Killer Waves in AKNS and KP-I hierarchies

Vladimir B. Matveev

[1]Institut de Mathématiques de Bourgogne (IMB), Dijon, France
[2]Sankt-Petersburg University of Aerospace Instrumentation (SUAI),Russia

Lecture delivered at the conference "December Readings" ,
S.L. Sobolev Mathematical Institut, Siberian branch of Russian Academy of Sciences, December 22-24, 2016 Novosibirsk , Russia:
http://math.nsc.ru/conference/dr/

December 23, 2016

Main Results

1. Multi-rogue wave solutions to the focusing NLS and Gross-Pitaevskii equation
2. General multi-rogue wave solution for n=3
3. Multi-rogue waves solutions of NLS equation and KP-I equation
4. Analogs of Peregrine breather solutions for AKNS hierarchy
5. Scaling invariance of the *n*-th RAKNS equation
6. Mixed equations generated by the RAKNS hierarchy
7. Conclusions

Abstract

This talk is based on the following articles:

P. Dubard and V. B. Matveev, NONLINEARITY **46** n.12 , R92-R125 (2013),
V. B. Matveev, P. Dubard and A. O. Smirnov , Russian journal " Nonlinear Dynamics " **2** :11 (2015) pp.217-238, available at http://nd.ics.org.ru
V. B. Matveev and A. O. Smirnov
"Some comments on continuous symmetries of AKNS hierarchy equations and their solutions ",
arXiv:1509.01134v2 [math-ph] 27 Sep.2015
V. B. Matveev and A. O. Smirnov
Theoretical and Mathematical Physics (TMF) **186**(2) , 156-182 (2016)

Focusing NLS equation reads

$$iv_t + 2|v|^2 v + v_{xx} = 0, \quad x, t \in \mathbb{R}.$$

Multi rogue waves solutions of the NLS equation are quasi rational solutions:

$$v = e^{2iB^2 t} R(x,t), \quad R(x,t) = \frac{N(x,t)}{D(x,t)}, \quad B > 0,$$

Here $N(x,t), D(x,t)$ are polynomials of x and t, and deg $N(x,t)$ = deg $R(x,t) = n(n+1)$,

$$|v^2| \to B^2, \quad x^2 + t^2 \to \infty$$

The rational function $R(x,t)$ satisfies the 1D Gross-Pitaevskii equation:

$$iR_t + 2R(|R|^2 - B^2) + R_{xx} = 0, \quad |R| = |v|.$$

Below we take $B = 1$, $V = 0, \phi = 0$. The whole set of solutions with any B can be obtained by applying the scaling transformation, phase transformation, and Galilean transformation:

$$v(x,t) \to Bv(Bx, B^2 t), \quad v(x,t) \to v(x,t)e^{i\phi},$$

$$v(x,t) \to v(x - Vt, t)\exp\left(iVx/2 - iV^2 t/4\right),$$

preserving the NLS equation.

$$q_{2n}(k) := \prod_{j=1}^{n} \left(k^2 + ictg\frac{\alpha_j}{2}\right), \quad \alpha_j := \frac{(2j-1)\pi}{2n+1}.$$

$$\Phi(k) := i \sum_{l=1}^{2n} \varphi_l (ik)^l, \quad \varphi_j \in \mathbf{R},$$

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2n}(k)}, \quad D_k := \frac{k^2}{k^2+1}\frac{\partial}{\partial k},$$

$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=1} \ ,$$

$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-1} \ , \quad j = 1\ldots,n.$$

Consider two Wronskians:

$$W_1 := W(f_1, \dots, f_{2n}) \equiv \det A, \quad A_{lj} := \partial_x^{l-1} f_j,$$

$$W_2 := W(f_1, \dots, f_{2n}, f).$$

Multi-rogue solutions to the focusing NLS equation I

Theorem

The function $v(x,t)$ defined by the formula

$$v(x,t) = -q_{2n}(0)e^{2it}\frac{W_2}{W_1}|_{k=0}\ , \tag{1}$$

represents a family of nonsingular (quasi)-rational solutions to the focusing NLS equation depending on $2n$ independent real parameters φ_j.

Peregrine solution

The case $n = 1$ is well known: we obtain the original Peregrine solution taking $\varphi_1 = 0$ and $\varphi_2 = \frac{\sqrt{3}}{4}$. It takes especially simple forme form if one use variables $X = 2x$, $T = 4t$:

$$v(x,t) = \left(1 - \frac{4(1+iT)}{X^2+T^2+1}\right) e^{iT/2}.$$

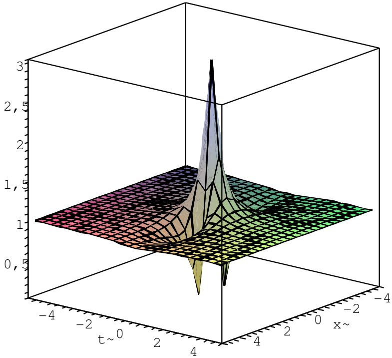


Figure: n=1 solution for $\varphi_1 = 0$ and $\varphi_2 = \frac{\sqrt{3}}{4}$.

