Dynamics of elastic curves

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Preliminaries

Notation

We deal with smooth immersions $f : \mathbb{S}^1 \to \mathbb{R}^d$, $d \geq 2$,

$$f = f(\theta), \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

Here θ is the angle variable, and f is a 2π-periodic function of θ. Denote by $\Gamma \subset \mathbb{R}^d$ the curve

$$\Gamma = f(\mathbb{S}^1) \tag{1}$$

Hereinafter we assume that the point $f(\theta)$ moves around Γ in the positive counterclockwise direction while the parameter θ increases.

Notation

The arc-length variable s on Γ is a function of θ. It is defined by the equality

$$s(\theta) = \int_0^\theta \sqrt{g(\sigma)}\, d\sigma, \quad \sqrt{g} = |\partial_\theta f|. \tag{2}$$

The element of the length of Γ equals

$$ds = \sqrt{g(\theta)}\, d\theta.$$

In this setting, the derivative with respect to the arc-length variable s,

$$\partial_s = \frac{1}{\sqrt{g}} \partial_\theta \tag{3}$$

becomes the nonlinear differential operator depending on f. We assume that

$$0 < c^{-1} \le g \le c < \infty, \quad g(\theta) = |\partial_\theta f(\theta)|^2. \tag{4}$$

Notation

The tangent vector $\boldsymbol{\tau}$ to Γ is defined by the equality

$$\boldsymbol{\tau}(\theta) = \partial_s f(\theta) := |\partial_\theta f|^{-1}\, \partial_\theta f(\theta), \tag{5}$$

The curvature vector $\mathbf{k}$ is defined by the equalities

$$\mathbf{k}(\theta) = \partial_s \boldsymbol{\tau}(\theta) = \partial_s^2 f(\theta).$$

Notice that $\mathbf{k}$ is orthogonal to $\boldsymbol{\tau}$.

Notation

For every smooth vector field $\phi : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^d$, the space and time normal connections ∇_s and ∇_t are defined by the equalities

$$\nabla_s \phi = \partial_s \phi - (\partial_s \phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}, \quad \nabla_t \phi = \partial_t \phi - (\partial_t \phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}, \qquad (6)$$

which can be written in the equivalent form

$$\nabla_s \phi = \Pi \partial_s \phi, \quad \nabla_t \phi = \Pi \partial_t \phi, \quad \Pi \phi = \phi - (\phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}.$$

Energy functionals and their gradients

Energy functionals

The Euler elastica energy $\mathcal{E}_e$ and the length $\mathcal{P}$ of Γ are defined by the formulae

$$\mathcal{E}_e = \frac{1}{2}\int_\Gamma |\mathbf{k}|^2\, ds, \quad \mathcal{P} = \int_\Gamma ds. \tag{7}$$

These quantities can be regarded as integral functionals of f. We take the total energy $\mathcal{E}$ of the curve Γ in the form

$$\mathcal{E} = \mathcal{E}_e + \gamma\mathcal{P} = \int_\Gamma \Big(\frac{1}{2}|\mathbf{k}|^2 + \gamma\Big)\, ds, \quad \gamma = \text{ const. } > 0.$$

Without loss of generality we may assume that $\gamma = 1$ and take the total energy in the form

$$\mathcal{E} = \mathcal{E}_e + \mathcal{P} = \int_\Gamma \Big(\frac{1}{2}|\mathbf{k}|^2 + 1\Big)\, ds. \tag{8}$$

Gradient of a geometric functional.

