac{\partial_{r_k} \omega_i}{\omega_k - \omega_i}, \qquad i \neq k.$$

Due to the generalized hodograph method, a local solution is given by the following system of algebraic equations:

$$\omega_i(r) = v_i(r) t + x.$$

Polynomial integrals of the geodesic flow on a 2-surface

Theorem (G. Abdikalikova, A.E. Mironov, 2019)
On a 2-surface introduce the coordinates $ds^2 = g^2(t,x)dt^2 + dx^2$. *The Hamiltonian takes the form* $H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$. *The corresponding geodesic flow has a local polynomial in momenta first integral of the fourth degree:*

$$F_4 = \frac{a_0}{g^4}p_1^4 + \frac{a_1}{g^3}p_1^3p_2 + \frac{a_2}{g^2}p_1^2p_2^2 + p_1p_2^3 + p_2^4.$$

Here

$$a_0(t,x) = \frac{3(c_2 + t + 3c_3^2)}{5c_3^2}, \quad a_2(t,x) = -\frac{6(2c_2 + 2t + c_3^2)}{5c_3^2},$$

$$a_1(t,x) = -\frac{3\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

$$g(t,x) = \frac{2\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

where c_1, c_2, c_3 *are arbitrary constants.*

Rational integrals of the geodesic flow on a 2-surface

Maciejewski A.J., Przybylska M. (2004): The following two functions commute

$$H = \frac{p_1^2 + p_2^2}{2} + f(p_1, p_2)(xp_1 - \alpha y p_2), \quad F = p_1^{\alpha} p_2, \quad \alpha \in \mathbb{R}.$$

- If $\alpha \in \mathbb{R}/\mathbb{Q}$, then F is not meromorphic.
- If $\alpha \in \mathbb{Q}$, then F is algebraic.
- If $\alpha \in -\mathbb{N}$, then F is rational.
- If $\alpha \in \mathbb{N}$, then F is polynomial.

Aoki A., Houri T., Tomoda K. (2016): Let $f(p_1, p_2) = p_1 + p_2,\ \alpha = -\frac{s}{r}$ with relatively prime, positive integers r, s. Then

$$H = \left(x + \frac{1}{2}\right) p_1^2 + (x - \alpha y)\, p_1 p_2 + \left(\frac{1}{2} - \alpha y\right) p_2^2, \quad \widetilde{F} = F^r = \frac{p_2^r}{p_1^s}.$$

So we obtain a rational first integral $\widetilde{F}$ of the geodesic flow on a 2-surface (with the exceptional flat case $\alpha = -1$).

Rational integrals of geodesic flows

Choose the conformal coordinates (x, y), such that $H = \frac{p_1^2+p_2^2}{2\Lambda(x,y)}$. Let U be a small neighborhood of a point $x = y = 0$. Denote P_r, Q_s — homogeneous in momenta p_1, p_2 polynomials of degrees r, s accordingly.

Theorem (V.V. Kozlov, 2014)
For any $r \geq 1, s \geq 1, r, s \in \mathbb{N}, r \geq s$ there exists an analytic function $\Lambda : U \to \mathbb{R}$ such that
1. the corresponding geodesic flow admits an irreducible rational in momenta first integral (independent on the Hamiltonian) of the form

$$F = \frac{P_r}{Q_s}$$

with analytic coefficients in U;
2. polynomials P_r, Q_s are irreducible a.e. in U;
3. the Hamiltonian system doesn't possess any rational first integrals (independent on the Hamiltonian) of the form $F = \frac{P'_r}{Q'_s}$, $r' + s' < r + s$.