For $n = 2$ the higher analog of Peregrine breather (which we call P_2 breather), found in 1995 by AEK reads:

$$P_2(x,t) = e^{iT/2}\left(1 - 12\frac{G(X,T) + iH(X,T)}{Q(X,T)}\right),$$

$$Q := (T^2 + X^2 + 1)^3 + 24(X^2 + 4T^2 - X^2T^2) + 8,$$

$$G := 5T^4 + X^4 + 6X^2T^2 + 6X^2 + 18T^2 - 3,$$

$$H := T^5 + 2T^3 + TX^4 - 15T + 2T^2 - 6TX^2.$$

It is clear that its magnitude reaches absolute maximum value 5 at the point $(0,0)$. Q and G are even functions both of X and T. His an odd function of T and an even function of X. This property is preserved for higher Peregrin breathers discussed below.

General structure of the P_n breathers of the rank n

$$P_n(x,t) = \left(1 - (-1)^{n-1}\, 2n(n+1)\frac{G(X,T,n) + iH(X,T,n)}{Q(X,T,n)}\right)\, e^{iT/2},$$

$$Q \sim (X^2+T^2)^{n(n+1)/2}, \quad H \sim T(X^2+T^2)^{n(n+1)/2-1}, \quad X^2+T^2 \to \infty,$$

$$deg\, Q(x,t) = n(n+1), \quad deg\, G = n(n+1)-2, \quad deg\, H = n(n+1)-1,$$

$$Q(n) \equiv Q(0,0,n) = 1^{2n}3^{2(n-1)}5^{2(n-2)}\dots(2n-1)^2, Q(n) > 0.$$

Numerical evaluations show that the central peak of the $|P_n(x,t)|$ is always surrounded by $n(n+1)-2$ smaller maxima and there are always $n(n+1)$ minima .

$$\frac{Q(n+1)}{Q(n)} = 1^2 3^2 \cdots (2n+1)^2 = \frac{2^{2(n+1)}}{\pi}\Gamma^2\left(n+\frac{3}{2}\right) = [(2n+1)!!]^2$$

$$Q(2n) = -(2n+1)\,G(2n), \quad Q(2n-1) = (2n-1)\,G(2n-1),$$

$$G(2n-1) > 0, G(2n) < 0, \forall n \geq 1$$

From this structure it follows that

$$P_n(0,0) = (-1)^n(2n+1), \quad n \geq 1.$$

Deformations of P_n breathers to the generic multi rogue waves solutions

In general case the P_n breathers (with a central maximum fixed at the origin) can be deformed in such a way that the phases φ_j can be represented as a linear combinations of the new bf real valued parameters $\alpha_j, \beta_j, \ j = 1, 2, \ldots, n-1$, in such a way that the polynomials $Q(n), G(n), H(n)$ become the polynomials of degree $2n-2$ with respect to these parameters having an advantage that the P_n breather itself corresponds to the choice $\alpha_j = \beta_j = 0$, which was not the case of a φ parametrisation described above.

Deformed polynomials $Q(n, x, t, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1})$, $H(n, \ldots)$ and $G(n, \ldots)$ always have the same leading term of asymptotics, when $X^2 + T^2 \to \infty$, as the non deformed polynomials .

From $\varphi_j \to (\alpha, \beta)$ parametrization: $n = 2$.

Let

$$\varphi_1 = 3\varphi_3, \quad \varphi_2 = 2\varphi_4 + \frac{3+\sqrt{5}}{16} \cdot \sqrt{10 - 2\sqrt{5}},$$

$$\alpha := 2(5+\sqrt{5})\sin(\pi/5) - 48\varphi_4,$$

$$\beta := 96\varphi_3$$

. Therefore α, β are fixed by the choice of φ_3, φ_4 and the condition $\alpha = \beta = 0$ is equivalent to

$$\varphi_1 = \varphi_3 = 0, \varphi_4 = \frac{1}{24}(5+\sqrt{5})\sin(\pi/5)$$

$$\varphi_2 = \frac{1}{6}(7 + 2\sqrt{5})\sin(\pi/5)$$

$$v_2(x,t) = e^{iT}\left(1 - 12\frac{G_d(X,T) + iH_d(X,T)}{Q_d(X,T)}\right),$$

$$\begin{aligned}
&G_d(X,T) := G(X,T) + 2\beta X - 2\alpha T,\\
&H_d(X,T) := H(X,T) + \alpha X^2 + 2\beta TX + \alpha(1 - T^2),\\
&Q_d(X,T) :=\\
&Q(X,T) - 2\beta X^3 - 6\alpha T^3 X^2 + 6\beta(T^2 + 1)X\\
&-2\alpha T^3 - 18\alpha T + \beta^2 + \alpha^2.
\end{aligned}$$

