Universal formal groups whose exponentials are elliptic functions of level n.

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Recall classical definitions

An **elliptic function** is a meromorphic function in $\mathbb{C}$ that is doubly periodic:

$$f(z+2\omega_1) = f(z), \quad f(z+2\omega_2) = f(z), \quad \mathrm{Im}\frac{\omega_2}{\omega_1} \neq 0.$$

Properties

For any nonconstant elliptic functions $f(z)$, $g(z)$,
— there exists a polynomial $P(x_1, x_2)$, such that $P(f(z), g(z)) \equiv 0$,
— any elliptic function $h(z)$ is a rational function of $f(z)$ and $f'(z)$.

Often one takes $f(z) = \wp(z; g_2, g_3)$ — the solution of

$$f'(z)^2 = 4f(z)^3 - g_2 f(z) - g_3.$$

Corollary

Any elliptic function $h(z)$ is a rational function of $\wp(z)$ and $\wp'(z)$.

Weierstrass functions

$\wp(z; g_2, g_3)$ is the unique elliptic function with periods $2\omega_1$, $2\omega_2$ and poles only in lattice points such that $\lim\limits_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0$.

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

The Weierstrass ζ-function is defined by

$$\zeta(z; g_2, g_3)' = -\wp(z; g_2, g_3), \qquad \lim_{z\to 0}(z\zeta(z)) = 1.$$

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\zeta(\omega_k).$$

The Weierstrass σ-function is defined by

$$\big(\ln\sigma(z; g_2, g_3)\big)' = \zeta(z; g_2, g_3), \qquad \lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right) = 1.$$

$$\sigma(z + 2\omega_k) = -\exp(2\zeta(\omega_k)(z + \omega_k))\sigma(z).$$

Definition: Elliptic function of level n

A meromorphic function $f_n(x)$ in $\mathbb{C}$ with periodic properties

$$f_n(x+2\omega_1) = f_n(x), \qquad f_n(x+2\omega_2) = \sqrt[n]{1} f_n(x), \qquad \omega_2/\omega_1 \notin \mathbb{R},$$

where $(\sqrt[n]{1})^n = 1$, $(\sqrt[n]{1})^k \neq 1$ for $0 < k < n$,
with one simple pole
on the parallelogram of periods with generators $2\omega_1$, $2\omega_2$.

Such a function will be a true elliptic function
for the parallelogram $2\omega_1$, $2n\omega_2$.

We assume

$$f_n(0) = 0, \quad f_n'(0) = 1.$$

Elliptic functions of level n exist for any $n \geqslant 2$.

Description: Elliptic function of level n

For a lattice L in $\mathbb{C}$ (with generators $2\omega_1$, $2\omega_2$) consider the elliptic function $g(x)$ with divisor $N \cdot 0 - N \cdot z$.
We demand that $g(x) = x^N + \ldots$.
Set $f(x) = g(x)^{1/N}$ where $f(x) = x + \ldots$.

The function $f(x)$ is elliptic with respect to a sublattice L' of L.

$$f(x) = \frac{\exp(\alpha x)}{\Phi(x; z)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z)x),$$

where $z = 2\frac{k}{N}\omega_1 + 2\frac{m}{N}\omega_2, \qquad \alpha = -2\frac{k}{N}\zeta(\omega_1) - 2\frac{m}{N}\zeta(\omega_2) + \zeta(z).$

Realization: Elliptic function of level n

Realization by I.M. Krichever
in terms of Baker–Akhiezer function $\Phi(x; z)$

$$f_n(x) = \frac{\exp(\alpha x)}{\Phi(x; z)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z)x), \qquad (1)$$

where

$$z = 2\frac{k}{n}\omega_1, \qquad \alpha = -2\frac{k}{n}\zeta(\omega_1) + \zeta(z),$$

$0 < k < n$, and k, n are coprime.
The periodic properties will follow from

$$\Phi(x + 2\omega_k; z) = \Phi(x; z) \exp\left(2\zeta(z)\omega_k - 2\zeta(\omega_k)z\right).$$

Thus, the pole of $f_n(x)$ is at $z = 2\frac{k}{n}\omega_1$.

Lemma

For the Baker-Akhiezer function $\Phi(x) = \Phi(x; z)$

$$\Phi(x)\Phi'''(x) - 3\Phi'(x)\Phi''(x) = -6\wp(z)\Phi(x)\Phi'(x) - 2\wp'(z)\Phi(x)^2.$$

Corollary

For
$$f(x) = \frac{\exp(\alpha x)}{\Phi(x; z)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z)x),$$

we have
$$f(x)f'''(x) - 3f'(x)f''(x) = C_1 f'(x)^2 + C_2 f(x)f'(x) + C_3 f(x)^2,$$

where $C_1 = -6\alpha$, $C_2 = 6(\alpha^2 - \wp(z))$,
$C_3 = 2(\wp'(z) + 3\alpha\wp(z) - \alpha^3)$.

$$f(x) = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} e^{\alpha x - \zeta(z)x} \in \mathbb{Q}[\alpha, \wp(z), \wp'(z), g_2/2][[x]].$$

Isogenies of elliptic curves.

