Идентификация коэффициентов гиперболических уравнений на основе прямой обработки данных

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Лаборатория прикладных обратных задач

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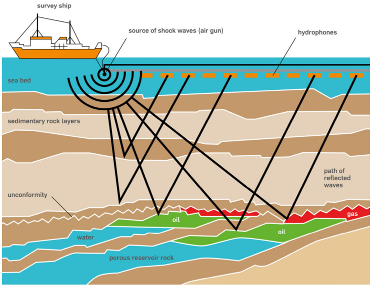
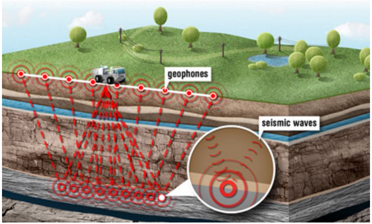
Aim of the approach: Development of numerical methods for solving inverse problem of determining the parameters of the medium in one-dimensional and multi-dimensional case.

Outline:

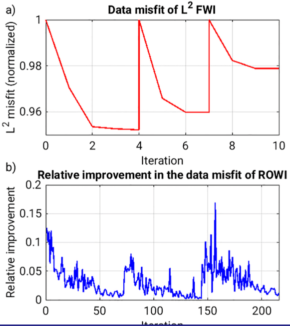
1) Introduction;
2) Formulation of the approach;
3) Numerical method;
4) Results


survey ship
source of shock waves (air gun)
hydrophones
sea bed
sedimentary rock layers
path of
reflected
waves
unconformity
oil
gas
oil
water
porous reservoir rock


geophones
seismic waves

Several suitable methods:
Ray-based seismic tomography;
Inversion of dispersion curves;
Full waveform inversion;
Reverse-time migration;

Important aspects:
- Requirement of the basic model of the medium (*a priori* information);
- Usage of optimization approach (fitting of the parameters)



Gelfand-Levitan-Krein approach

History:
I.M. Gelfand, B.M. Levitan (1951), M.G. Krein (1954) - spectral IP
V.A. Marchenko (1955) - scattering inverse problem
A.S. Alekseev, A.S. Blagoveschenskiy, V.G. Yakhno - seismic IP
V.E. Zakharov, A.B. Schabat (1971) - inverse scattering method for solving KdV equation
M.I. Belishev (1987), S.I. Kabanikhin (1988) – multi-dimensional case
And many more (V.G. Romanov, B. Gopinath, M. Sondi, R. Burridge, W.W. Symes, A.V. Baev etc.)

Features:
1) Nonlinear IP reduces to the family of linear integral equations
2) No need to solve direct problem multiple times;
3) No need of *a priori* information

Other methods: BC - method (M. Belishev),
GCM - method (M. Klibanov).

1D problem for oscillation equation

$$u_{tt}(x,t) = u_{xx} - q(x)u;\ \ x,t > 0$$
$$u|_{t<0} \equiv 0;$$
$$u_x|_{x=0} = H(t).$$

Direct problem: to calculate $u(x,t)$, when $q(x), H(t)$ is known.

$$u_{tt}(x,t) = u_{xx} - q(x)u;\ \ x,t > 0$$
$$u|_{t<0} \equiv 0;$$
$$u_x|_{x=0} = H(t);$$
$$u|_{x=0} = F(t).$$

Inverse problem: to calculate $q(x)$, when $H(t), F(t)$ is known.