Let

$$J = J(\Gamma) \text{ or equivalently } J = J(\mathbf{f})$$

be a geometric functional. Let X be an arbitrary periodic vector field. If the derivative of J admits the Hadamard representation

$$\lim_{\sigma \to 0} \frac{1}{\sigma}\big(J(f + \sigma X) - J(f)\big) = \int_\Gamma \Phi n \cdot X \, ds, \quad \Phi \in L^1(\Gamma),$$

then the gradient dJ of J is defined by the equality

$$dJ = \Phi \, n.$$

Gradient of the Euler elastica energy.

$$d\mathcal{E}(f) = \nabla_s \nabla_s k + \frac{1}{2}|k|^2 k - k. \tag{9}$$

Recall

$$\nabla_s \Phi = \partial_s \Phi - (\partial_s \Phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}, \tag{10}$$

or

$$\nabla_s \Phi = \Pi \partial_s \Phi, \quad \Pi \Phi = \Phi - (\Phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau} \tag{11}$$

Gradient flow equation

Gradient flow equation

The gradient flow equation for the energy functional $\mathcal{E}(f)$ reads

$$\partial_t f = -d\mathcal{E}(f), \quad f(0) = f_0.$$

It can be written in the form of the evolutionary nonlinear partial differential equation

$$\partial_t f + \nabla_s^2 \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2 \mathbf{k} - \mathbf{k} = 0, \quad f(0) = f_0, \tag{12}$$

or equivalently in the form of the operator equation

$$\mathbf{\Phi}(f) \equiv \partial_t f + \mathcal{A}(f) = 0, \quad f(0) = f_0. \tag{13}$$

$$\mathcal{A}(f) = \nabla_s^2 \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2 \mathbf{k} - \mathbf{k}.$$

Equation (12) is named straightening equation or $1D$ Willmore flow.

Wen (1995), Koiso (1996), Polden (1996), Dzuik, Kuwert& Shatzle (2002), Lin(2012), Wheeler (2012), Acqua, Pozzi(2014), Abels, Garke& Muller (2016), Menzel (2020), Rupp, Spener (2020), Mantegazza, Pozetta (2021).

Remark

For nonlinear evolutionary problems, the standard proof of the existence theorem consists of three steps.
The first is the proof of the local solvability of problem on the small time intervals.
The second step is the proof of the global a priori estimates for smooth solutions in Sobolev or Hölder classes.
The third step is the application the time continuation method

Graph concept solutions

The alternative approach is based on the concept of graph solutions, Koiso, Abels, Garke Muller, Rupp, Spener, The theory of graph solutions operates with the modified straightening equation

$$\nabla_t f + \nabla_s^2 \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2\mathbf{k} - \mathbf{k} = 0, \quad f(0) = f_0, \tag{14}$$

This equation is equivalent to the operator equation

$$\mathbf{\Psi}(f) \equiv \nabla_t f + \mathcal{A}(f) = 0, \quad f(0) = f_0. \tag{15}$$

Iteration scheme

Functional spaces.

Sobolev spaces of periodic functions. For every integer $r \geq 0$, denote by $H^r_\sharp$, the Sobolev space of all 2π -periodic mappings with the finite norm

$$\|f\|^2_{H^r_\sharp} = \int_0^{2\pi} (|f|^2 + |\partial^r_\theta f|^2)\, d\theta. \tag{16}$$

For real $r \geq 0$, the space $H^r_\sharp$ is defined by the interpolation. Note that the equivalent norm in $H^r_\sharp$ may be defined by the equality

$$\|f\|^2_{H^r_\sharp} = \sum_{m \in \mathbb{Z}} (1 + |m|^2)^r |f_m|^2,$$

where the Fourier coefficients

$$f_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-m\, i}\ f(\theta)\, d\theta.$$

Functional spaces

$$\mathcal{H}^{r,s}(0,T) = \{\, u : S^1 \times (0,T) \to \mathbb{R}^d : \ \|u\|^2_{r,s} < \infty \,\},$$

$$\|u\|^2_{r,s} = \int_0^T (\|\partial_t u(t)\|^2_{H^r_\sharp} + \|u(t)\|^2_{H^s_\sharp}) dt$$

$$\mathcal{X}^s = \mathcal{H}^{s,s+4}(0,T), \quad \mathcal{Y}^s = L^2(0,T; H^s_\sharp)$$