Recall Weierstrass $\wp$-function with periods $2\omega_1, 2\omega_2$ satisfies

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3 \quad \text{for}$$

$$g_2 = 60 \sum_{(k,m)\neq(0,0)} (2k\omega_1 + 2m\omega_2)^{-4}, \quad g_3 = 140 \sum_{(k,m)\neq(0,0)} (2k\omega_1 + 2m\omega_2)^{-6}.$$

Lemma (for $2\widetilde{\omega}_1 = 2\omega_1, 2\widetilde{\omega}_2 = 4\omega_2$)

$$\wp(x; g_2, g_3) = \frac{(\wp(x;\widetilde{g}_2,\widetilde{g}_3) - a_1)(\wp(x;\widetilde{g}_2,\widetilde{g}_3) - a_2)}{\wp(x;\widetilde{g}_2,\widetilde{g}_3) - b_1}$$

$$b_1 = a_1 + a_2,$$

$$g_2 = 4(a_1 + 3a_2)(3a_1 + a_2), \qquad \widetilde{g}_2 = g_2 - 20a_1a_2,$$

$$g_3 = -8(a_1 + a_2)(a_1 - a_2)^2, \qquad \widetilde{g}_3 = g_3 - 28a_1a_2b_1,$$

$$\Delta = 256a_1a_2(9b_1^2 - 4a_1a_2)^2, \qquad \widetilde{\Delta} = 16a_1^2a_2^2(9b_1^2 - 4a_1a_2).$$

Corollary

Any elliptic function $h(z)$ is a rational function of $\wp(z)$ and $\wp'(z)$.

Consider

$$f_2(x) = \frac{\sigma(x)\sigma(\omega_2)}{\sigma(\omega_2 - x)} \exp(-\zeta(\omega_2)x).$$

For the parallelogram of periods with generators $2\omega_1, 2\widetilde{\omega}_2 = 4\omega_2$:
Zeroes: 0, $2\omega_2$; poles: ω_1, $\omega_1 + 2\omega_2$. Thus for $c = \wp(2\omega_2; \widetilde{g}_2, \widetilde{g}_3)$

$$f_2(x) = \frac{-2(\wp(x; \widetilde{g}_2, \widetilde{g}_3) - c)}{\wp'(x; \widetilde{g}_2, \widetilde{g}_3)}$$

For $\beta = \wp(\omega_1; g_2, g_3)$

$$4\beta^3 - g_2\beta - g_3 = 0, \qquad \widetilde{g}_2 = -\frac{1}{4}g_2 + 15c^2,$$

$$c = -\frac{1}{2}\beta, \qquad \widetilde{g}_3 = 4c^3 - c\widetilde{g}_2.$$

For the parallelogram of periods with generators $2\omega_1, 2\widetilde{\omega}_2 = 4\omega_2$:

$$f_2(x) = \frac{-2(\wp(x;\widetilde{g}_2,\widetilde{g}_3) - c)}{\wp'(x;\widetilde{g}_2,\widetilde{g}_3)} = \frac{\wp'(x;\widetilde{g}_2,\widetilde{g}_3)}{-2(\wp(x;\widetilde{g}_2,\widetilde{g}_3) - a)(\wp(x;\widetilde{g}_2,\widetilde{g}_3) - b))}.$$

For $\beta = \wp(\omega_1; g_2, g_3)$, $c = \wp(2\omega_2;\widetilde{g}_2,\widetilde{g}_3)$, $a = \wp(\omega_1;\widetilde{g}_2,\widetilde{g}_3)$,
$b = \wp(\omega_1 + 2\omega_2;\widetilde{g}_2,\widetilde{g}_3)$

$$4\beta^3 - g_2\beta - g_3 = 0, \qquad \widetilde{g}_2 = -\frac{1}{4}g_2 + 15c^2,$$
$$c = -\frac{1}{2}\beta, \qquad \widetilde{g}_3 = 4c^3 - c\widetilde{g}_2.$$

and additional relations

$$c = -a - b, \quad \widetilde{g}_2 = 4(a^2 + ab + b^2), \quad \widetilde{g}_2 = 4abc.$$

Elliptic function of level 4

$$f_4(x) = \frac{(\wp(x) - a_1)(-\frac{1}{2}\wp'(x)(\wp(x) - a_2) + \alpha(\wp(x) - a_3)(\wp(x) - a_4))}{(\wp(x) - b_1)(\wp(x) - b_2)(\wp(x) - b_3)(\wp(x) - b_4)},$$

where $a_k \neq b_k$.

$$f_4(x) = \frac{1}{32}\frac{\alpha(4\wp(x) + 3\alpha^2 - \beta)(2\wp(x) - \alpha^2 + \beta)(4\wp(x) - 7\alpha^2 + 5\beta)}{\wp(x)^4 + c_1\wp(x)^3 + c_2\wp(x)^2 + c_3\wp(x) + c_4} - \\ - \frac{1}{32}\frac{\wp'(x)(4\wp(x) + 3\alpha^2 - \beta)^2}{\wp(x)^4 + c_1\wp(x)^3 + c_2\wp(x)^2 + c_3\wp(x) + c_4}.$$

N-th special Hirzebruch functional equation

Elliptic functions of level n for $n \mid N+1$ are solutions of N-*th special Hirzebruch functional equation*

$$\sum_{j=1}^{N+1} \prod_{i \neq j} \frac{1}{f(x_i - x_j)} = 0. \qquad (2)$$

(F. Hirzebruch comparing a paper by P.S. Landweber with a paper by S. Ochanine)

We assume $f(0) = 0, f'(0) = 1$.