This new parametrization is now free of irrational factors. When $\alpha^2 + \beta^2 \to \infty$

$$v_2(x,t,\alpha,\beta) \to e^{iT/2} \equiv= e^{2it}$$

$$v_2(x,t,0,0) = P_2(x,t).$$

Symmetry of the deformed solution $u_2(x, t, \alpha, \beta)$.

$$u_2(x, t, \alpha, \beta) = u_2(-x, t, -\alpha, \beta) = \bar{u}_2(x, -t, \alpha, -\beta)$$

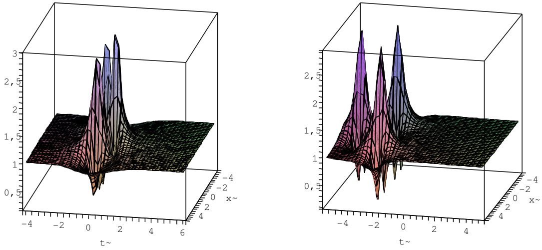
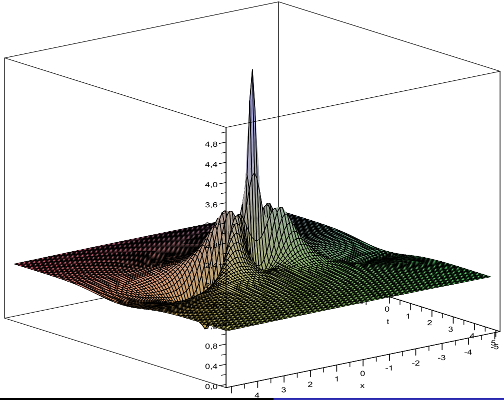


Figure: Amplitude of the solution to the NLS equation for $n = 2$ with $\varphi_2 = 1$ and $\varphi_1 = \varphi_3 = \varphi_4 = 0$ on the left, and $\varphi_4 = 1$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$ on the right.

Plot of the Magnitude of P_2 breather.



When the parameters α, β are small enough the related deformation of the higher Peregrine breather keeps its extreme rogue wave character i.e. the maximum of its magnitude is very close to 5 and a plot of the solution is quite similar to what we have when $\alpha = \beta = 0$.

$n = 3 : \alpha, \beta$-parametrisation.

$$\alpha_1 := 48(\varphi_3 - 5\varphi_5), \quad \alpha_2 = 480(\varphi_3 - 13\varphi_5),$$

$$\beta_1 := 8(12(\,4\varphi_6 - \varphi_4) + Im[\omega(1+\omega)^2])$$

$$\beta_2 := 32(60(\,8\varphi_6 - \varphi_4) + Im[\bar{\omega}(1-2\bar{\omega})^2]).$$

$$\omega := e^{-i\pi/7}.$$

$$
\begin{aligned}
\varphi_1 &= 3\varphi_3 - 5\varphi_5 \\
\varphi_2 &= 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1-\cos(\pi/7))} \\
768\varphi_3 &= 26\alpha_1 - \alpha_2 \\
1920\varphi_4 &= -40\beta_1 + \beta_2 + 96(3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)) \\
3840\varphi_5 &= 10\alpha_1 - \alpha_2 \\
7680\varphi_6 &= -20\beta_1 + \beta_2 + 32(4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)),
\end{aligned}
$$

$$v_3(x,t,\alpha_1,\beta_1,\alpha_2,\beta_2)=\left(1-24\frac{G_3(X,T)+iH_3(X,T)}{Q_3(X,T)}\right)e^{iT/2},$$

$$\begin{aligned}
G_3(X,T) &= X^{10}+15(T^2+1)X^8+\textstyle\sum_{n=0}^{6} g_n(T)X^n \\
H_3(X,T) &= TX^{10}+5(T^3-3T+\beta_1)X^8+\textstyle\sum_{n=0}^{6} h_n(T)X^n \\
Q_3(X,T) &= (1+X^2+T^2)^6-20\alpha_1X^9-60(2T^2-\beta_1T-2)X^8+4\textstyle\sum_{n=0}^{7} q_n(T)X^n.
\end{aligned}$$