1D Gelfand-Levitan method

The coefficient inverse problem for the 1D equation is equivalent to the following system of equations:

$$\int_{-x}^{x} F(t-s)\tilde{W}(x,s)ds + \int_{-x}^{x} H(t-s)\tilde{V}(x,s)ds =$$
$$= -\frac{1}{2}\left[F(t-x)+F(t+x)\right] - \frac{1}{2}\int_{-x}^{x} H(t-s)ds,\ \ t \in (-x,x)$$

Basic idea of the method:
— Studying the properties of direct problem's solution $u(x,t)$;
— Using the connection between $u(x,t)$ and some special auxiliary functions $\tilde{W},\tilde{V}$;
— The value of $\tilde{W},\tilde{V}$ on the characteristic lines $s = x - 0$ is determined by $q(x)$

If $H(t) \to \delta(t)$, then we obtain classic Gelfand—Levitan equation:

$$\tilde{W}(x,t)+\int_{-x}^{x} F'(t-s)\tilde{W}(x,s)ds = -\frac{1}{2}\left[F'(t-x)+F'(t+x)\right],\ t \in (-x,x)$$

For every fixed x this is a linear integral equation of the Fredholm type of the second kind.

Solution of inverse problem can be obtained as follows:

$$q(x) = 4\frac{d}{dx}\tilde{w}(x,x-0), \quad x > 0$$

1D problem for acoustic equation

$$u_{tt} = \sigma(x)\frac{\partial}{\partial x}\left(\frac{u_x}{\sigma(x)}\right) = u_{xx} - \frac{\sigma'(x)}{\sigma(x)}u_x, x > 0, t > 0;$$

$$u|_{t<0} \equiv 0, x > 0;$$

$$u_x|_{x=0} = H(t), t > 0;$$

$$u|_{x=0} = F(t), t > 0.$$

Inverse problem: to calculate $\sigma(x)$, when $H(t), F(t)$ is known.
The inverse problem is reduced to the following system:

$$-\int_0^t \frac{H(\tau)}{\sigma(0)}d\tau = \int_{-x}^{x} V_1(x,s)F'(t-s)ds + \int_{-x}^{x} V_2(x,s)H'(t-s)ds, t \in (-x, x)$$

New analogue of M.G. Krein equation

The coefficient inverse problem for the 1D equation is equivalent to the following system of equations:

$$-2\int_0^{T_G} H(\tau)d\tau - \int_{-x}^{x} [H + F'](t-s)ds =$$
$$= \int_{-x}^{x} \Big([H - F'](t-s) - [H + F'](t+s) \Big) \Phi(x,s)ds, \quad t \in (-x, x).$$

The solution of inverse problem:

$$\sigma(x) = \frac{\sigma(0)}{(1 - \Phi(x, x-0))^2}.$$

1D seismics

System of equations of the theory of elasticity:

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} =$$

$$= (\lambda + \mu)\text{grad div}\mathbf{U} + \mu\Delta\mathbf{U} + \text{grad}\lambda\text{div}\mathbf{U} + \sum_{i=1}^{3} \text{grad}\mu(\frac{\partial \mathbf{U}}{\partial x_i} + \text{grad}U_{x_i})\mathbf{e}_i; \quad (1)$$

Here $x_1 = x, x_2 = y, x_3 = z$, $\mathbf{U} = (U_x, U_y, U_z)^T$ - displacement vector.
λ, μ are Lame parameters, ρ is the density.
We suppose that $\lambda = \lambda(z), \mu = \mu(z), \rho = \rho(z)$.
$v_s^2(z) = \frac{\mu}{\rho}$ - shear wave's velocity, $v_p^2(z) = \frac{\lambda+2\mu}{\rho}$ - plane wave's velocity.

Initial and boundary conditions:

$$\mathbf{U}(x,y,z,t)|_{t<0} \equiv 0; \tag{2}$$

$$\sigma_z|_{z=0} = \lambda_0 \left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right] + 2\mu_0 \frac{\partial U_z}{\partial z} = g_1(x,y,t);$$

$$\tau_{xz}|_{z=0} = \mu_0 \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) = g_2(x,y,t);$$

$$\tau_{yz}|_{z=0} = \mu_0 \left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) = g_3(x,y,t); \tag{3}$$

Data of inverse problem:

$$U_x(x,y,0,t) = f_1(x,y,t);\; U_y(x,y,0,t) = f_2(x,y,t);\; U_z(x,y,0,t) = f_3(x,y,t) \tag{4}$$

The problem is to recover $\lambda(z), \mu(z), \rho(z)$ from (1)-(4) by given f_1, f_2, f_3.

