Integrability of certain restricted systems of material points

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Outline

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Open n-chain

- $(n+1)$ points $(m_i, \boldsymbol{q}_i)$, $i = 0, \ldots, n$,
- n holonomic constraints:

$$\|\boldsymbol{q}_i - \boldsymbol{q}_{i-1}\| = l_i > 0, \quad \text{for} \quad i = 1, \ldots, n.$$

- configuration space:

$$(\boldsymbol{r}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times \cdots \times \mathbb{S}^{d-1}$$

  where

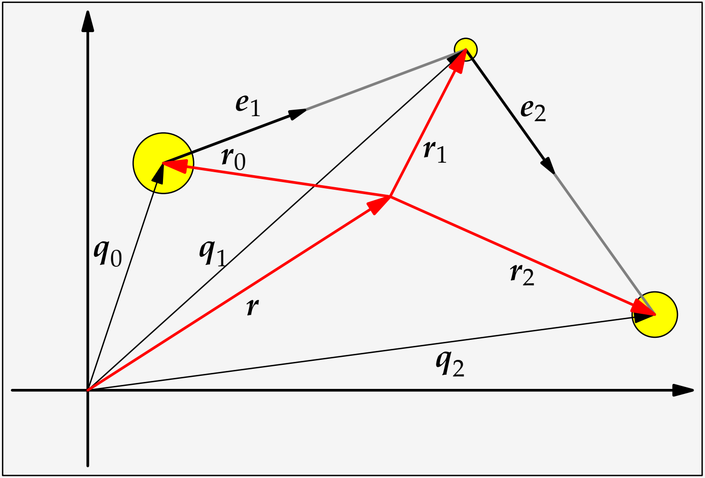
$$\boldsymbol{r} = \frac{1}{m} \sum_{i=0}^{n} m_i \boldsymbol{q}_i, \quad \text{where} \quad m = \sum_{i=0}^{n} m_i.$$

  end

$$\boldsymbol{e}_i = \frac{1}{l_i} \left(\boldsymbol{q}_i - \boldsymbol{q}_{i-1} \right), \quad \text{for} \quad i = 1, \ldots, n,$$

- number of degrees of freedom: $d + n(d-1)$

A 2-chain in $\mathbb{R}^2$



Lagrange function

Kinetic energy

$$T = \frac{1}{2}m\|\dot{\boldsymbol{r}}\|^2 + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} I_{\alpha,\beta}\,\dot{\boldsymbol{e}}_\alpha \cdot \dot{\boldsymbol{e}}_\beta.$$

where

$$I_{\alpha,\beta} := l_\alpha l_\beta\left(\sigma_\gamma - \frac{\sigma_\alpha\sigma_\beta}{m}\right) \quad \text{with} \quad \gamma = \max(\alpha,\beta).$$

and

$$\sigma_\alpha := \sum_{k=\alpha}^{n} m_k.$$

Example: n-chain in constant gravity field

$$V = -g\sum_{i=0}^{n} m_i \boldsymbol{n}\cdot\boldsymbol{q}_i = -g\sum_{i=0}^{n} m_i \boldsymbol{n}\cdot(\boldsymbol{r}+\boldsymbol{r}_i) = -mg\,\boldsymbol{n}\cdot\boldsymbol{r}$$

Equations of motion:

$$m\ddot{\boldsymbol{r}} = -mg\boldsymbol{n},$$

and

$$\sum_{\beta=1}^{n} I_{\alpha,\beta}\ddot{\boldsymbol{e}}_\beta + \frac{\partial V}{\partial \boldsymbol{e}_\alpha} = \lambda_\alpha \boldsymbol{e}_\alpha, \qquad \alpha = 1,\ldots,n,$$

Integrability of free n-chain

1. For a free n-chain $\boldsymbol{r}$ is cyclic.
2. Reduced system has $n \cdot (d-1)$ degrees of freedom.
3. Reduced system has $\mathrm{SO}(d, \mathbb{R})$ symmetry group.

