Non-commutative normal forms and inverse spectral problems.

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Schrödinger operator with potential well

The main object is the one-dimensional Schrödinger operator:

$$\hat{H} = \frac{\hat{p}^2}{2} + V(x), \quad \hat{p} = -ih\frac{d}{dx}$$

where the potential $V(x)$ is an even degree polynomial s.t. $V(x) > 0$ for $x > 0$, $V(0) = V'(0) = 0$ and $V''(0) = 1$. Its spectrum is discrete, its 'low lying eigenvalues' (i.e. lying in $[0, Ch]$) are

$$\hat{H}\psi_\nu = E_\nu\psi_\nu, \quad E_\nu \sim h\big(\nu + 1/2\big) + \sum_{k=2}^{\infty} \mathcal{E}_\nu^k h^k$$

as $h \to 0$. Number $\nu \in \mathbb{Z}_+$ is fixed.

The goal

$$\hat{H}\psi_\nu = E_\nu \psi_\nu, \quad E_\nu \sim h\big(\nu + 1/2\big) + \sum_{k=2}^{\infty} \mathcal{E}_k^\nu h^k$$

1. New method for calculating $\mathcal{E}_k^\nu$ via Taylor coefficients of V based on an analog of Birkhoff normal forms in a non-commutative algebra.
2. New results about the growth of coefficients $\mathcal{E}_k^\nu$.
3. New inverse results: how to recover V from the knowledge of $\mathcal{E}_k^\nu$?

Formal graded Heisenberg algebra

Definition

The formal graded Heisenberg algebra $\mathbb{H}$ is an algebra of formal non-commutative series $\mathbf{F} = \sum_{\mathbf{z}=\mathbf{z}_1\ldots\mathbf{z}_k} f_{\mathbf{z}}\mathbf{z}, \quad \mathbf{z}_j \in \{\mathbf{p}, \mathbf{x}\}, \quad f_{\mathbf{z}} \in \mathbb{C}$, with identities $\mathbf{rp} = \mathbf{pr}$, $\mathbf{rx} = \mathbf{xr}$, where $\mathbf{r} = \mathbf{px} - \mathbf{xp}$.

Remark

The grading: $\mathbb{H} = \prod_{k=0}^{\infty} \mathbb{H}_k$ (or informally $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$). Here $\mathbb{H}_k$ is the space of degree k homogeneous forms. E.g. $\deg \mathbf{p} = \deg \mathbf{x} = 1$, $\deg \mathbf{r} = 2$.

Remark

In quantum mechanics $\mathbf{p} = -ih\frac{d}{dx}$, $\mathbf{r} = -ih$.

Lie algebra structure

Proposition

For any $\mathbf{F}, \mathbf{G}$ *there is a unique* $\mathbf{W}$, *s.t.* $[\mathbf{F}, \mathbf{G}] \equiv \mathbf{FG} - \mathbf{GF} = \mathbf{r}\mathbf{W}$.

Definition

(Quantum) Poisson bracket: $\{\mathbf{F}, \mathbf{G}\} := \mathbf{W}$ (or informally $\{\mathbf{F}, \mathbf{G}\} = \frac{\mathbf{FG}-\mathbf{GF}}{\mathbf{r}}$.)

Remark

$\deg \mathbf{F} = n$, $\deg \mathbf{G} = m \Rightarrow \deg \{\mathbf{F}, \mathbf{G}\} = n + m - 2$.

Definition

Involution: $\left(\sum f_{\mathbf{z}} \mathbf{z}_1 \ldots \mathbf{z}_k\right)^* = \sum \overline{f_{\mathbf{z}}} \mathbf{z}_k \ldots \mathbf{z}_1$. $\mathbf{F}$ is Hermitian, if $\mathbf{F} = \mathbf{F}^*$.

Remark

There is a 'homomorphism into classics' $\pi : \mathbb{H} \to \mathbb{O}$. Here $\mathbb{O}$ is an algebra of formal Taylor series with standard product and formal Poisson bracket.