1. Recovering of $v_s(z), \rho(z)$ by using SH-waves :
- Boundary conditions: $\sigma_z = 0, \tau_{rz} = 0, \tau_{\theta z} = \delta(t)b(r)$ - surface rotation moment of intensity $\delta(t)$.

$$\downarrow$$

$$\frac{1}{v^2}u_{tt} = u_{xx} - \frac{\mu'(x)}{\mu(x)}u_x - k^2 u \;\; \rightarrow \;\; u_{tt} = u_{xx} - q(x)u$$

$$\uparrow$$

2. Recovering of $v_p(z)$ by using P-waves :
- Boundary conditions: $\sigma_z|_{z=0} = \delta(x)\delta(y)\delta(t), \tau_{xz}|_{z=0} = 0, \tau_{xz}|_{z=0} = 0$
- instant directioned concentrated force.

Inverse problem for 2D acoustic equation

Let us consider the set of *direct problems*:

$$\frac{1}{c^2(z)}u_{tt} - \Delta u + \nabla \ln \rho(z,y)\,\nabla u = 0, \quad x > 0, \quad y \in \mathbb{R}, \quad t > 0, \qquad (5)$$

$$u\big|_{t<0} \equiv 0, \qquad (6)$$

$$u_x\big|_{x=0} = \delta(t)\delta(y-y_0). \qquad (7)$$

Inverse problem: to recover functions $\rho(z,y), c(z)$ by using the additional data:

$$u(+0,y,t) = f(y,t;y_0), \;\; t > 0, y \in \mathbb{R}, \qquad (8)$$

2D analogue of M.G. Krein equation

Inverse problem (5)-(8) is reduced to the following system of linear integral equations:

$$\Phi^{(y_0)}(x,t) - \frac{1}{2}\int_{-x}^{x}\left[\int_{\mathbb{R}} dr\, f'(r,t-s;y_0)\Phi^{(r)}(x,s)\right] ds = \frac{1}{2}\frac{1}{\rho(0,y_0)}, \tag{9}$$

Here $|t| < x$, $k \in \mathbb{Z}$.

The acoustic impedance can be recovered from the function $\Phi^{(y_0)}(x,t)$:

$$\sigma(x,r) = \rho(x,r)c(x) = \frac{1}{\rho(0,r)}\left[\Phi^{(r)}(x,x-0)\right]^{-2} \tag{10}$$

When $\sigma(x,r)$ is computed, the speed of sound can be found using the Gelfand - Levitan approach:

$$u_{tt} - \Delta u - q(x,y)u \quad x>0, \quad y\in\mathbb{R}, \quad t>0,$$
$$u\big|_{t<0} \equiv 0,$$
$$u_x\big|_{x=0} = \delta(t)\delta(y-y_0).$$
$$u(+0,y,t) = f(y,t;y_0),\ \ t>0, y\in\mathbb{R},$$

Here:

$$q(x,y) = \left[\frac{1}{2}\left(\frac{\sigma_x}{\sigma}\right)_x - \frac{1}{4}\left(\frac{\sigma_x}{\sigma}\right)^2\right] + c^2(x)\left[\frac{1}{2}\left(\frac{\sigma_y}{\sigma}\right)_y - \frac{1}{4}\left(\frac{\sigma_y}{\sigma}\right)^2\right] \quad (11)$$

Projection method of solving IP

Inverse problem (5)-(8) can be approximated by the following system:

$$u^{(k)}_{tt} - \Delta_{x,y} u^{(k)} + \nabla_{x,y} \ln \sigma(x,y)\, \nabla_{x,y} u^{(k)} = 0,$$
$$x > 0, \qquad y \in (-\pi, \pi), \qquad t > 0, \qquad |k| \le N;$$
$$u^{(k)}\big|_{t<0} \equiv 0;$$
$$\frac{\partial u^{(k)}}{\partial x}(+0, y, t) = e^{iky}\, \delta(t);$$
$$u^{(k)}\big|_{y=\pi} = u^{(k)}\big|_{y=-\pi}.$$