$$
\begin{aligned}
g_6 &= 50T^4 - 60T^2 + 80\beta_1 T + 210\\
g_5 &= 120\alpha_1 T^2 - 18\alpha_2 + 300\alpha_1\\
g_4 &= 70T^6 - 150T^4 + 200\beta_1 T^3 + 450T^2 + 30\beta_2 T - 450 + 150\alpha_1^2 - 50\beta_1^2\\
g_3 &= 400\alpha_1 T^4 + (3000\alpha_1 - 60\alpha_2)T^2 - 800\alpha_1\beta_1 T - 600\alpha_1 - 60\alpha_2\\
g_2 &= 45T^8 + 420T^6 + 6750T^4 - (6000\beta_1 - 180\beta_2)T^3 - (300\alpha_1^2 - 900\beta_1^2 + 13500)T^2\\
&\quad +(3600\beta_1 + 180\beta_2)T - 675 - 300\alpha_1^2 - 300\beta_1^2\\
g_1 &= 280\alpha_1 T^6 + (150\alpha_2 - 2100\alpha_1)T^4 + 800\alpha_1\beta_1 T^3 - (3600\alpha_1 - 540\alpha_2)T^2\\
&\quad +(120\beta_2\alpha_1 + 1200\alpha_1\beta_1 - 120\alpha_2\beta_1)T - 200\alpha_1\beta_1^2 - 900\alpha_1 - 90\alpha_2 - 200\alpha_1^3\\
g_0 &= 11T^{10} + 495T^8 - 120\beta_1 T^7 + 2190T^6 - (42\beta_2 + 1200\beta_1)T^5\\
&\quad +(350\alpha_1^2 + 150\beta_1^2 - 7650)T^4 + (6600\beta_1 - 420\beta_2)T^3\\
&\quad -(2100\beta_1^2 + 2025 - 120\beta_2\beta_1 - 120\alpha_2\alpha_1 + 900\alpha_1^2)T^2 + (200\alpha_1^2\beta_1 + 200\beta_1^3 - 90\beta_2)T\\
&\quad +675 + 150\alpha_1^2 + 6\alpha_2^2 + 150\beta_1^2 + 6\beta_2^2.
\end{aligned}
$$

$$
\begin{aligned}
h_6 &= 10T^5 - 140T^3 + 40\beta_1 T^2 - 150T + 60\beta_1 - 5\beta_2\\
h_5 &= 40\alpha_1 T^3 + (60\alpha_1 - 18\alpha_2)T + 40\alpha_1\beta_1\\
h_4 &= 10T^7 - 210T^5 + 50\beta_1 T^4 - 450T^3 + 15\beta_2 T^2 - (50\beta_1^2 + 1350 - 150\alpha_1^2)T\\
&\quad +150\beta_1 - 15\beta_2\\
h_3 &= 80\alpha_1 T^5 + (1000\alpha_1 - 20\alpha_2)T^3 - 400\alpha_1\beta_1 T^2 - (1800\alpha_1 - 60\alpha_2)T\\
&\quad +200\alpha_1\beta_1 + 20\beta_2\alpha_1 - 20\alpha_2\beta_1\\
h_2 &= 5T^9 - 60T^7 + 1710T^5 + (45\beta_2 - 2100\beta_1)T^4 + (300\beta_1^2 - 6300 - 100\alpha_1^2)T^3\\
&\quad +(1800\beta_1 - 90\beta_2)T^2 + (4725 + 300\alpha_1^2 + 300\beta_1^2)T - 135\beta_2 - 100\beta_1^3\\
&\quad -100\alpha_1^2\beta_1 - 900\beta_1\\
h_1 &= 40\alpha_1 T^7 + (30\alpha_2 - 1140\alpha_1)T^5 + 200\alpha_1\beta_1 T^4 - (2400\alpha_1 - 60\alpha_2)T^3\\
&\quad +(60\beta_2\alpha_1 - 60\alpha_2\beta_1 + 600\alpha_1\beta_1)T^2 - (900\alpha_1 + 450\alpha_2 + 200\alpha_1^3 + 200\alpha_1\beta_1^2)T\\
&\quad +60\alpha_2\beta_1 - 60\beta_2\alpha_1\\
h_0 &= T^{11} + 25T^9 - 15\beta_1 T^8 - 870T^7 + (40\beta_1 - 7\beta_2)T^6 + (70\alpha_1^2 - 9630 + 30\beta_1^2)T^5\\
&\quad +(5850\beta_1 - 75\beta_2)T^4 + (40\beta_2\beta_1 + 40\alpha_2\alpha_1 - 2475 - 900\alpha_1^2 - 1300\beta_1^2)T^3\\
&\quad +(100\alpha_1^2\beta_1 + 495\beta_2 + 100\beta_1^3)T^2 + (6\alpha_2^2 + 4725 - 240\alpha_2\alpha_1 - 240\beta_2\beta_1\\
&\quad +750\beta_1^2 + 6\beta_2^2 + 750\alpha_1^2)T - 20\alpha_1^2\beta_2 - 675\beta_1 - 45\beta_2 - 100\alpha_1^2\beta_1 - 100\beta_1^3\\
&\quad +40\alpha_2\alpha_1\beta_1 + 20\beta_1^2\beta_2
\end{aligned}
$$