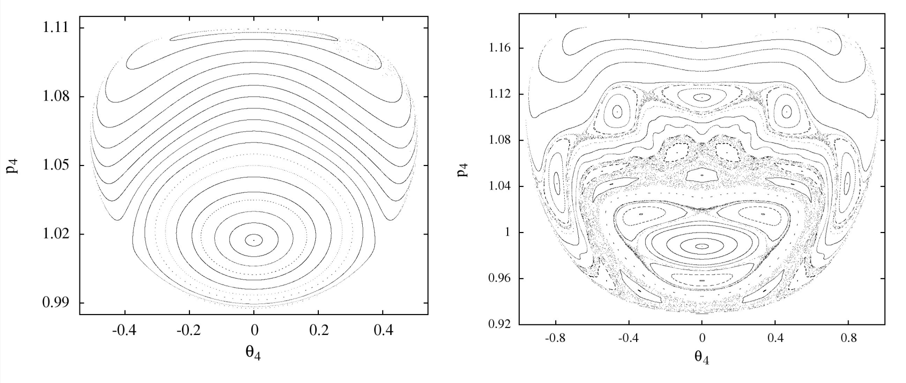
The simplest case: planar n-chain

Conjecture

Arbitrary planar n-chain with $n \geq 3$ is not integrable.

Problem: number of parameters: $n(n+1)/2$.

Examples



1.11
1.08
1.05
1.02
0.99
p_4
-0.4
-0.2
0
0.2
0.4
θ_4
1.16
1.12
1.08
1.04
1
0.96
0.92
p_4
-0.8
-0.4
0
0.4
0.8
θ_4

n-chain in a circular orbit

Lagrange function

$$L = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} I_{\alpha,\beta}\left(\dot{\boldsymbol{s}}_\alpha + \boldsymbol{\omega}\times\boldsymbol{s}_\alpha\right)\cdot\left(\dot{\boldsymbol{s}}_\beta + \boldsymbol{\omega}\times\boldsymbol{s}_\beta\right) - V,$$

where

$$V = -\frac{1}{2}\omega^2\sum_{\alpha,\beta=1}^{n} I_{\alpha,\beta}\left[3(\boldsymbol{s}\cdot\boldsymbol{s}_\alpha)(\boldsymbol{s}\cdot\boldsymbol{s}_\beta) - \boldsymbol{s}_\alpha\cdot\boldsymbol{s}_\beta\right],$$

Planar 2-chain in orbit

Coordinates (φ_1, φ_2):

$$\boldsymbol{s}_1 = (\cos\varphi_1, \sin\varphi_1, 0)^T, \qquad \boldsymbol{s}_2 = (\cos\varphi_2, \sin\varphi_2, 0)^T.$$

$$L = \frac{1}{2}\left[I_{1,1}\left(\dot{\varphi}_1 + 1\right)^2 + I_{2,2}\left(\dot{\varphi}_2 + 1\right)^2\right] + \\ I_{1,2}\cos(\varphi_1 - \varphi_2)\left(\dot{\varphi}_1 + 1\right)\left(\dot{\varphi}_2 + 1\right) - V(\varphi_1, \varphi_2),$$

where

$$V(\varphi_1, \varphi_2) = -\frac{1}{2}\sum_{\alpha,\beta=1}^{2} I_{\alpha,\beta}\left[3\cos\varphi_\alpha\cos\varphi_\beta - \cos(\varphi_\alpha - \varphi_\beta)\right].$$

Planar 2-chain in orbit

Parameters:

$$I_{1,1} = \frac{m_0(m_1 + m_2)l_1^2}{m_0 + m_1 + m_2}, \qquad I_{2,2} = \frac{m_2(m_0 + m_1)l_2^2}{m_0 + m_1 + m_2},$$