Inverse problem: to recover the function $\sigma(x,y)$ by using:

$$u^{(k)}(0,y,t) = f^{(k)}(y,t).$$

Let us suppose, that

$$u^{(k)}(x,y,t) = \sum_{|n|\le N} u^{(k)}_n(x,t) e^{iny}; \quad \ln \rho(x,y) = \sum_{|n|\le N} a_n(x) e^{iny}.$$

Using the finite number $N-$approximation of M.G. Krein equation (9):

$$2\Phi^{(k)}(x,t) - \sum_{|m|\le N} \int_{-x}^{x} f_m^{(k)\prime}(t-s)\,\Phi^{(m)}(x,s)ds = G^{(k)}. \tag{12}$$

Here $|t| < x$, $|k| \le N$.
Matrix form of the equation (12):

$$2\mathbb{E}\vec{\Phi}(x,t) - \int_{-x}^{x} \mathbb{F}'(t-s)\,\vec{\Phi}(x,s)ds = \vec{G}, \qquad |t| < x. \tag{13}$$

Here $\vec{\Phi}(x,t) = \left(\Phi^{(-N)}(x,t),\dots,\Phi^{(0)}(x,t),\dots,\Phi^{(N)}(x,t)\right)^{\top}$ - unknown vector function, $\mathbb{E}$ - identity matrix, $\vec{G} = \left(G^{(-N)},\dots,G^{(0)},\dots,G^{(N)}\right)^{\top}$. $\mathbb{F}(t) = \left[f_m^{(k)}(t)\right]_{k,m=-N,\dots N}$ - matrix form of the data of inverse problem.

Block-Toeplitz matrix inversion

Let us consider Toeplitz matrix

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & a_0 & \cdots & a_{-n+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}$$

Our goal is to compute first and last columns of inverse matrix A^{-1}. Let us denote A^k - the principal submatrix of the order $k+1$. We assume, that we know vectors x_k, y_k, such that :

$$A^k x_k = e_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, A^k y_k = e_n = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}$$

Let us calculate the multiplication:

$$A^{k+1}\begin{bmatrix} x_k \\ 0 \end{bmatrix} = \begin{bmatrix} & A^k & & \begin{matrix} a_{-k} \\ a_{-k+1} \\ \vdots \end{matrix} \\ a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}\begin{bmatrix} x^k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \ldots \\ \epsilon_k^x \end{bmatrix}$$

Here the error term $\epsilon_k^x = a_k x_0^k + \cdots + a_1 x_{k-1}^k$.
In the same manner we obtain:

$$A^{k+1}\begin{bmatrix} 0 \\ y_k \end{bmatrix} = \begin{bmatrix} \epsilon_k^y \\ 0 \\ \ldots \\ 1 \end{bmatrix}$$

Therefore, we can construct x_{k+1}, y_{k+1} using the following structure:

$$x_{k+1} = \alpha \begin{bmatrix} x_k \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ y_k \end{bmatrix}, y_{k+1} = \gamma \begin{bmatrix} x_k \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ y_k \end{bmatrix}$$

Coefficients $\alpha, \beta, \gamma, \delta$ are determined from the following ratios:

$$\begin{bmatrix} 1 & \epsilon_k^y \\ \epsilon_k^x & 1 \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-2\int_0^{T_G} H(\tau)d\tau - \int_{-x}^{x} \left[H + F'\right](t-s)ds =$$
$$= \int_{-x}^{x} \Big(\left[H - F'\right](t-s) - \left[H + F'\right](t+s) \Big) \Phi(x,s)ds, \quad t \in (-x, x).$$

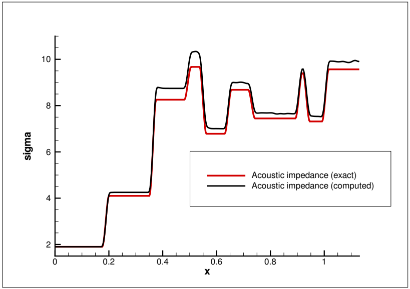
Discretization of the equation leads to ill-conditioned system with Toeplitz-plus-Hankel matrix:
— Permutation of rows and columns allows to consider the block-Toeplitz matrix;
— Special form of Tikhonov regularization deals with ill-conditioning (by doubling the size of the system);
— Additional permutation of rows and columns provides well-conditioned block-Toeplitz matrix.