Rational integrals of the geodesic flow on a 2-surface

Suppose that the geodesic flow of the metric $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ on a 2-surface with coordinates x, y admits a rational in momenta first integral:

$$F = \frac{f_1(x,y)p_1 + g_1(x,y)p_2}{f_2(x,y)p_1 + g_2(x,y)p_2}.$$

Hamiltonian takes the form $H = \frac{p_1^2+p_2^2}{2\Lambda}$. F conserves which is equivalent to

$$2\Lambda(f_2 f_{1x} - f_1 f_{2x}) + \Lambda_y(f_2 g_1 - f_1 g_2) = 0,$$

$$2\Lambda(g_2 f_{1x} - g_1 f_{2x} + f_2(f_{1y} + g_{1x}) - f_1(f_{2y} + g_{2x})) + \Lambda_x(f_1 g_2 - f_2 g_1) = 0,$$
$$2\Lambda(f_2 g_{1y} - f_1 g_{2y} + g_2(f_{1y} + g_{1x}) - g_1(f_{2y} + g_{2x})) - \Lambda_y(f_1 g_2 - f_2 g_1) = 0,$$
$$2\Lambda(g_2 g_{1y} - g_1 g_{2y}) + \Lambda_x(f_1 g_2 - f_2 g_1) = 0.$$

Examples of rational first integrals

A., Shubin (2020)

1. $ds^2 = e^{2x}(J_0^2(y) + J_1^2(y))(dx^2 + dy^2)$,

$$F = x - y\frac{J_0(y)p_1 - J_1(y)p_2}{J_1(y)p_1 + J_0(y)p_2}.$$

2. $ds^2 = e^{4x}(1 + 3\cos^2(y))(dx^2 + dy^2)$,

$$F = e^x \frac{\sin(y)p_1 + 2\cos(y)p_2}{3\sin(y)\cos(y)p_1 + (2 - 3\sin^2(y))p_2}.$$

3. $ds^2 = e^{6x}(4\cos^4(y) + \sin^4(y))(dx^2 + dy^2)$,

$$F = 2e^x \frac{2\sin(2y)p_1 + (1 + 3\cos(2y))p_2}{(\sin(y) + 5\sin(3y))p_1 + (3\cos(y) + 5\cos(3y))p_2}.$$

4. $ds^2 = \left(\frac{x^2}{y^2} + \ln^2(y)\right)(dx^2 + dy^2)$,

$$F = \frac{(x^2 + y^2\ln(y))p_1 - xy(\ln(y) - 1)p_2}{x\,p_1 - y\ln(y)p_2}.$$

Examples of rational first integrals

A. Galajinsky (2021)

These examples were constructed via generalized Killing vectors and tensors. All the examples are derived from finite-dimensional Lie algebras.

1. $ds^2 = \frac{(1+y^2)dx^2+(1+k^2+(x+y)^2)dy^2-2(1+y(x+y))dxdy}{x^2+k^2(1+y^2)}, \quad F = \frac{p_2}{p_1} - \ln p_1.$

2. $ds^2 = \frac{((\alpha y-x)^2+\lambda^2+\alpha^2 k^2)dx^2+((\alpha x-y)^2+k^2+\alpha^2\lambda^2)dy^2-2((\alpha x-y)(\alpha y-x)-\alpha(k^2+\lambda^2))dxdy}{(\alpha^2-1)^2(\lambda^2 y^2+k^2(\lambda^2+x^2))},$

$$F = (p_1+p_2)^{1+\alpha}(p_1-p_2)^{1-\alpha}.$$

3. $ds^2 = \frac{((\alpha y-x)^2+\lambda^2+\alpha^2 k^2)dx^2+((\alpha x+y)^2+k^2+\alpha^2\lambda^2)dy^2-2((\alpha x+y)(\alpha y-x)+\alpha(k^2-\lambda^2))dxdy}{(\alpha^2+1)^2(\lambda^2 y^2+k^2(\lambda^2+x^2))},$

$$F = e^{-2\alpha \arctan\left(\frac{\alpha p_2+p_1}{\alpha p_1+p_2}\right)}(p_1^2+p_2^2).$$

Magnetic geodesic flow (systems with gyroscopic forces)

$$\frac{d}{dt} y^i = \{y^i, H(y)\}_{mg}, \qquad i = 1, \dots, N.$$

In coordinates $(y^1, \dots, y^N) = (x^1, \dots, x^n, p_1, \dots, p_n)$, $N = 2n$ magnetic Poisson bracket is given by

$$\{x^i, p_j\}_{mg} = \delta^i_j, \qquad \{x^i, x^j\}_{mg} = 0, \qquad \{p_i, p_j\}_{mg} = \Omega_{ij}(x),$$