$$I_{1,2} = \frac{m_0 m_2 l_1 l_2}{m_0 + m_1 + m_2},$$

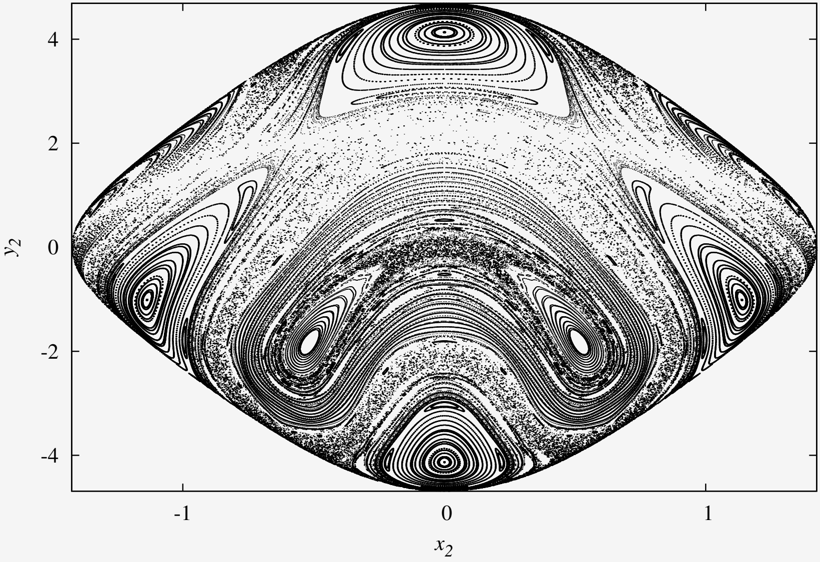
$$\lambda_1 = \frac{I_{1,1}}{I_{1,2}}, \qquad \lambda_2 = \frac{I_{2,2}}{I_{1,2}}, \qquad \lambda = \lambda_1 + \lambda_2,$$

Planar 2-chain in orbit

Non-canonical variables $(x_1, x_2) := (\varphi_1, \varphi_2 - \varphi_2)$, $(y_1, y_2) := (\dot{x}_1, \dot{x}_2)$. Energy integral

$$H = \frac{1}{2}\left[(\lambda + 2\cos x_2)y_1^2 + 2y_1y_2(\lambda_2 + \cos x_2) + \lambda_2 y_2^2\right] -$$
$$\frac{3}{4}\left[\lambda + \lambda_1 \cos(2x_1) + \lambda_2 \cos(2(x_1 + x_2)) + 2\cos x_2 + 2\cos(2x_1 + x_2)\right]$$

Planar 2-chain in orbit



Main result

Theorem

The system does not admit an additional meromorphic first integral.

Main tool: differential Galois theory.

Theorem (Morales-Ramis)

If a Hamiltonian system is integrable in the Liouville sense with meromorphic first integrals, then the differential Galois group of variational equations along a particular solution is virtually Abelian.

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.
- Audin, M., *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés 8, Collection SMF, SMF et EDP Sciences, Paris, 2001.

Morales-Ramis theorem for Hamiltonian systems

Hamiltonian system

$$\dot{\boldsymbol{z}} = JH'(\boldsymbol{z}), \quad J = \begin{bmatrix} \mathbf{0} & \boldsymbol{E}_n \\ -\boldsymbol{E}_n & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{z} = [\boldsymbol{q}, \boldsymbol{p}]^T \in \mathbb{C}^{2n}$$

$H(\boldsymbol{z})$ – a holomorphic function
VE along a particular solution $\boldsymbol{\varphi}(t)$

$$\dot{\boldsymbol{y}} = JH''(\boldsymbol{\varphi}(t))\boldsymbol{y}.$$

The differential Galois group of VEs is a subgroup of $\mathrm{Sp}(2n, \mathbb{C})$.

Particular solutions

Two invariant planes

$$\mathcal{N}_k = \left\{ (x_1, x_2, y_1, y_2) \in \mathbb{C}^4 \mid \ x_2 = (k-1)\pi, \quad y_2 = 0 \right\},$$

where $k = 1, 2$. The system restriction to $\mathcal{N}_k$ reads

$$\dot{x} = y, \qquad \dot{y} = -\frac{3}{2}\sin(2x), \qquad (x, y) = (x_1, y_1).$$

These equations have the energy integral

$$h = \frac{y^2}{2} - \frac{3}{4}\cos(2x).$$

Phase curve $\Gamma(\eta)$ corresponding to level $h(x, y) = \eta$, and $\Gamma_k(\eta) \subset \mathcal{N}_k$,

Variational equations

$$\ddot{X} - b_k(x,y)X = 0, \qquad (x,y) \in \Gamma_k(\eta), \quad k = 1,2.$$

where:

$$b_k(x,y) = c_k\,(3 + 2y(2+y)) + 3(c_k - 1)\cos(2x),$$

and

$$c_k = \frac{\varepsilon_k \lambda - 2}{2(\lambda_1\lambda_2 - 1)}, \qquad \varepsilon_k = (-1)^k,$$