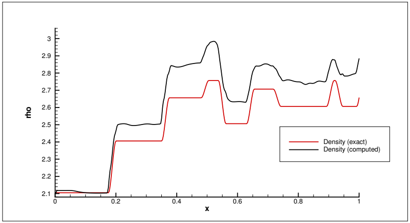
Numerical experiments

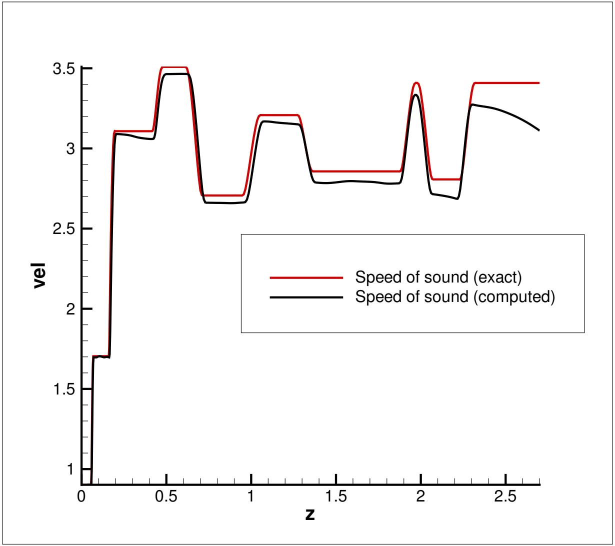
Numerical experiment (restoration of 1d seismic parameters, Yurubchenko-Tokhomskoe gas field, simulated data)

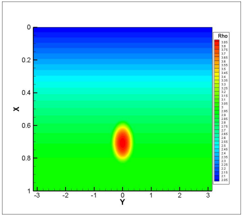
Depth, km	0.17	0.47	0.87	1.070	1.320	1.600	2.100	2.2
$v_s(z)$, km/s	0.9	1.7	3.1	3.5	2.7	3.2	2.85	3
$\rho(z)$, 10^3 kg /m^3	2.1	2.4	2.65	2.75	2.5	2.7	2.6	2.

Таблица: 1D model - parameters of the layers

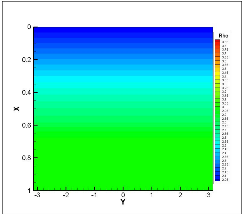

sigma
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1
x
Acoustic impedance (exact)
Acoustic impedance (computed)


rho
3
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
0
0.2
0.4
0.6
0.8
1
x
Density (exact)
Density (computed)

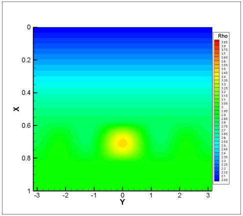

3.5
3
2.5
2
1.5
1
vel
0
0.5
1
1.5
2
2.5
z
Speed of sound (exact)
Speed of sound (computed)



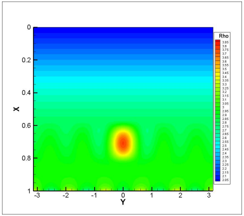
Exact solution



Reconstruction (1 source)

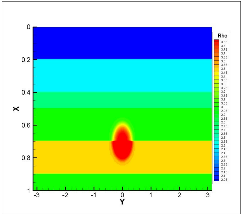


Reconstruction (5 sources)

