ON DISCRETE STRUCTURES IN PHASE PORTRAITS OF SOME NON-LINEAR DYNAMICAL SYSTEMS

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Аннотация. We consider one class of multidimensional non-linear dynamical systems as models of functioning of some gene networks. Cycles of these systems describe important oscillatory biological processes.

We construct discrete structures (structure graphs) in phase portraits of these systems and show some connections of these graphs with geometry of phase portraits. Detection of possible locations of the cycles is one of the main aims of our studies, and here the main question is: Let two dynamical systems in the considered class have the same structure graph, how different can they be?

Oscillatory processes are very important in gene networks analysis: they control circadian rhythms and play a crucial role in natural biological clock.

Functioning of gene networks can be described by kinetic dynamical systems, and we consider here the case when their right-hand sides are smooth. Sometimes, we can obtain useful information on structures of their phase portraits from their discretization: if we divide them to subareas with unambiguous transitions from part to part, we can imitates the original system, and detect areas where periodic trajectories can appear. Such approach was successfully used in analysis of some biological models, see [2-4].

We apply this approach to non-linear dynamical systems of the type

$$\dot{x_1} \equiv \frac{dx_1}{dt} = f_1(x_n) - x_1, \qquad \dot{x_i} \equiv \frac{dx_i}{dt} = f_i(x_{i-1}) - x_i. \tag{1}$$

Here $i = 2, 3, \dots n$, $x_0 = x_n$, $x_{n+1} = x_1$, and all $f_j \geq 0$ are smooth monotonic functions. Denote monotonically increasing functions f_j as f_+, and monotonically decreasing functions f_j as f_-; they correspond to positive, respectively, negative feedbacks in gene networks. Typical examples used in (bio)chemical kinetics are $f_+(x) = ax^\gamma/(1+x^\gamma)$ and $f_-(x) = a/(1+x^\gamma)$.

We define the **profile** S of the system (1) as a binary vector:

$$S_i = 0, \text{ if } f_i \text{ is } f_-, \quad S_i = 1, \text{ if } f_i \text{ is } f_+, \quad i = 1, 2, \dots n.$$

The **parity** $p(S)$ of the system (1) is defined as $p(S) = \sum_1^n S_i \pmod 2$.

The system (1) has at least one equilibrium point $\tilde{x}$. Let Θ be its invariant neighborhood. Actually, detection of such neighborhoods is a difficult task, but in some particular cases they can be easily constructed.

We decompose Θ by 2^n disjoint domains by n hyperplanes containing the equilibrium point $\tilde{x}$ and parallel to the coordinate hyperplanes. We also use the word **domain** to denote binary vector v:

$$v_i = 0, \quad \text{if} \quad x_i^* \leq \tilde{x}_i; \quad v_i = 1, \quad \text{if} \quad x_i^* > \tilde{x}_i; \quad i = 1, 2, \dots n,$$

where $x^* \in \Theta$. Two domains u and v are incident by the i-th coordinate ($(u, v) \in A_i(G)$ and $u_i \neq v_i$) if they differ only in the i-th coordinate. Each pair of incident domains has a common face. Similar constructions in other situations were realized in [1-4].

Lemma 1. *For any pair of domains u and v that are incident by the i-th coordinate, the vector field on the common face of u and v is oriented in one direction along the i-th coordinate axis.*

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We define an oriented graph G as follows: its vertices correspond to domains of Θ, and an edge exits from a domain u to v if they are incident and the vector field of (1) is directed from u to v. We use the notations:

(1) All vertices that have both, incoming and outcoming edges are called **transit vertices**. Let $V(G)$ be the set of all vertices of G, $T(G)$ be the set of its transit vertices, and $A(G) = \bigcup_1^n A_i(G)$.
(2) $E_i(G)$ is a subset of edges of G parallel to the i-th coordinate axis. Let $E(G) = \bigcup_1^n E_i(G)$, then $E_i(G) = A_i(G) \cap E(G)$.
(3) The **valency** $P(u)$ of a domain u is the number of outcoming edges from corresponding vertex. The **valency level** $V_i(G)$ is the set of all domains: $u \in V_i(G)$, and $P(u) = i$. A valency level that contains transit domains only is called **transit valency level**.
(4) A **subsequence** of a vector u is a vector consisting of some of the components of u. If vector v is a subsequence of u, we say that u **contains** v and denote this as $v \subset u$.

There is a direct connection between cyclic trajectories of the system (1) and loops in G: if a cyclic trajectory passes through domains $v_1, v_2, \ldots, v_k$, there is also a loop consisting of corresponding vertices in the graph G.

Lemma 2. *For any system* (1), *vertices of a loop in G belong to the same valency level. Valencies of all domains have the same parity.*

Consider now the system (1) where all f_i are the f_- functions. Let Q_v be a domain of such a system that satisfies following conditions:
1. $(0) \subset v$ and $(1) \subset v$; 2. $(00) \subset v$ or $(11) \subset v$.
Thus, all domains which have valency greater than zero and less than n satisfy both conditions. We call them **transit domains**, and the valency levels that contain them are called **transit levels**. We show that all transit valency levels (and only these levels) contain loops in such graphs.

Proposition 1. *In a system with negative feedbacks, each transit domain contains at least one of subsequences* (001) *or* (110). *For any transit domain, there is always an edge to another transit domain on the same valency level. Eeach transit valency level contains a loop.*

Now, let S be a profile of a system (1) and $I(S,i)$ is defined as follows: $I(S,i)_j = S_j + 1 \pmod 2$ if $j = (i-1)$ or $j = i$; and $I(S,i)_j = S_j$ otherwise. This operator inverts the $(i-1)$-th and the i-th elements of each profile and preserves the parity of the system.

Theorem 1. *Domain graphs of systems of the type* (1) *with profile vectors S and $I(S,\ i)$ are isomorphic for any i.*

Corollary 1. *The graphs of two such systems with the same parity are isimorphic.*

References

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