INTEGER POINTS IN A PARAMETERISED POLYHEDRON

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The classical parameterised integer feasibility problem is as follows. Given a rational matrix $A \in \mathbb{Q}^{m\times n}$ and a rational polyhedron $Q \subseteq \mathbb{R}^m$, decide whether there exists a vector $b \in Q$ such that the system $Ax \leqslant b$ is infeasible in integer variables. Our main result is a polynomial algorithm to solve a slightly more general parameterised integer feasibility problem if the number n of columns of A is fixed. This extends a result of Kannan [2], who provided such an algorithm for the case, in which—additionally to n—also the affine dimension of the polyhedron Q must be fixed. More precisely, we prove the following theorem:

Theorem 1. *There exists an algorithm that, given rational matrices $A \in \mathbb{Q}^{m+n}$ and $B \in \mathbb{Q}^{k+n}$, rational affine transformations $\Phi : \mathbb{R}^l \to \mathbb{R}^m$ and $\Psi : \mathbb{R}^l \to \mathbb{R}^k$ and a rational polyhedron $Q \subseteq \mathbb{R}^l$, finds $b \in Q$ such that the system $Ax \leqslant \Phi(b)$ has an integer solution, while the system $Bx \leqslant \Psi(b)$ has no integer solution, or asserts that no such b exists. The algorithm runs in polynomial time if n is fixed.*

As an application of our result, we describe an algorithm to find the maximum difference between the optimum values of an integer program

$$\max\left\{cx : Ax \leqslant b,\, x \in \mathbb{Z}^n\right\}$$

and its linear programming relaxation, as the right-hand side b varies over all vectors, for which the integer program is feasible. The latter is an extension of a recent result of Hoşten and Sturmfels [1], who gave such an algorithm for integer programs in standard form.

REFERENCES

1. S. Hoşten and B. Sturmfels. Computing the integer programming gap. To appear in *Combinatorica*.
2. R. Kannan. Lattice translates of a polytope and the Frobenius problem. *Combinatorica*, 12(2):161–177, June 1992.

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