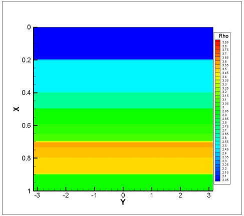


Reconstruction (11 sources)

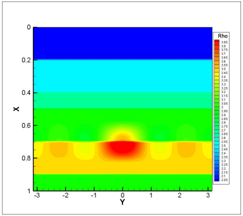
Рис.: Test problem: inclusion and smooth trend



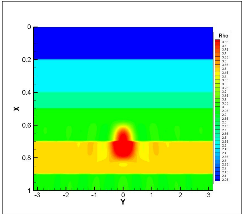
Exact solution



Reconstruction (1 source)

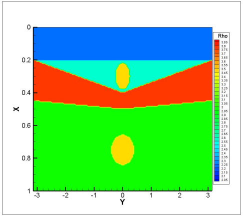


Reconstruction (7 sources)

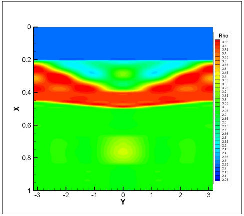


Reconstruction (15 sources)

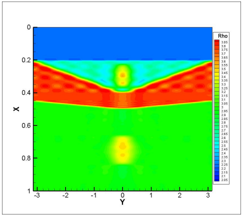
Рис.: Test problem: inclusion and non-smooth trend



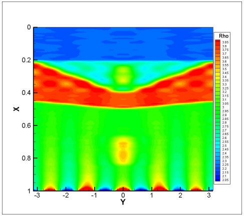
Exact solution



Reconstruction (7 sources)



Reconstruction (15 sources)



Reconstruction (15 sources, noise 5%)

	$N_x = 100$			$N_x = 200$		
	$N_F = 6$	$N_F = 9$	$N_F = 12$	$N_F = 6$	$N_F = 9$	$N_F = 12$
$Error$	0.335	0.337	0.337	0.301	0.300	0.298
$T(sec)$	7.32	16.2	26.9	29.0	59.6	95.3
$T_{(\mathsf{CVM})}$	72.1	156.4	288.1	639.0	-	-

Таблица: Numerical results - Time cost comparison

$$\rho = \begin{cases} \exp\{1 + \frac{0.04}{r^2 - 0.04}\}, & 0 \le r \le 0.2; \\ \exp\{1 + \frac{0.01}{(r-0.35)^2 - 0.01}\}, & 0.25 \le r \le 0.45 \\ 0, else. \end{cases}$$

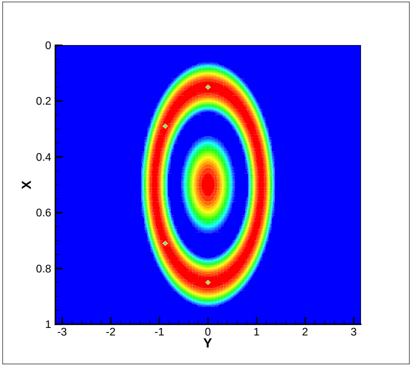


Рис.: Exact solution of the inverse problem.

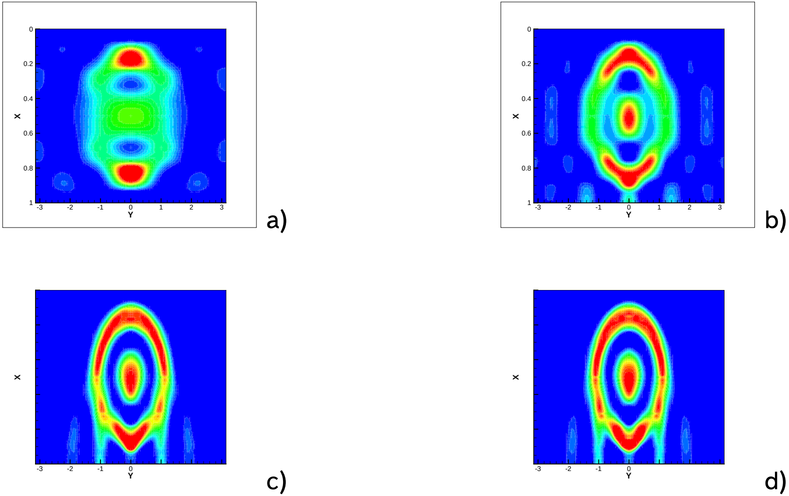


Рис.: Solution of the inverse problem by block-Toeplitz matrix inversion : a) $N_f = 3$, b)$N_f = 5$, c) $N_f = 11$, d) $N_f = 15$.