The norm in $\mathcal{X}^s$ is denoted as $\|\cdot\|_s$
The norm in $\mathcal{X}^s$ is denoted as $|\cdot|_s$
Mollifier

$$\mathbf{T}_N u = \frac{1}{\sqrt{2\pi}} \sum_{|m|\le N} f_m e^{m\,i}$$

Let $r \geq 6$, $\|f\|_r \leq R$. Then for any $l \geq 0$,

$$|\mathbf{\Phi}(f)|_{r+l} \leq c(R)(1 + \|f\|_{r+l}).$$

Let

$$\mathbf{\Phi}'(f)\delta f = \lim_{\sigma \to 0} \frac{1}{\sigma} \big(\mathbf{\Phi}(f + \sigma\delta f) - \mathbf{\Phi}(f)\big).$$

Then

$$|\mathbf{\Phi}'(f)\delta f|_{r+l} \leq c(R)\|\delta f\|_{r+l} + c(R)(1 + \|f\|_{r+l})\|\delta f\|_r$$

Nash-Moser scheme

$$\mathbf{\Phi}(f) = 0$$

We know the approximate solution $f = f_0$.

$$f_{n+1} = f_n + T_{N_n}\delta f_n, \quad \mathbf{\Phi}'(f_n)\delta f_n = -\mathbf{\Phi}(f_n),\ N_n = N_{n-1}^{3/2}.$$

Remark. If

$$\mathbf{\Phi}'(f)\delta f = \mathcal{L}(f)\delta f + O(\mathbf{\Phi}(f))\,\delta f,$$

then we can take

$$\mathcal{L}(f_n)\delta f_n = -\mathbf{\Phi}(f_n).$$

Linearized equations

Straightening equation

Recall the formulation of straightening equation

$$\mathbf{\Phi}(f) = \partial_t f + \mathcal{A}(f) = 0, \quad \mathcal{A}(f) = \nabla_s^2 \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2\mathbf{k} - \mathbf{k}$$

Let us calculate

$$\mathbf{\Phi}'(f)\delta f.$$

Variation

Represent the variation δf in the form

$$\delta f(\theta, t) = u(\theta, t) + v(\theta, t)\boldsymbol{\tau}(\theta, t), \quad u(\theta, t) \perp \boldsymbol{\tau}(\theta, t). \tag{17}$$

Here u is an *infinitesimal normal flow*,
$v\boldsymbol{\tau}$ is an *infinitesimal tangent flow*.

The operator $\mathbf{\Phi}'$

Then

$$\mathbf{\Phi}'(f)\,\delta f = \mathbf{\Omega}\,\delta f + (\Theta\,\delta f)\,\boldsymbol{\tau} \tag{18}$$

where the normal part $\mathbf{\Omega}$ is given by

$$\begin{aligned} \mathbf{\Omega}\,\delta f \quad &= \quad \nabla_t u + \nabla_s^4 u + \frac{3}{2}|\mathbf{k}|^2 \nabla_s^2 u - \nabla_s^2 u + \\ & \quad \mathbf{L}_1 u + \mathbf{L}_0 u + v\,\nabla_s \mathbf{\Phi}, \end{aligned} \tag{19}$$

and the tangent part Θ is given by

$$\Theta\,\delta f = \partial_t v - (\mathcal{A}\cdot\nabla_s u) - v(\mathcal{A}\cdot\mathbf{k}) - u\cdot\nabla_s\partial_t f. \tag{20}$$

The operator $\mathbf{\Phi}'$

The operators $\mathbf{L}_i$ are defined by

$$\begin{aligned}
\mathbf{L}_0\, u &= 3(u\cdot\mathbf{k})\,\mathcal{A} + 3(u\cdot\nabla_s\mathbf{k})\nabla_s\mathbf{k} + (u\cdot\nabla_s^2\mathbf{k})\,\nabla_s\mathbf{k} - (u\cdot\mathbf{k})\,\mathbf{k},\\
\mathbf{L}_1\, u &= \frac{3}{2}\partial_s|\mathbf{k}|^2\nabla_s u + 2(\nabla_s u\cdot\mathbf{k})\nabla_s k + (\nabla_s u\cdot\mathbf{k})\,\mathbf{k}.
\end{aligned}$$