Lemma

Elliptic function of order n is odd if and only if $n = 2$.

Corollaries from Hirzebruch functional equations

N-th: $$\sum_{j=1}^{N+1} \prod_{i \neq j} \frac{1}{f(x_i - x_j)} = 0, \qquad f(0) = 0, f'(0) = 1.$$

Corollary

Each odd solution of 3-rd special Hirzebruch functional equation is elliptic function of level 2.

Corollary

Each solution of 2-nd special Hirzebruch functional equation is elliptic function of level 3.

Corollary

Each non-odd solution of 3-rd special Hirzebruch functional equation is elliptic function of level 4.

Formal group

Let R be a commutative ring with unity 1.
A commutative one-dimensional **formal group** over R is a formal series

$$F(u,v) = u + v + \sum_{i,j>0} a_{i,j} u^i v^j, \qquad a_{i,j} \in R,$$

with conditions

$$F(v,u) = F(u,v), \qquad F(u, F(v,w)) = F(F(u,v),w).$$

It's **exponential** $f(x) \in R \otimes \mathbb{Q}[[x]]$, $f(0) = 0, f'(0) = 1$, is determined by the addition law

$$f(x+y) = F(f(x), f(y)).$$

Buchstaber's formal group

$$F(u, v) = \frac{u^2 A(v) - v^2 A(u)}{uB(v) - vB(u)}, \quad A(0) = B(0) = 1. \qquad (3)$$

Here $A(u), B(u)$ are formal series over some coefficient ring R. As (3) does not depend on $B'(0)$ and $A''(0)$, set $B'(0) = A''(0) = 0$.

Theorem (V.M. Buchstaber, 1990)

The exponential of the universal formal group of the form (3) is

$$f(x) = \frac{\exp(\alpha x)}{\Phi(x; z)} = \frac{\sigma(x)\sigma(z)}{\sigma(z - x)} e^{\alpha x - \zeta(z) x}.$$

Elliptic function of level 2.

Elliptic function of level 2 is Jacobi's elliptic sine $sn(x)$.

$$f'(x)^2 = 1 - 2\delta f(x)^2 + \varepsilon f(x)^4, \quad f(0) = 0, f'(0) = 1.$$

It determines Oshanine-Witten elliptic genus.

We have

$$f_2(x) = \frac{\sigma(x)\sigma(\omega_2)}{\sigma(\omega_2 - x)} \exp(-\zeta(\omega_2)x).$$

Theorem

$f_2(x)$ is the exponential of the universal formal group of the form

$$F(u,v) = \frac{u^2 - v^2}{uB(v) - vB(u)}, \quad B(0) = 1.$$

Note: here $A(u) = 1$.

Elliptic function of level 3.

Elliptic function of level 3 is the solution of

$$f'(x)^2(f'(x)+3af(x)) = \left((a^3+3b)f(x)^3+1\right)^2 - 12a^3bf(x)^6$$

with $f(0)=0, f'(0)=1$.
We have

$$f_3(x) = \frac{\sigma(x)\sigma(\frac{2}{3}\omega_1)}{\sigma(\frac{2}{3}\omega_1 - x)}\exp(\alpha x - \zeta(\frac{2}{3}\omega_1)x),$$

Theorem (V.M. Buchstaber, E. Yu. Bunkova)

$f_3(x)$ is the exponential of the universal formal group of the form

$$F(u,v) = \frac{u^2A(v) - v^2A(u)}{uA(v)^2 - vA(u)^2}, \qquad A(0)=1, \quad A''(0)=0.$$

Note: here $B(u) = A(u)^2 - 2A'(0)u$.

Universal formal groups for elliptic functions of level n.

Set $A(0) = B(0) = 1, B'(0) = A''(0) = 0, A'(0) = A_1, B''(0) = B_2$.

Theorem

$f_n(x)$ is the exponential of the universal formal group of the form

$$F(u, v) = \frac{u^2 A(v) - v^2 A(u)}{uB(v) - vB(u)},$$

with:

$$\begin{aligned} &\textit{Level 2}, & 1 &= A(u), \\ &\textit{Level 3}, & B(u) &= A(u)^2 - 2A_1 u, \\ &\textit{Level 4}, & (2B(u) + 3A_1 u)^2 &= 4A(u)^3 - (3A_1^2 - 8B_2)u^2 A(u)^2. \end{aligned}$$

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