Conclusion

Advantages of the approach:
- over-determined formulation of the problem allows to reduce nonlinear inverse problem to a family of linear integral equations
- usage of Levinson-Durbin method for solving SLAE with block-Toeplitz matrix allows to increase the efficiency of the method (we restore whole solution of IP when dealing with only one linear system)
- we do not need multiple solution of direct problem

Future work:
- Improvement of the stability of the method
- Implementation of real data
- Development of the approach for more realistic models

Ways to continue the work:
2D wave equation:

$$c^{-2}(x,y)u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)}, \qquad x \in R, \qquad t > 0;$$
$$u^{(k)}|_{t=0} = 0, \qquad u_t^{(k)}|_{t=0} = e^{iky}\,\delta(x).$$
$$u^{(k)}(0,y,t) = f^{(k)}(y,t), \qquad u_x^{(k)}(+0,y,t) = 0.$$

GLK-type ratio:

$$\sum_m S^{(m)}(z,y) f_m^{(k)\prime}(t-z) + \tilde{w}^{(k)}(z,y,t) +$$
$$+ \sum_m \int_{-z}^{z} f_m^{(k)\prime}(t-s)\tilde{w}^{(m)}(z,y,s)ds = 0$$

Ways to continue the work:
Add dissipation:

$$u_{tt}(x,t) - 2\mu(x)u_t = u_{xx} - q(x)u, \qquad x > 0, \quad t > 0;$$
$$u|_{t<0} \equiv 0, \quad x > 0;$$
$$u_x|_{x=0} = \delta(t), \quad t > 0;$$
$$u|_{x=0} = f_+(t), \quad t > 0.$$

GLK-type ratio:

$$-2w(x,t) + \int\limits_{-x}^{\infty} f'(t-s)\tilde{w}(x,s)ds = -\frac{1}{2}\left(E^+(x)f'(t-x) + E^-(x)f'(t+x)\right)$$

Thank you for your time!

References:

- S. Kabanikhin, N. Novikov, M. Shishlenin, *On the analogue of the M.G. Krein equation for the 2D acoustic inverse problem in case of general source function*, Journal of Inverse and Ill-posed Problems, 2022 (to appear)
- S. Kabanikhin, K. Sabelfeld, N. Novikov, M. Shishlenin, *Numerical solution of an inverse problem of coefficient recovering for a wave equation by a stochastic projection methods*, Monte Carlo methods and applications, Volume 21, Issue 3, p. 189–203
- S. Kabanikhin, K. Sabelfeld, N. Novikov, M. Shishlenin, *Numerical solution of the multidimensional Gelfand–Levitan equation*, Journal of Inverse and Ill-posed Problems. Volume 23, Issue 5, p. 439-450
- S. Kabanikhin, N. Novikov, I. Oseledets, M. Shishlenin , *Fast Toeplitz linear system inversion for solving two-dimensional acoustic inverse problem*, Journal of Inverse and Ill-posed Problems. Volume 23, Issue 6, p. 687-700

- A.S. Alekseev, *Inverse dynamical problems of seismology*, Some Methods and Algorithms for Interpretation of Geophysical Data. Nauka, Moscow, 1967, pp. 9-84 (in Russian).
- I.M. Gel'fand and B.M. Levitan, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSSR. Ser. Mat. 15, (1951), pp. 309-360 (in Russian).
- M.G. Krein, *Solution of the inverse Sturm-Liouville problem*, Dokl. Akad. Nauk SSSR 76 (1951), pp. 21-24 (in Russian).
- M.G. Krein, *On a method of effective solution of an inverse boundary problem*, Dokl. Akad. Nauk SSSR 94 (1954), pp. 987-990 (in Russian).
- A.S. Blagoveschenskii.*The local method of solution of the non-stationary inverse problem for an inhomogeneous string*, Proceedings of the Mathematical Steklov Institute 115 (1971), pp. 28-38 (in Russian).