Linearized equation

In the iteration scheme the linearized equation

$$\mathcal{L}(f)\delta f = \mathbf{F}.$$

is in the form

$$\begin{aligned} \nabla_t u + \nabla_s^4 u + \frac{3}{2}|\mathbf{k}|^2\nabla_s^2 u - \nabla_s^2 u &\quad + \\ \mathbf{L}_1 u + \mathbf{L}_0 u &\quad = \quad \Pi\mathbf{F}, \end{aligned}$$

$$\partial_t v - v(\mathcal{A}\cdot\mathbf{k}) - (\mathcal{A}\cdot\nabla_s u) - u\cdot\nabla_s\partial_t f = \mathbf{F}\cdot\boldsymbol{\tau}.$$

The operator $\mathbf{\Psi}'$

$$\mathbf{\Psi}'(f)\,\delta f = \boldsymbol{\Upsilon}\,\delta f + (\Lambda\,\delta f)\,\boldsymbol{\tau}$$

where the normal part $\boldsymbol{\Upsilon}$ is given by

$$\begin{aligned} \boldsymbol{\Upsilon}\delta f \quad = \quad & \nabla_t u + \nabla_s^4 u + \Big(\frac{3}{2}|\mathbf{k}|^2 - 1\Big)\nabla_s^2 u + \\ & \mathbf{M} u + v\,\nabla_s \mathbf{\Psi}. \end{aligned}$$

The tangent part Λ is given by

$$\Lambda\,\delta f = -\mathbf{\Psi}\cdot\nabla_s u - v(\mathbf{\Psi}\cdot\mathbf{k}). \tag{21}$$

Here

$$\mathbf{M}\,u = \mathbf{L}_1\,u + \mathbf{L}_0 u - (\partial_t f\cdot\boldsymbol{\tau})\,\nabla_s u,$$

Linearized equation

$$\mathcal{L}(f)\delta f = \mathbf{F}.$$

is in the form

$$\begin{aligned}\nabla_t u + \nabla_s^4 u + \Big(\frac{3}{2}|\mathbf{k}|^2 - 1\Big)\nabla_s^2 u &+ \\ \mathbf{M}u &= \Pi\mathbf{F}.\end{aligned}$$

The solvability condition

$$\mathbf{F}\cdot\tau = 0$$

in the Nash-Moser iteration scheme is satisfied automatically since

$$\mathbf{F}\cdot\boldsymbol{\tau} = -\mathbf{\Psi}(f_n)\cdot\boldsymbol{\tau}_n \equiv 0.$$

In this case we may take

$$\delta f_n = u_n$$

Moving frame

Moving frame

Let

$$\boldsymbol{\tau} : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^d, \quad |\boldsymbol{\tau}| = 1, \quad \tau \in \mathcal{H}^{r,s}(0, T), \quad r, s \geq 1.$$

Assume that

$$\|\boldsymbol{\tau}\|_{1,1} \leq R.$$

Then there is an orthogonal frame

$$(\boldsymbol{\tau}, \mathbf{n}_1, \dots, \mathbf{n}_{d-1}), \quad \boldsymbol{\tau} \cdot \mathbf{n}_i = 0, \quad \mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij},$$

such that $\mathbf{n}_i \in \mathcal{H}^{r,s}(0, T)$ and

$$\|\mathbf{n}_i\|_{r,s} \leq c(R)(1 + \|\boldsymbol{\tau}\|_{r,s}).$$

Moving frame

Set

$$u = \sum_{i=1}^{d-1} \pi_i \, \mathbf{n}_i, \quad \delta f = \sum_{i=1}^{d-1} \pi_i \, \mathbf{n}_i + v\boldsymbol{\tau},$$

$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_{d-1}) : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^{d-1}$$