Consider a Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

on the 2-torus in presence of a magnetic field with $H = \frac{1}{2} g^{ij} p_i p_j$ and the Poisson bracket:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial x^i} \right) + \Omega(x^1, x^2) \left(\frac{\partial F}{\partial p_1} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial p_1} \right).$$

The only known examples of integrable geodesic flows on the 2-torus on all energy levels

An integrable geodesic flow

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad F_1 = p_1;$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

An integrable magnetic geodesic flow

$$ds^2 = dx^2 + dy^2, \quad \omega = B dx \wedge dy, \quad B = const \neq 0, \quad F_1 = \cos\left(\frac{p_1}{B} - y\right);$$

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y) dx \wedge dy, \qquad F_1 = p_1 + u(y).$$

Quadratic integrals on several energy levels

Theorem (I.A. Taimanov, 2016)
Consider the magnetic flow of the Riemannian metric $ds^2 = \Lambda(x,y)(dx^2 + dy^2)$ *with the non-zero magnetic form* ω. *Suppose the magnetic flow admits a quadratic in momenta first integral* F_2 *(with analytic periodic coefficients) on 2 different energy levels. Then in some coordinates we have*

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y)dx \wedge dy$$

so there exists another integral F_1 *which is linear in momenta:* $F_1 = p_1 + u(y)$.

Polynomial integrals on several energy levels

Taimanov (2016) — a polynomial integral of degree $N = 2$ at $M = 2$ energy levels.

Recently this result was generalized on the case of an arbitrarily high degree integrals.

Butler, Naqvi (2020) — $N = 3$, $M = 2$.

A., Valyuzhenich (2019) and **A., Valyuzhenich, Shubin (2021)** — in case of arbitrary N and $M = \frac{N+1}{2}$ or $M = \frac{N+2}{2}$ (depending on the parity of N).

The proof is more or less straightforward but requires complicated calculations, certain combinatoric technique and the analysis of solutions to the system of equations on two unknown functions $f(x, y), g(x, y)$ of the following kind:

$$f_x = g_y, \; P(f, g) = 0.$$

Quadratic integrals on a fixed energy level

For a Riemannian metric $ds^2 = \Lambda(x,y)(dx^2+dy^2)$ and quadratic in momenta first integral on the 2-torus on a fixed energy level we obtain the following system

$$A(U)U_x + B(U)U_y = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \Lambda \\ u_0 \\ f \\ g \end{pmatrix}.$$

Magnetic field has the form: $\Omega = \frac{1}{4}(g_x - f_y).$

M. Bialy, A.E. Mironov (2012): This system is proved to be semi-Hamiltonian.

The only known explicit non-trivial solution

Dorizzi B., Grammaticos B., Ramani A. and Winternitz P. (1985):

$$A(U)U_x + B(U)U_y = 0, \qquad U = (\Lambda, u_0, f, g)^T, \qquad \Omega = \frac{1}{4}(g_x - f_y).$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}.$$

Explicit solution:

$$U_0(x,y) = \begin{pmatrix} \Lambda(x,y) \\ u_0(x,y) \\ f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2E - 2h(x,y) \\ -8q(x,y) - 4(E - h(x,y)) \\ -4R'(y) \\ 4S'(x) \end{pmatrix}, \quad \Omega = S''(x) + R''(y),$$

$$h(x,y) = \frac{1}{2}(S')^2 + \frac{1}{2}(R')^2 + SR'' + RS'' + \mu_1 - \mu_2, \quad q(x,y) = \frac{1}{2}(S')^2 + SR'' + \mu_2,$$

here $\mu_1(x,y) = (S')^2 + \frac{1}{2}\beta_2 S^2 - \beta_3 S, \quad \mu_2(x,y) = -(R')^2 - \frac{1}{2}\beta_1 R^2 - \beta_3 R$ and

$$S'' = \alpha S^2 + \beta_1 S + \gamma_1, \quad R'' = -\alpha R^2 + \beta_2 R + \gamma_2.$$

Quadratic integrals on a fixed energy level

Theorem (A., Bialy, Mironov, 2017)
There exist real analytic Riemannian metrics on the 2-torus which are arbitrary close to the Liouville metrics (and different from them) and a non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level $\{H = \frac{1}{2}\}$ *have polynomial in momenta first integral of degree two.*

The key idea is as follows. We managed to deform the Liouville metric on the 2-torus in such a way that it becomes nomore Liouville one, and a non-zero magnetic field also appears.