Variational equations

Change of dependent variables:

$$t \longmapsto z = \frac{1}{\sqrt{3}} y(t)$$

$$X'' + P(z)X' + Q_k(z)X = 0, \tag{1}$$

where

$$P(z) = \frac{\ddot{z}}{(\dot{z})^2} = \frac{2z}{2z^2 - \eta + 1} + \frac{2z}{2z^2 - \eta - 1},$$

$$Q_k(z) = -\frac{b_k}{(\dot{z})^2} = \frac{4\left[-6z^2 + c_k(3 + 4z(\sqrt{3} + 3z) - 3\eta) + 3\eta\right]}{3\left[(\eta - 2z^2)^2 - 1\right]}.$$

Variational equations

Set $\eta = 1$ and change independent variable

$$X = w \exp\left[-\frac{1}{2}\int^{z} P(v)dv\right], \tag{2}$$

which convert equations (1) into its reduced form

$$w'' = r_k(z)w, \qquad r_k(z) = \frac{1}{2}P'(z) + \frac{1}{4}P(z)^2 - Q_k(z).$$

$$r_k(z) = \frac{3/4}{z^2} - \frac{3/16}{(z-1)^2} - \frac{3/16}{(z+1)^2} + \frac{4c_k}{\sqrt{3}z} + \frac{39 - 32c_k(3+\sqrt{3}z)}{24(z^2-1)}$$

r_k is a symmetric function of λ_1 and λ_2 as

$$c_k = \frac{\varepsilon_k\lambda - 2}{2\sigma}, \quad \text{where} \quad \sigma := \lambda_1\lambda_2 - 1, \quad \lambda := \lambda_1 + \lambda_2.$$

Main steps

Differential Galois group $\mathcal{G}_k$ of $w'' = r_k(z)w$ is algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$.

Lemma (A)

If $\mathcal{G}$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$, then (up to conjugation):

1. $$\mathcal{G} \subset \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$$
2. $$\mathcal{G} \subset \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \cup \begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix}$$
3. *$\mathcal{G}$ is classical finite group (tetrahedral,octahedral or icosahedral);*
4. $\mathcal{G} = \mathrm{SL}(2,\mathbb{C})$

Logarithmic term

Equations $w'' = r_k(z)w$ have four themed singular point $\{0, \pm 1, \infty\}$ (i.e., they are Heun equations). The differences of exponents at these points are:

$$\Delta_0 = 2, \qquad \Delta_{\pm 1} = \frac{1}{2}, \quad \Delta_\infty = \delta_k,$$

where

$$\delta_1 = \sqrt{9 + 8\frac{2+\lambda}{\sigma}}, \qquad \delta_2 = \sqrt{9 + 8\frac{2-\lambda}{\sigma}}.$$

Solutions near $z = 0$:

$$w_1(z) = z^{3/2} f(z), \qquad w_2(z) = g w_1(z) \ln z + z^{-1/2} h(z),$$

where $f(z)$ and $h(z)$ are holomorphic at $z = 0$ and $f(0) \neq 0$.

Logarithmic term

$$w_1(z) = z^{3/2} f(z), \qquad w_2(z) = g w_1(z) \ln z + z^{-1/2} h(z),$$

where

$$g = \frac{1}{6}\left(8c_k^2 - 6c_k + 3\right). \tag{3}$$

so, $g > 0$ for arbitrary $c_k \in \mathbb{R}$. Monodromy around $z = 0$

$$M_0 := \begin{bmatrix} -1 & -2\pi\mathrm{i} \\ 0 & -1 \end{bmatrix}.$$

This excludes case 2 and 3 from Lemma A for both variational equations.

Elimination of reducible case.

If the first case of Lemma A occurs then the considered variational equation has a solution of the form

$$w = P(z) \exp\left[\int^{z} \omega(s)\mathrm{d}s\right], \qquad P \in \mathbb{C}[z], \quad \omega(z) \in \mathbb{C}(z).$$

Analysis around singularities gives restriction $\delta_1, \delta_2 \in \mathbb{Z}$.
Physical restrictions on (λ, σ), limits $\delta_2 \in \{0, 1, 2\}$ but for these case solution of prescribed form does not exist.
Conclusion: $\mathcal{G}_k = \mathrm{SL}(2, \mathbb{C})$.
Maciejewski A., J. and M. Przybylska, *Non-integrability of multibody chain in circular orbit*, submitted for publication.

