CRITERIA OF THE EXISTENCE OF RATIONAL INTEGRALS OF TWO-DIMENSIONAL GEODESIC FLOWS

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Geodesic flow on a two-dimensional surface with a Riemannian metric $ds^2 = g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2$ is described by a pair of second-order ODEs in x, y. Its projection on the (x,y)-plane has the form of scalar equation

$$\begin{gathered}\frac{d^2y}{dx^2} = S(x,y)\left(\frac{dy}{dx}\right)^3 + 3R(x,y)\left(\frac{dy}{dx}\right)^2 + 3Q(x,y)\frac{dy}{dx} + P(x,y),\\ S = \Gamma^1_{22}, \quad 3R = 2\Gamma^1_{12} - \Gamma^2_{22}, \quad 3Q = \Gamma^1_{11} - 2\Gamma^2_{12}, \quad P = -\Gamma^2_{11}.\end{gathered} \tag{1}$$

If geodesic flow possesses rational integral

$$F = \Big(\sum_{j=1}^{n} a_j(x,y)\Big(\frac{dx}{ds}\Big)^{n-j}\Big(\frac{dy}{ds}\Big)^{j}\Big)\Big(\sum_{j=1}^{n} b_j(x,y)\Big(\frac{dx}{ds}\Big)^{n-j}\Big(\frac{dy}{ds}\Big)^{j}\Big)^{-1}$$

then its projection (1) has the first integral

$$F = \Big(\sum_{j=1}^{n} a_j(x,y)y'^j\Big)\Big(\sum_{j=1}^{n} b_j(x,y)y'^j\Big)^{-1}, \quad y' = \frac{dy}{dx}. \tag{2}$$

When $n = 1$, 2 the criteria of existence of integral (2) in terms of relative invariants of the family of equations (1) are obtained (see, e.g., the case $n = 1$ in [1]).

We consider examples of the metric $ds^2 = \lambda(x,y)dxdy$ and integrals of (1)

$F_1 = \dfrac{\alpha'}{\beta' y'}(\beta+\gamma) - 2\alpha$ when $\lambda = \alpha'\beta'(\beta+\gamma)$;

$F_2 = \dfrac{\alpha'(\alpha+\beta)f}{\alpha' f + 2(\alpha+\beta)f_y y'} - \alpha$ when $\lambda = 2ff_y$, $f_{xy} = \left(\dfrac{\alpha''}{2\alpha'} - \dfrac{\alpha'}{\alpha+\beta}\right)f_y + \dfrac{\alpha'\beta'}{4(\alpha+\beta)^2}f$;

$F_3 = \dfrac{\beta f_y^2}{y'^2} + \dfrac{\alpha' f}{f_y y'} - \alpha$ when $\lambda = f_y$, $(\beta f_y^4)_y + \alpha'' f f_y + \alpha'(f_x f_y - 2ff_{xy})$.

Here the notation $\alpha = \alpha(x)$, $\beta = \beta(y)$, $\gamma = \gamma(x)$, $f = f(x,y)$ is used.

References

[1] Bagderina Yu.Yu. “Invariants of a family of scalar second-order ODEs for Lie symmetries and first integrals”, *J. Phys. A: Math. Theor.*, Vol. 49 (2016).

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