Normal form in $\mathbb{H}$

Definition

$e^{\mathrm{ad}\mathbf{W}}\mathbf{H} := \sum_{k=0}^{\infty} \frac{\mathrm{ad}\mathbf{W}^k}{k!}\mathbf{H} \equiv \mathbf{H} + \{\mathbf{W},\mathbf{H}\} + \frac{1}{2}\{\mathbf{W},\{\mathbf{W},\mathbf{H}\}\} + \ldots,$

Proposition (D. V. Treschev, 2005)

If $\mathbf{W} \in \prod_{k=3}^{\infty} \mathbb{H}_k$, *then* $e^{\mathrm{ad}\mathbf{W}} : \mathbb{H} \to \mathbb{H}$ *is an automorphism of non-commutative algebra and Lie algebra.*

Proposition (A., 2009)

Assume that

$\mathbf{H} = \mathbf{H}^* = \mathbf{H}_2 + \sum_{k=3}^{\infty} \mathbf{H}_k$, $\mathbf{H}_2 = \frac{1}{2}\left(\mathbf{p}^2 + \mathbf{x}^2\right), \deg \mathbf{H}_k = k$. *There is a Hermitian* $\mathbf{W} = \sum_{k=3}^{2N+1} \mathbf{W}_k$ *s.t.*

$$e^{\mathrm{ad}\mathbf{W}}\mathbf{H} = \mathbf{H}_2 + \mathbf{G} + \mathbf{R}, \quad \{\mathbf{G},\mathbf{H}_2\} = 0, \quad \mathbf{R} \in \prod_{k=2N+2}^{\infty} \mathbb{H}_k.$$

Spectrum via normal form (idea)

Let us think that we deal with operators.

$$e^{\mathrm{ad}_{\mathbf{W}}}\mathbf{H} = U^*\mathbf{N}U,\ \mathbf{N} = \mathbf{H}_2 + \mathbf{G} + \mathbf{R},\ U = e^{-\frac{i}{h}\mathbf{W}}$$

U is unitary $\Rightarrow$ $\mathbf{N}$ is self-adjoint with the same spectrum as $\mathbf{H}$. Let us truncate: $\widetilde{\mathbf{N}} = \mathbf{H}_2 + \mathbf{G}$. Since $[\widetilde{\mathbf{N}}, \mathbf{H}_2] = 0$, the eigenfunctions ψ_ν of $\mathbf{H}_2$ are eigenfunctions of $\widetilde{\mathbf{N}}$.

Lemma (Easy exercise)

If $\mathbf{R}_k \in \mathbb{H}_k$*, then* $\|\mathbf{R}\psi_n u\|_{L^2(\mathbb{R})} = O(h^{k/2})$.

Let $\mathbf{G} = \sum_{k=2}^{N} \mathbf{G}_{2k}$, then $\mathbf{G}_{2k}\psi_\nu = a_k(\nu)h^k\psi_\nu$, and we arrive at

$$E_\nu = h\big(\nu + 1/2\big) + \sum_{k=2}^{N} a_k(\nu)h^k + O(h^{N+1}).$$

Spectrum via normal form

Theorem

Let $\widehat{H} = \frac{\hat{p}^2}{2} + V(\hat{x})$, where $V(x) \geq 0$ is an even degree polynomial s.t. $V(0) = V'(0) = 0$, and $V''(0) = 1$. Let $\widehat{G}(\hat{p}, \hat{x})$ be the differential operator of degree $2N$ that corresponds to the normal form of $\hat{H}$. Then $\widehat{H}$ has a series of eigenvalues $E_\nu = \lambda_\nu + O(h^{N+1})$, where $\hat{G}\psi_\nu = \lambda_\nu \psi_n u$, and ψ_ν are eigenfunctions of $\hat{H}_2 = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}$.

The idea of proof (suggested by V.E. Nazaikinskii) is based on lifting a formal normal form into a pseudo-differential algebra of ordered symbols.

Remark

In a different language: S. vu Ngoc, L. Charles, 2008.

Explicit formulas

Let us expand $\mathbb{H}_k = \bigoplus_{s=0}^{k} \mathbb{H}_{(s,k-s)}$ ($\mathbb{H}_{(s,k-s)}$ is generated by words, where $\mathbf{p}$ appears s times, and $\mathbf{x}$ appears $k-s$ times).

Proposition

The following system is a basis in $\mathbb{H}_{(s,r)}$:

$$\mathbf{p}^s\mathbf{x}^r,\ \mathbf{p}^{s-1}\mathbf{x}^r\mathbf{p},\ \ldots, \mathbf{p}\mathbf{x}^r\mathbf{p}^{s-1},\ \mathbf{x}^r\mathbf{p}^s,\ \textit{if } s \le r.$$
$$\mathbf{x}^r\mathbf{p}^s,\ \mathbf{x}^{r-1}\mathbf{p}^s\mathbf{x},\ \ldots, \mathbf{x}\mathbf{p}^s\mathbf{x}^{r-1},\ \mathbf{p}^s\mathbf{x}^r\ \textit{if } r \le s.$$

Definition

This basis is called primitive.