Outline

A dumbbell on a sphere

Consider two points $(m_1, \boldsymbol{r}_1)$ and $(m_2, \boldsymbol{r}_2)$ with $\boldsymbol{r}_i \in \mathbb{S}^2$, and holonomic constrain $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = c_1$.
This system is equivalent to a (flat) rigid body with a fixed point.

Add a third point $(m_3, \boldsymbol{r}_3)$ with constrain $\boldsymbol{r}_2 \cdot \boldsymbol{r}_3 = c_2$.
This system is equivalent to a rigid body with a fixed point and one point moving along a circle fixed in the body.

Outline

Anisotropic Kepler problems

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2 + \mu y^2}}$$

- Introduced in [1]
- Investigated in many papers e.g. [2] [3]
- non-integrability proofs using Morales-Ramis theory in [4]

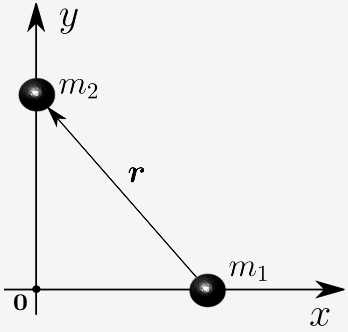
[1] Gutzwiller M.C., Bernoulli sequences and trajectories in the anisotropic Kepler problem *J. Math. Phys* 18(4):806–823, 1977.

[2] Casasayas J., Llibre J, Qualitative Analysis of the Anisotropic Kepler Problem, *Memoirs of Amer. Math. Soc.* 52, no. 312, 1984.

[3] Devaney R. L., Blowing up Singularities in Classical Mechanical Systems, *Amer. Math. Monthly*, 89:535–552, 1982.

[4] Arribas M., Elipe A., Riaguas, A, Non-integrability of anisotropic quasi-homogeneous Hamiltonian systems, *Mech. Res. Comm.*, 30(3):209–216, 2003.

Relation to anisotropic Kepler problem



$$H = \frac{p_x^2}{2m_1} + \frac{p_y^2}{2m_2} - \frac{a}{\sqrt{x^2+y^2}}.$$

after rescaling

$$p_x = \sqrt{m_1}p_1, \quad p_y = \sqrt{m_2}p_2,$$
$$x = \frac{1}{\sqrt{m_1}}q_1, \quad y = \frac{1}{\sqrt{m_2}}q_2$$

$$\widetilde{H} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{a}{\sqrt{\mu_1 q_1^2 + \mu_2 q_2^2}},$$

$$\mu_1 = 1/\sqrt{m_1}, \quad \mu_2 = 1/\sqrt{m_2}.$$

Generalised anisotropic Kepler problem

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \left(\mu_1 q_1^2 + \mu_2 q_2^2\right)^{-n}, \qquad 2n \in \mathbb{Z},$$

Theorem

Hamiltonian system generated by H is integrable in the Liouville sense with meromorphic first integrals in three cases:

1. *radial case when* $\mu_1 = \mu_2$,
2. *uncoupled linear oscillators case when* $n = -1$,
3. *degree* $k = -2$ *potential when* $n = 1$.

In the remaining cases the system is not integrable.

Difficult case

Proposition

Hamiltonian system given by

$$\widetilde{H} = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \left(q_1^2 - q_2^2\right)^{-3/2}, \tag{4}$$

is not integrable.

Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

V — homogeneous of degree k

Particular solutions

Definition

Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}$$

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d},\ \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

On the energy level:

$$H(\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d}) = e \in \mathbb{C}^{\star},$$

hyperelliptic curve

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \quad \varepsilon = ke \in \mathbb{C}^{*}.$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2} V''(\boldsymbol{d})\boldsymbol{x},$$

where $V''(\boldsymbol{d})$ is the Hessian of V calculated at $\boldsymbol{d}$. If $V''(\boldsymbol{d})$ is diagonalisable

$$\ddot{\eta}_i = \lambda_i \varphi(t)^{k-2} \eta_i, \qquad i = 1, \ldots, n,$$

where λ_i for $i = 1, \ldots, n$ are eigenvalues of V''.