$$
\begin{aligned}
q_7 &= 3\alpha_2 - 30\alpha_1\\
q_6 &= -60T^4 + 40\beta_1 T^3 + 120T^2 - (15\beta_2 - 60\beta_1)T + 35\beta_1^2 + 15\alpha_1^2 + 580\\
q_5 &= 30\alpha_1 T^4 - (27\alpha_2 - 90\alpha_1)T^2 + 120\alpha_1\beta_1 T - 27\alpha_2 + 540\alpha_1\\
q_4 &= 30\beta_1 T^5 - 360T^4 + (15\beta_2 + 600\beta_1)T^3 + (3360 + 225\alpha_1^2 - 75\beta_1^2)T^2\\
&\quad +(135\beta_2 - 1350\beta_1)T + 225\beta_1^2 - 30\alpha_2\alpha_1 + 525\alpha_1^2 - 30\beta_2\beta_1 + 840\\
q_3 &= 40\alpha_1 T^6 + (1950\alpha_1 - 15\alpha_2)T^4 - 400\alpha_1\beta_1 T^3 + (90\alpha_2 + 4500\alpha_1)T^2\\
&\quad +(60\beta_2\alpha_1 - 1800\alpha_1\beta_1 - 60\alpha_2\beta_1)T - 450\alpha_1 + 100\alpha_1^3 + 100\alpha_1\beta_1^2 - 135\alpha_2\\
q_2 &= 60T^8 + 3360T^6 - (1620\beta_1 - 27\beta_2)T^5 + (225\beta_1^2 - 75\alpha_1^2 + 19560)T^4\\
&\quad -(16200\beta_1 - 270\beta_2)T^3 + (450\alpha_1^2 - 9120 + 4050\beta_1^2)T^2\\
&\quad +(675\beta_2 + 2700\beta_1 - 300\beta_1^3 - 300\alpha_1^2\beta_1)T + 3036 + 9\alpha_2^2 - 180\alpha_2\alpha_1\\
&\quad +225\beta_1^2 + 225\alpha_1^2 + 9\beta_2^2 - 180\beta_2\beta_1\\
q_1 &= 15\alpha_1 T^8 + (15\alpha_2 - 90\alpha_1)T^6 + 120\alpha_1\beta_1 T^5 + (405\alpha_2 - 5400\alpha_1)T^4\\
&\quad +(3000\alpha_1\beta_1 - 60\alpha_2\beta_1 + 60\beta_2\alpha_1)T^3 + (1485\alpha_2 - 300\alpha_1\beta_1^2 - 1350\alpha_1 - 300\alpha_1^3)T^2\\
&\quad +(540\beta_2\alpha_1 - 540\alpha_2\beta_1)T + 300\alpha_1^3 - 120\alpha_1\beta_1\beta_2 - 60\alpha_2\alpha_1^2 + 135\alpha_2 + 60\alpha_2\beta_1^2\\
&\quad +300\alpha_1\beta_1^2 + 2025\alpha_1\\
q_0 &= 30T^{10} - 5\beta_1 T^9 + 930T^8 - (240\beta_1 + 3\beta_2)T^7 + (15\beta_1^2 + 3820 + 35\alpha_1^2)T^6\\
&\quad +(1710\beta_1 - 153\beta_2)T^5 + (30\beta_2\beta_1 + 30\alpha_2\alpha_1 - 975\beta_1^2 + 35940 - 75\alpha_1^2)T^4\\
&\quad +(100\beta_1^3 + 100\alpha_1^2\beta_1 + 135\beta_2 - 23400\beta_1)T^3\\
&\quad +(9\beta_2^2 + 23286 + 9\alpha_2^2 - 360\beta_2\beta_1 - 360\alpha_2\alpha_1 + 4725\alpha_1^2 + 8325\beta_1^2)T^2\\
&\quad +(120\alpha_2\alpha_1\beta_1 - 60\alpha_1^2\beta_2 - 1500\alpha_1^2\beta_1 + 60\beta_1^2\beta_2 - 7425\beta_1 - 675\beta_2 - 1500\beta_1^3)T\\
&\quad +506 + 9\beta_2^2 + 100\beta_1^4 + 675\alpha_1^2 + 100\alpha_1^4 + 9\alpha_2^2 + 90\beta_2\beta_1 + 200\alpha_1^2\beta_1^2\\
&\quad +675\beta_1^2 + 90\alpha_2\alpha_1.
\end{aligned}
$$

Solutions of the NLS equation above provide $2n$-parametric family of the smooth rational solutions to the KP-I equation:

$$\partial_x(4u_t + 6uu_x + u_{xxx}) = 3u_{yy}.$$

Replace t by y and φ_3 by t. Obviously the function

$$f(k,x,y,t) := \exp(kx + ik^2y + k^3t + \phi(k)),$$

where

$$\phi(k) := \Phi(k) - \varphi_3 k^3,$$

satisfies the system

$$-f_t = f_{xxx}, \quad f_y = -if_{xx} = 0.$$

The same is true for the functions f_j, defined above if we denote t by y and φ_3 by $-t$. Now from (Matveev LMP 1979 p.214-216) we get following result:

Theorem

$$u(x, y, t) = 2\partial_x^2 \log\, W(f_1, \ldots, f_{2n}) = 2(|v|^2 - B^2)$$

is smooth rational solution to the KP-I equation. It is obvious that

$$\int_{-\infty}^{\infty} u(x, y, t) d\,x = 0,$$

and

$$u(x, y, t) \geq -2B^2$$

THEOREM:

For given B and n the maximal value the solutions of KP-I equation described by the theorem above is given by the formula:

$$\max_{x,y,t\in R} u(x,y,t) = 8B^2 n(n+1).$$

The solutions of KP-I equation given by the previous theorem depend on $2n$ real parameters $\varphi_j, , j \neq 3$, representing the action of the KP-I hierarchy flows. The phases φ_1, φ_2 correspond respectively to space and time translations.

This maximum value (i.e. 48) for $n = 2, B = 1$ is attended at the point $x = y = t = 0$ provided that

$$\varphi_1 = 0, \varphi_3 = t, \varphi_4 = \frac{1}{24}(5 + \sqrt{5}) \sin(\pi/5)$$

$$\varphi_2 = \frac{1}{6}(7 + 2\sqrt{5}) \sin(\pi/5)$$

.

For n=3 the related absolute maximum of u equals 96 attended at $x = y = t = 0$, with

$$\varphi_1 = \varphi_5 = 0,$$

$$\varphi_4 = 3\sin(\pi/7) + 8\sin(2\pi/7) + 2\sin(3\pi/7)/20,$$

$$\varphi_6 = (4\sin(\pi/7) + 14\sin(2\pi/7) + \sin(3\pi/7)/240,$$

$$\varphi_2 = 2\varphi_4 - 3\varphi_6 + \frac{\sin(\pi/7)}{4(1-\cos(\pi/7))}$$

.

The AKNS hierarchy

$$\begin{cases} \Psi_x = U\Psi, \\ \Psi_{t_k} = V_k\Psi, \end{cases}$$

where

$$U := \lambda J + U^0, \quad V_1 := 2\lambda U + V_1^0, \quad V_{k+1} := 2\lambda V_k + V_{k+1}^0, \quad k \geqslant 1$$

$$J := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^0 := \begin{pmatrix} 0 & i\psi \\ -i\phi & 0 \end{pmatrix}.$$

The compatibility condition $(\Psi_x)_{t_k} = (\Psi_{t_k})_x$ implies that :

$$[J, V_1^0] = 2(U^0)_x, \quad [J, V_{k+1}^0] = 2(V_k^0)_x + 2[V_k^0, U^0], \quad k \geqslant 1,$$

$$(U^0)_{t_k} = (V_k^0)_x + [V_k^0, U^0] = \frac{1}{2}[J, V_{k+1}^0].$$

The later equation is equivalent to the system of two coupled scalar nonlinear PDE's
representing the k -th member of the (non reduced) AKNS hierarchy.

Under the condition $\phi = -\bar{\psi}$ this system becomes a single scalar equation for the ψ. The k-th member of the reduced AKNS (RAKNS) hierarchy can be written in the form

$$\psi_{t_k} = i^k H_k(\psi, \bar{\psi}),$$

the first five of these equations obviously can be obviously written as

$$i\psi_{t_1} + H_1(\psi) = 0, \quad \psi_{t_2} + H_2(\psi) = 0, \quad -i\psi_{t_3} + H_3(\psi) = 0,$$
$$-\psi_{t_4} + H_4(\psi) = 0, \quad i\psi_{t_5} + H_5(\psi) = 0$$

where H_k are explicit differential polynomials of ψ and $\bar{\psi}$, described below.

The first three equations of the RAKNS hierarchy are widely known: the first one is the focusing NLS equation with $t_1 = t$, the second one is a modified complex KdV equation :

$$\psi_{t_2} + \psi_{xxx} + 6|\psi|^2\psi_x = 0.$$

The 3-d member of RAKNS hierarchy (known as Lakshmanan-Porcesian-Daniel equation) with $t_3 = -t$:

$$i\psi_t + \psi_{xxxx} + 8|\psi|^2\psi_{xx} + 2\psi^2\bar{\psi}_{xx} + 6\psi_x^2\bar{\psi} + 4\psi|\psi_x|^2 + 6|\psi|^4\psi = 0.$$

The less known is a 4-th RAKNS equation:

$$\psi_t + \psi_x^{(5)} + 10|\psi|^2\psi_{xxx} + 20\psi_{xx}\psi_x\bar{\psi} + 10(|\psi_x|^2\psi)_x + 30|\psi|^4\psi_x = 0.$$

Next , i.e. the 5-th equation of the RAKNS hierarchy is much longer: the related function H_5 contains 16 different terms:

RAKNS-5 equation

$$
\begin{gathered}
i\psi_{t_5} + \psi_x^{(6)} + 12|\psi|^2\psi_{xxxx} + 2\psi^2\bar{\psi}_{xxxx} + \\
30\psi_{xxx}\psi_x\bar{\psi} + 18\psi_{xxx}\psi\bar{\psi}_x + 8\psi_x\psi\bar{\psi}_{xxx} + \\
+ 50\psi_{xx}|\psi_x|^2 + 50\psi_{xx}|\psi|^4 + \\
20\psi_{xx}^2\bar{\psi} + 22|\psi_{xx}|^2\psi + 20\psi_x^2\bar{\psi}_{xx} + 20|\psi|^2\psi^2\bar{\psi}_{xx} + \\
+ 10\psi^3(\bar{\psi}_x)^2 + 70\psi_x^2|\psi|^2\bar{\psi} + 60|\psi|^2|\psi_x|^2\psi + 20|\psi|^6\psi = 0.
\end{gathered}
$$

It is easy to check that the *n*-th equation of the RAKNS hierarchy is covariant with respect to a following scaling transformation

$$\psi(x,t) \to q\psi(qx, q^{n+1}t), \quad q > 0.$$

In particular, this means that the study of the solutions of the n-th RAKNS equation with asymptotic magnitude q can always be reduced to the case $q = 1$.
A priori, it is not obvious that the analogs of Peregrine breathers,
corresponding to all RAKNS equations, having the analytical form
as simple as for the NLS equation do exist. This is what we'll explain now.

Suppose that t_k are the time variables of the RAKNS hierarchy and

$$\varphi_1 = 0, \quad \varphi_2 = \frac{\sqrt{3}}{4},$$

$$t = t_1,\ \varphi_3 = -2t_2,\ \varphi_4 = -3t_3,\ \varphi_5 = -6t_4,\ \varphi_6 = -10t_5,$$

$$\varphi_n = -\frac{n(n-1)}{2}\, t_{n-1}, \quad n = 1, \ldots$$

$$\hat{X} := 2(x - \varphi_1 + 3\varphi_3 - 5\varphi_5 + \ldots) = 2(x - 6t_2 + 30t_4 + \ldots)$$

$$\hat{T} := 4(t + \sqrt{3}/4 - \varphi_2 + 2\varphi_4 - 3\varphi_6 + \ldots) = 4(t_1 - 6t_3 + 30t_5 + \ldots).$$

P_n breathers for first 5 members of AKNS hierarchy

THEOREM. *The function*

$$\Psi := \left(1 - \frac{4(1+i\hat{T})}{\hat{X}^2 + \hat{T}^2 + 1}\right) e^{2i(t_1 - 3t_3 + 10t_5)}$$

is a joint rational solution of the first 5 equations of reduced AKNS hierarchy.

Obviously this gives the description of the analogs of Peregrine breather for these 5 equations .

$$P_1^1 = \left(1 - \frac{4(1+4it)}{4x^2+16t^2+1}\right) e^{2it},$$
$$P_1^2 = \left(1 - \frac{4(1+iT)}{4(x-6t)^2+T^2+1}\right),$$
$$P_1^3 = \left(1 - \frac{4(1+24it)}{4x^2+576t^2+1}\right) e^{6it},$$
$$P_1^4 = \left(1 - \frac{4(1+iT)}{4(x-30t)^2+T^2+1}\right),$$
$$P_1^5 = \left(1 - \frac{4(1+120it)}{4x^2+14400t^2+1}\right) e^{20it}.$$

T above means an arbitrary real number. Observe that only P_1^3 and P_1^5 have the behavior similar to the genuine Peregrine

Mixed RAKNS equation I :Hirota equation

An idea to generate them is to set $t = t_k$ for certain values of k simultaneously.
Hirota equation (known since 1973) has a form

$$i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) = 0.$$

It is easy to see that it has a solution given by

$$\Psi(x, \alpha t, -\beta t, t_4, \dots, t_k),$$

where $\Psi(x, t_1, t_2, \dots, t_k)$ is any solution of first k flows of AKNS hierarchy.

Next mixed RAKNS model equations

$RAKNS_{1-3}$ equation:

$$i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) = 0.$$

Its solution is given by $\Psi(x, \alpha t, -\beta t, -\gamma_1 t, \dots, t_k)$, where the times $t_k, k \geq 4$ play the role of parameters . (this equation was partially analyzed by many authors). The $RAKNS_{1-4}$ and $RAKNS_{1-5}$ models are described by the following two nonlinear PDE's : [1]

$$i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) - i\gamma_2 H_4(\psi) = 0,$$

$$i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) - i\gamma_2 H_4(\psi) + \gamma_3 H_5(\psi) = 0.$$

[1] $\beta = 0, \gamma_1 = 0$ case of the first equation below was considered in Phys.Rev.E **91** (2015) by A.Chowdury , N.Akhmediev, other cases were never studied.