Linear equation

Again consider equation

$$\mathcal{L}\,\delta f = \mathbf{F}.$$

It is equivalent to the system of differential equations of mixed type

$$\partial_t \boldsymbol{\pi} + a_0\ \partial_\theta^4 \boldsymbol{\pi} + \sum_{k=1}^{3} \mathbf{A}_k\, \partial_\theta^k \boldsymbol{\pi} = \overline{\mathbf{F}}$$

$$\partial_t v - v(\mathcal{A}\cdot\mathbf{k}) - (\mathcal{A}\cdot\nabla_s u) - u\cdot\nabla_s\partial_t f = \mathbf{F}\cdot\boldsymbol{\tau}$$

Here

$$a_0 = |\partial_\theta f|^{-4}, \quad (\overline{\mathbf{F}})_i = \mathbf{F}\cdot\mathbf{n}_i,$$

$(d-1)\times(d-1)$ matrices $\mathbf{A}_i$ are polynomial od $\partial_\theta^\alpha f$, $\partial_\theta^\alpha(|\partial_\theta f|^{-1}$, and $\partial_t\partial_\theta f$ with $1\le\alpha\le 4$.

Solvability of linear equation

Let

$$f \in \mathcal{X}^{r+l}, \quad r \geq 10, \quad l \geq 0, \quad \|f\|_r \leq R$$

Then for every

$$\mathbf{F} \in \mathcal{Y}^{r+l} \quad \text{and} \quad \rho \leq r - 5,$$

the equation

$$\mathcal{L}(f)\delta f = \mathbf{F}, \quad \delta f\Big|_{t=0} = 0,$$

has a solution. This solution admits the estimates

$$\|\delta f\|_{\rho+l} \leq c(R,l)(1 + \|f\|_{r+l})\,|\mathbf{F}|_r + c(R,l)\,|\mathbf{F}|_{r+l}$$

Conclusion

Let

$$f_0 \in H^r_\sharp \quad \text{for some} \quad r \geq 14.$$

Then there is $T > 0$ such that the problem

$$\partial_t f + \mathcal{A}(f) = 0, \;\; t \in (0, T), \quad f(0) = f_0$$

has a solution

$$f \in \mathcal{X}^\rho, \quad 0 \leq \rho \leq 9.$$

Application. Viscoelastic rod equation

$$\nabla_s^4 \nabla_t f + \nabla_s^2 k + \frac{1}{2}|k|^2 k - \gamma k = 0 \ \text{ in } \ \mathbb{S}^1 \times (0, T),$$

or

$$-\nabla_s^2 \nabla_t f + \nabla_s^2 k + \frac{1}{2}|k|^2 k - \gamma k = 0 \ \text{ in } \ \mathbb{S}^1 \times (0, T),$$

Shape optimization

A cost function $J(f)$
The steepest descent method

$$f_{n+1} = f_n - \delta dJ(f)$$

leads to the gradient flow

$$\partial_t f = -dJ(f)$$

Regularization

$$\partial_t + \mathcal{A}(f) + dJ(f) = 0.$$

Sokolowski & P., Geometric aspects of shape optimization, J. Geom. Anal. (2023)

Problems

Equation

$$\boldsymbol{\Psi}(f) \equiv \nabla_t f + \mathcal{A}(f) = 0.$$

Iteration scheme

$$f_{n+1} = f_n + \mathbf{T}_{N_n}\delta f_n,$$

$$\mathcal{L}(f_n)\delta f_n = -\boldsymbol{\Psi}(f_n)$$

$$\delta f_n = u_n, \quad u_n \perp \boldsymbol{\tau}_n$$

or more generally

$$\delta f_n = u_n + v_n\boldsymbol{\tau}_n, \quad u_n \perp \boldsymbol{\tau}_n$$

with an arbitrary fast decreasing sequence v_n

Problems

$$f : \mathbb{S}^2 \to \mathbb{R}^3$$

The Willmore flow

$$\partial_t f + \Delta_f H + 2(|H|^2 - K)\, H = 0.$$