By homogeneity of V, $\lambda_n = k - 1$.

Other Morales-Ramis Theorem

If the Hamiltonian system with homogeneous potential is meromorphically integrable, then each (k, λ_i) belong to the following list:

1. $\left(k, p + \frac{k}{2}p(p-1)\right)$,
2. $\left(k, \frac{1}{2}\left[\frac{k-1}{k} + p(p+1)k\right]\right)$,
3. $(2, \text{arbitrary})$,
4. $(-2, \text{arbitrary})$,
5. $\left(3, -\frac{1}{24} + \frac{1}{6}(1+3p)^2\right)$,
6. $\left(3, -\frac{1}{24} + \frac{3}{32}(1+4p)^2\right)$,
7. $\left(3, -\frac{1}{24} + \frac{3}{50}(1+5p)^2\right)$,
8. $\left(3, -\frac{1}{24} + \frac{3}{50}(2+5p)^2\right)$,

$$9.\quad \left(4, -\frac{1}{8} + \frac{2}{9}\,(1+3p)^2\right), \qquad 10.\quad \left(5, -\frac{9}{40} + \frac{5}{18}\,(1+3p)^2\right),$$

$$11.\quad \left(5, -\frac{9}{40} + \frac{1}{10}\,(2+5p)^2\right), \quad 12.\quad \left(-3, \frac{25}{24} - \frac{1}{6}\,(1+3p)^2\right),$$

$$13.\quad \left(-3, \frac{25}{24} - \frac{3}{32}\,(1+4p)^2\right), \quad 14.\quad \left(-3, \frac{25}{24} - \frac{3}{50}\,(1+5p)^2\right),$$

$$15.\quad \left(-3, \frac{25}{24} - \frac{3}{50}\,(2+5p)^2\right), \quad 16.\quad \left(-4, \frac{9}{8} - \frac{2}{9}\,(1+3p)^2\right),$$

$$17.\quad \left(-5, \frac{49}{40} - \frac{5}{18}\,(1+3p)^2\right), \quad 18.\quad \left(-5, \frac{49}{40} - \frac{1}{10}(2+5p)^2\right),$$

Easy part

Potential $V = -\left(\mu_1 q_1^2 + \mu_2 q_2^2\right)^{-n}$ has two Darboux points

$$\boldsymbol{d}_1 = \left(0, (2n)^{\frac{1}{2+2n}} \mu_2^{-\frac{n}{2+2n}}\right), \qquad \boldsymbol{d}_2 = \left((2n)^{\frac{1}{2+2n}} \mu_1^{-\frac{n}{2+2n}}, 0\right).$$

Hessians of potential calculated at these points are diagonal matrices

$$V''(\boldsymbol{d}_1) = \operatorname{diag}\left(\frac{1}{\mu}, -2n-1\right), \qquad V''(\boldsymbol{d}_1) = \operatorname{diag}\left(-2n-1, \mu\right),$$

where $\mu := \mu_2/\mu_1$. If the system is integrable then pairs $(-2n, \mu)$ and $(-2n, 1/\mu)$ belong to the Morales-Ramis table.
Case $\mu = -1, n = 3/2$ in item 1, with $p = 2$.

Variational Equations VE_i

- System

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^m, \quad t \in \mathbb{C}. \tag{N}$$

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- Expansion: $\boldsymbol{x} = \boldsymbol{\varphi}(t) + \varepsilon\boldsymbol{x}_1 + \frac{1}{2!}\varepsilon^2\boldsymbol{x}_2 + \cdots$

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- Expansion: $\boldsymbol{x} = \boldsymbol{\varphi}(t) + \varepsilon \boldsymbol{x}_1 + \frac{1}{2!}\varepsilon^2 \boldsymbol{x}_2 + \cdots$
- Variational equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_1 = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{x}_1, \qquad \boldsymbol{x}_1 \in \mathbb{C}^m, \tag{VE$_1$}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{x}_i + \boldsymbol{f}_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}), \tag{VE$_i$}$$

with $i = 2, 3, \ldots$.

Variational Equations VE_i

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$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^m, \quad t \in \mathbb{C}. \tag{N}$$
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  with $i = 2, 3, \ldots$.
- Differential Galois Group: $\mathrm{Gal}(\mathrm{VE}_i) = \mathrm{Gal}(L_i/K)$.