I. Gohberg and A. Semencul, *On the inversion of finite Toeplitz matrices and their continuous analogs*, Mat. Issled. 2, 201–233 (1972).

V.V. Voevodin, E.E. Tyrtyshnikov, *Computational processes with Toeplitz matrices*, Moscow, Nauka, 1987 (in Russian)

K. Sabelfeld, N. Loshchina, *Stochastic iterative projection methods for large linear systems*, Monte Carlo Methods Appl. 16 (2010), 343–359

T. Strohmer and R. Vershynin, *A randomized Kaczmarz algorithm with exponential convergence*, Journal of Fourier Analysis and Applications, vol. 15 (2009), 262–278.

L. Nguyen,L. Beilina, M. Klibanov *Reconstruction of the refractive index from experimental backscattering data using a globally convergent inverse method*, Siam Journal of Scientific Computing, Volume 36, Issue 3, pp. 273-293, 2014

L. Beilina and M.V. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*, Springer, New York, 2012.

M.I. Belishev, *Recent progress in the boundary control method.* Inverse problems, Volume: 23, Issue: 5 Pages: R1-R67, 2007.

M.I. Belishev , *Boundary control in reconstruction of manifolds and metrics (the BC method)*, Inverse Problems, Volume: 13, Issue: 5, pp. R1-R45, O 1997.

V.A. MARCHENKO *Restoration of the potential energy by the phase of the dissipated waves*, Dokl. Akad. Nauk SSSR 104 (1955), pp. 695-698 (in Russian)

S.I. KABANIKHIN AND M.A. SHISHLENIN, *Boundary control and Gel'fand-Levitan-Krein methods in inverse acoustic problem*, J. Inv. Ill-Posed Problems 12 (2), (2004), pp. 125-144.

S.I. KABANIKHIN. *Inverse and ill-posed problems: Theory and applications*, De Gruyter, Berlin–Boston, 2011

S.I. KABANIKHIN AND M.A. SHISHLENIN. *Numerical algorithm for two-dimensional inverse acoustic problem based on Gelfand-Levitan-Krein equation*, Journal of inversed and ill-Posed problems, 18 (2010), 802–814

S.I. KABANIKHIN, *On linear regularization of multi-dimentional inverse problems for hyperbolic equations*, Sov. Math. Dokl. 1990, vol 40, 3, pp. 579-583.

V. G. Romanov and S. I. Kabanikhin *Inverse Problems for Maxwell's Equations* Utrecht, The Netherlands, VSP, 1994, 249 pp.

S.I. Khabanikhin, O. Scherzer, M.A. Shishlenin *Iteration method for a solving a two-dimensional problem for hyperbolic equation*, J. Inv. Ill-posed Problems, 2003, V.11, N 1, p. 1-23

A. Karchevskiy, M. Klibanov, L. Nguyen, N. Pantong, A. Sullivan, *The Krein method and the globally convergent method for experimental data*, Applied Numerical Mathematics 74 (2013), pp. 111–127

N.S. Novikov, M.A. Shishlenin, *Comparative analysis of two numerical methods of solving Gelfand–Levitan–Krein equation*, Siberian Electronics Mathematical Reports, Proceedings of the 2nd International scientific school-conference for young scientists "Theory and numerical methods for solving inverse and ill-posed problems pp. 379-393, 2011

S. Kabanikhin, M. Shishlenin, N. Novikov, *Algorithms of determination of the elastic parameters by using the areal data*