Remark

A traditional approach of h^k expansions and symbol maps means using the basis: $\mathbf{p}^s\mathbf{x}^r$, $\mathbf{r}\mathbf{p}^{s-1}\mathbf{x}^{r-1}, \ldots,$ $\mathbf{r}^{s-1}\mathbf{p}\mathbf{x}^{r-s+1}$, $\mathbf{r}^s\mathbf{x}^{r-s}$.

Birkhoff normal form: reminder

To reduce a classical Hamiltonian $H=\sum_{k=2}^{\infty}H_k,\ H_2=\frac{p^2+x^2}{2}$ to a normal form, we use steps as follows:

(1) Linear change of variables: $p=\frac{a_-+ia_+}{\sqrt{2}}, \quad x=\frac{ia_-+a_+}{\sqrt{2}}$.

(2) A sequence of near-identity maps: $e^{\mathrm{ad}_{W_3}}, e^{\mathrm{ad}_{W_4}},\ldots$:
$e^{\mathrm{ad}_{W_k}}:\ a_-=\tilde{a}_-+O(\tilde{a}_-,\tilde{a}_+)^{k-1}, a_+=\tilde{a}_++O(\tilde{a}_-,\tilde{a}_+)^{k-1}$

(3) Inverse change of variables to step (1): $(a_-,a_+)\to(p,x)$.

$$H(p,x)=\frac{p^2+x^2}{2}+\sum_{k=3}^{\infty}H_k\overset{(1)}{\to} ia_-a_++\sum_{k=3}^{\infty}\widetilde{H}_k\overset{(2)}{\to} ia_-a_++\sum_{k=3}^{N}\widetilde{G}_k+$$

$$O(a_-,a_+)^{N+1}\overset{(3)}{\to}\frac{p^2+x^2}{2}+\sum_{k=3}^{N}G_k+O(a_-,a_+)^{N+1},$$

$$G_{2k}=g_{2k}(p^2+x^2)^k,\ \widetilde{G}_{2k}=\tilde{g}_{2k}(a_-a_+)^k,\ G_{2k+1}=\widetilde{G}_{2k+1}=0.$$

Non-commutative analog

After the changes of coordinates reducing $\mathbf{H} \in \mathbb{H}$ to a normal form: $(\mathbf{p}, \mathbf{x}) \to (\mathbf{a}_-, \mathbf{a}_+) \to (\tilde{\mathbf{a}}_-, \mathbf{a}_+)$ $\mathbf{H}$ maps to the form:

$$\widetilde{\mathbf{H}} = \frac{i}{2}(\mathbf{a}_-\mathbf{a}_+ + \mathbf{a}_+\mathbf{a}_-) + \sum_{k=2}^{N} \mathbf{G}_{2k}(\mathbf{a}_-, \mathbf{a}_+) + O(\mathbf{a}_-, \mathbf{a}_+)^{2N+2}$$

, where $\mathbf{G}_{2k} = \sum_{s=0}^{2k} \alpha_s^{2k} \mathbf{a}_+^s \mathbf{a}_-^{2k} \mathbf{a}_+^{2k-s}$.

Theorem

The low lying eigenvalues of $\hat{H} = \frac{\hat{p}^2}{2} + V(x)$ *are*
$E_\nu = h\big(\nu + 1/2\big) + \sum_{k=2}^{N} \mathcal{E}_k^\nu h^k + O(h^{N+1})$, *where*

$$\mathcal{E}_k^\nu = k!(-i)^k \sum_{s=0}^{k} \binom{\nu + k - s}{k} \alpha_s^{2k}.$$

Application 1: Growth of coefficients

Proposition (D. V. Treschev, 2005)

For any monomial $\mathbf{z}$ *of type* (r,k) *the linear combination* $\mathbf{z} = \sum_{s=0}^{k} \alpha_s \mathbf{x}^s \mathbf{p}^r \mathbf{x}^{k-s}$ *is convex, i.e.* $\alpha_s \geq 0$ *and* $\sum_{s=0}^{k} \alpha_s = 1$.