Rank 1 solutions of the mixed models

$$P_1^{1,2} = \left(1 - \frac{4(1+4i\alpha t)}{4(x+6\beta t)^2 + 16(\alpha t)^2 + 1}\right) e^{2i\alpha t},$$

$$P_1^{1-3} = \left(1 - \frac{4(1+4i\alpha_1 t)}{4(x+6\beta t)^2 + 16(\alpha_1 t)^2 + 1}\right) e^{2i(\alpha+3\gamma_1)t},$$

$$P_1^{1-4} = \left(1 - \frac{4(1+4i\alpha_1 t)}{4(x+6\beta_1 t)^2 + 16(\alpha_1 t)^2 + 1}\right) e^{2i(\alpha+3\gamma_1)t},$$

$$P_1^{1-5} = \left(1 - \frac{4(1+4i\alpha_2 t)}{4(x+6\beta_1 t)^2 + 16(\alpha_2 t)^2 + 1}\right) e^{2i(\alpha+3\gamma_1+10\gamma_3)t},$$

$$\alpha_1 := \alpha + 6\gamma_1, \quad \beta_1 := \beta + 5\gamma_2, \quad \alpha_2 := \alpha + 6\gamma_1 + 30\gamma_3.$$

Rank 2 solutions for first five *RAKNS* equations

Rank 2 solution already discussed above for NLS case can be extended in order to give the solutions of RAKNS equations with $k \leq 5$:

$$\Psi_2(X, T_1, T_2, T_3) = \left(1 - 12\frac{G(X, T_1, T_2, T_3) + iH(X, T_1, T_2, T_3)}{Q(X, T_1, T_2, T_3)}\right) e^{2it_1 - 6it_3 + 20it_5},$$

$$\begin{aligned}
G(X, T_1, T_2, T_3) =&(X^2 + 3T_1^2 + 3)^2 - 4T_1^4 + 2XT_2 + 2T_1T_3 - 12,\\
H(X, T_1, T_2, T_3) =&T_1(X^2 + T_1^2 + 1)^2 + 2XT_1T_2 + T_3(T_1^2 - X^2 - 1) - 8T_1(X^2 + 2),\\
Q(X, T_1, T_2, T_3) =&(X^2 + T_1^2 + 1)^3 + T_2^2 + 2XT_2(3T_1^2 - X^2 + 3) + T_3^2 + 2T_1T_3(T_1^2 - 3X^2 + 9) +\\
&+ 24T_1^4 - 24T_1^2X^2 + 96T_1^2 + 24X^2 + 8.
\end{aligned}$$

This formula provides the solutions of first 5 RAKNS equations under the following choice of X, T_1, T_2 and T_3 and of phases:

$$
\begin{aligned}
X &= 2x + 6\varphi_3 - 10\varphi_5,\\
T_1 &= 4t_1 + 8\varphi_4 - 12\varphi_6,\\
T_2 &= 96\varphi_3 - 480\varphi_5,\\
T_3 &= 192\varphi_4 - 768\varphi_6.
\end{aligned}
$$

$$
\varphi_1 = 3t_2, \quad \varphi_2 = 4t_3 - 10t_5, \quad \varphi_3 = -t_2 + 10t_4, \quad \varphi_4 = -t_3 + 10t_5.
$$

$$X = 2(x - 6t_2 + 30t_4 + \dots,)$$
$$T_1 = 4(t_1 - 6t_3 + 30t_5 + \dots,)$$
$$T_2 = -96(t_2 - 10t_4 + \dots,)$$
$$T_3 = -192(t_3 - 10t_5 + \dots)$$

rank 2 solution of the mixed RAKNS_{1-5} , *equation*

$$\Psi_2(X, T_1, T_2, T_3) = \left(1 - 12\frac{G(X, T_1, T_2, T_3) + iH(X, T_1, T_2, T_3)}{Q(X, T_1, T_2, T_3)}\right) e^{2i(\alpha+3\gamma_1+10\gamma_3)t},$$

$$X = 2[x + (6\beta + 30\gamma_2)t], \qquad T_2 = T_2(0) + 96(\beta + 10\gamma_2)t,$$

$$T_1 = T_1(0) + 4(\alpha + 6\gamma_1 + 30\gamma_3)t, \qquad T_3 = T_3(0) + 192(\gamma_1 + 10\gamma_3)t,$$

The formula above contains all (quasi)-rational solutions of rang 2 of $\text{RAKNS}_{(1-5)}$ equation

The main new result is that the same formula for NLS with a functions Φ containing, for a given rank n, more then n phases ,(possibly depending on time variables of the RAKNs hierarchy in a linear way) , - can be used as an universal tool for description of the MRW solutions of any rank for all members of AKNS hierarchy , thus simplifying their analysis and providing further opportunities for new experiments in nonlinear optics. Here we illustrated this point only by the applications to the first 5 members of RAKNS hierarchy, restricting ourselves by the small rank solutions, but similar results can be easily obtained for higher ranks.

THANK YOU FOR YOUR ATTENTION !