Quadratic integrals and the generalized hodograph method

A., Potashnikov (2021): Let $U_\tau = A_1(U)U_x + B_1(U)U_y$ be a commuting flow. The generalized hodograph method produces

$$\Lambda = -\frac{1}{2}\left(f^2 + g^2\right) + 8\alpha f - 8\beta g, \qquad u_0 = 4(\epsilon + 2\alpha f + 2\beta g),$$

and $f(x,y), g(x,y)$ satisfy:

$$-f^3 + 26\alpha f^2 - f(192\alpha^2 + 12\beta g + g^2) + 2y + \gamma + 64\alpha\beta g + 6\alpha g^2 = 0,$$

$$g^3 + 26\beta g^2 + g(192\beta^2 - 12\alpha f + f^2) + 2x + \delta - 64\alpha\beta f + 6\beta f^2 = 0,$$

here $\alpha, \beta, \gamma, \delta, \epsilon$ are arbitrary constants.
The magnetic field $\Omega = \frac{1}{4}(g_x - f_y)$.

Theorem (A., Potashnikov (2021))
In a small neighborhood of certain points (x_0, y_0) this system admits smooth solutions $f(x,y)$, $g(x,y)$. Moreover, by an appropriate choice of constants α and β one may obtain the solutions which correspond to a positive conformal factor $\Lambda(x,y)$ of the metric with a nonzero curvature and a nonzero magnetic field.

Quadratic integrals and the generalized hodograph method

There is the following "formal" partial solution: $\alpha = \beta = 0$,

$$\Lambda(x,y) = -\frac{1}{2}\sqrt[3]{(2x+\delta)^2 + (2y+\gamma)^2}, \quad u_0(x,y) = \epsilon,$$

$$f(x,y) = \frac{2y+\gamma}{\sqrt[3]{(2x+\delta)^2 + (2y+\gamma)^2}}, \quad g(x,y) = \frac{-(2x+\delta)}{\sqrt[3]{(2x+\delta)^2 + (2y+\gamma)^2}}.$$

This metric is flat. The magnetic field has the form

$$\Omega(x,y) = -\frac{8}{3\sqrt[3]{(2x+\delta)^2 + (2y+\gamma)^2}}.$$

Explicit solutions

Making the change of variables $(x, y) \to (f(x, y), g(x, y))$, in new coordinates f, g we obtain the magnetic field $\omega(f, g)$ and the metric $g_{ij}(f, g)$ such that

$$\omega = -((f - 2\alpha)(f - 6\alpha) + (g + 2\beta)(g + 6\beta))df \wedge dg;$$

$$g_{11} = -\frac{1}{2}\{9f^4 + 10f^2g^2 + g^4 - 156\alpha f^3 - 76\alpha f g^2 + 964f^2\alpha^2 + 132g^2\alpha^2$$
$$-2496f\alpha^3 + 2304\alpha^4 + 4g(3g^2 + (5f - 24\alpha)(3f - 8\alpha))\beta + 4(9g^2 + (3f - 8\alpha)^2\beta^2)\}\times$$
$$(f^2 - 8\alpha f + g^2 + 8g\beta),$$
$$g_{12} = -4(fg - 3\alpha g + 3f\beta - 8\alpha\beta)(f^2 - 8\alpha f + g^2 + 8g\beta)\times$$
$$((f - 6\alpha)(f - 2\alpha) + (g + 2\beta)(g + 6\beta)),$$
$$g_{22} = -\frac{1}{2}(f^2 - 8\alpha f + g^2 + 8g\beta)\{f^4 - 12f^3\alpha - 4f\alpha(3g + 8\beta)(5g + 24\beta)$$
$$+2f^2(5g^2 + 18\alpha^2 + 38g\beta + 66\beta^2) + (3g + 8\beta)^2(4\alpha^2 + (g + 6\beta)^2)\}.$$

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Thank you for your attention!