Remark

The expansions in h^k are much worse. E.g. in the sum $\mathbf{x}^k \mathbf{p}^k = \sum_{s=0}^{k} \beta_s \mathbf{r}^s \mathbf{p}^{r-s} \mathbf{x}^{k-s}$ also $\sum_{s=0}^{k} \beta_s = 1$ but β_s may be $\sim k!$.

Definition

A norm in $\mathbb{H}_{(r,k)}$ is $\|\mathbf{F}\| := \inf_{\mathbf{z}_s} \sum_{s=0}^{k} |\beta_s|$, where $\mathbf{F} = \sum_{s=0}^{k} \beta_s \mathbf{z}_s$.

Remark

$$\|\mathbf{F}\| := \sum_{s=0}^{k} |\alpha_s|$$

Application 1: Growth of coefficients

Definition

An element $\mathbf{F} = \sum_{r,k=0}^{\infty} \mathbf{F}_{(r,k)}$, where $\mathbf{F}_{(r,k)} \in \mathbb{H}_{(r,k)}$ is analytic, if $\|\mathbf{F}_{(r,k)}\| \leq \beta\gamma^{r+k}$ for some constants $\beta, \gamma > 0$.

Remark

If $\mathbf{F}$ is analytic, then $\pi\mathbf{F}$ is a convergent Taylor series.

Theorem (A., 2009)

In one degree of freedom a normal form of an analytic element is analytic.

Corollary

Coefficients of the normal form in primitive basis: $|\alpha_s^{2k}| \leq \beta\gamma^k$

Application 1: Growth of coefficients

Recall that

$$E_\nu(h) \sim \sum_{k=1}^{\infty} h^k \mathcal{E}_k^\nu, \quad \mathcal{E}_k^\nu = k!(-i)^k \sum_{s=0}^{k} \binom{\nu + k - s}{k} \alpha_s^{2k}$$

Corollary

$|\mathcal{E}_k^\nu| \le (\nu + 1) \dots (\nu + k)\beta\gamma^k \le k!\beta\gamma^k$.

Example: $\mathcal{E}_k^0 = k!(-i)^k \alpha_k^{2k}$.

Question

What can be said about the remainder $\left| E_\nu(h) - \sum_{k=1}^{N} h^k \mathcal{E}_k^\nu \right|$*?*

Recall that $V(x)$ is an even degree polynomial s.t. $V(0) = V'(0) = 0$ and $V''(0) = 1$.

Proposition

Denote $\gamma^{\nu}_{k,s} = \binom{\nu+k-s}{k}\binom{k}{s}$ *and* $\tau^k_s = 2k^2 + 7k + 16s(k-s)$, *then*

$$\mathcal{E}^{\nu}_k = \frac{(2k)!}{2^{2k}k!}\left[u_{2k}\sum_{s=0}^{k}\gamma^{\nu}_{k,s} - \left(\sum_{s=0}^{k}\gamma^{\nu}_{k,s}\tau^k_s\right)\frac{u_3 u_{2k-1}}{3k}\right] + \widetilde{\mathcal{E}}^{\nu}_k$$

where $\widetilde{\mathcal{E}}^{\nu}_k$ *depends only on* $u_3, \ldots, u_{2k-2}$.

Proposition

$$\begin{vmatrix} \sum_{s=0}^{k}\gamma^{\nu}_{k,s} & \sum_{s=0}^{k}\gamma^{\nu}_{k,s}\tau^k_s \\ \sum_{s=0}^{k}\gamma^{\mu}_{k,s} & \sum_{s=0}^{k}\gamma^{\mu}_{k,s}\tau^k_s \end{vmatrix} \neq 0 \text{ for all } \mu \neq \nu \text{ and } k \geq 2.$$

Application 2: Inverse spectral problem

Theorem

Fix $\nu \neq \mu$. In the class of potentials s.t. $V^{(3)}(0) \neq 0$, sequences $\mathcal{E}_k^\mu$ and $\mathcal{E}_k^\nu$ determine $V(x)$ uniquely up to the $x \to -x$ symmetry.

Remark

Known results: $V(x)$ is determined by $\mathcal{E}_k^\mu$ for all $\mu \geq 0$ and $k \geq 2$ (Y. Colin de Verdiere, H. Hezari, 2008). (The assumption $V^{(3)}(0) \neq 0$ is also imposed).

Remark

In the degenerate case $V^{(3)}(0) = 0$ one should consider first $V^{(5)}(0) \neq 0$. This leads also to a 2×2 determinant but more complicated.