Non-Hamiltonian Systems

System

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^m. \tag{N}$$

Theorem of M. Ayoul and T. Zung (2010)

Theorem

If the system (N) is B-integrable, then for each $i \in \mathbb{N}$ group $\mathrm{Gal}(\mathrm{VE}_i)$ is virtually Abelian.

Hamiltonian Systems

- Hamiltonian function: $H = H(\boldsymbol{q}, \boldsymbol{p})$, $\boldsymbol{q}, \boldsymbol{p} \in \mathbb{C}^n$.

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- Equations:

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- Theorem of J. J. Morales, J.-P. Ramis, and C. Simó (2007)

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Natural Hamiltonian Systems

- Hamiltonian function:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}).$$

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- Equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} q_i = p_i, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} p_i = -\dfrac{\partial V}{\partial q_i}, \qquad i = 1, \dots, n. \end{cases} \tag{H}$$

or just

$$\ddot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q}), \quad \text{where} \quad \boldsymbol{F}(\boldsymbol{q}) = -V'(\boldsymbol{q}).$$

Natural Hamiltonian Systems

Natural Hamiltonian Systems

- If $V(\boldsymbol{q})$ is homogeneous of degree $k \in \mathbb{Z}^{\star}$, then

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \qquad \boldsymbol{q}(t) = \dot{\varphi}(t)\boldsymbol{d},$$

where

$$V'(\boldsymbol{d}) = \boldsymbol{d}, \quad \text{and} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Natural Hamiltonian Systems

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$$\ddot{x}_i = -\lambda_i \varphi(t)^{k-2} x_i, \quad i = 1, \ldots, n.$$

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- Morales-Ramis Theorem

VE_p and Taylor Expansion

$$\ddot{\boldsymbol{x}}_1 = -\varphi(t)^{k-2}V''(\boldsymbol{d})\cdot\boldsymbol{x}_1, \tag{VE_1}$$

$$\ddot{\boldsymbol{x}}_2 = -\varphi(t)^{k-2}V''(\boldsymbol{d})\cdot\boldsymbol{x}_2 - \varphi(t)^{k-3}V^{(3)}(\boldsymbol{d})\cdot(\boldsymbol{x}_1,\boldsymbol{x}_1), \tag{VE_2}$$

$$\begin{aligned}\ddot{\boldsymbol{x}}_3 = &-\varphi(t)^{k-2}V''(\boldsymbol{d})\cdot\boldsymbol{x}_3 - \varphi(t)^{k-3}V^{(3)}(\boldsymbol{d})\cdot(\boldsymbol{x}_1,\boldsymbol{x}_2) - \\ &-\varphi(t)^{k-4}V^{(4)}(\boldsymbol{d})\cdot(\boldsymbol{x}_1,\boldsymbol{x}_1,\boldsymbol{x}_1).\end{aligned} \tag{VE_3}$$

$K = \mathbb{C}(\varphi, \dot{\varphi})$

VE_2 splitting

If $V''(\boldsymbol{d}) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ then we select following subsystems of VE_2

$$\left.\begin{aligned} \ddot{x} &= -\lambda_\alpha \varphi(t)^{k-2} x, \\ \ddot{y} &= -\lambda_\gamma \varphi(t)^{k-2} y + \varphi(t)^{k-3} \theta^\gamma_{\alpha,\alpha} x^2, \end{aligned}\right\} \qquad (\mathrm{VE}^\gamma_{2,\alpha})$$

and

$$\left.\begin{aligned} \ddot{x} &= -\lambda_\alpha \varphi(t)^{k-2} x, \\ \ddot{y} &= -\lambda_\beta \varphi(t)^{k-2} y, \\ \ddot{z} &= -\lambda_\gamma \varphi(t)^{k-2} z + 2\varphi(t)^{k-3} \theta^\gamma_{\alpha,\beta} xy. \end{aligned}\right\} \qquad (\mathrm{EX}^\gamma_{2,(\alpha,\beta)})$$

1. Duval G. and Maciejewski A. J., 2014, Integrability of Hamiltonian systems with homogeneous potentials of degree ± 2. An application of higher order variational equations, *Discrete Contin. Dyn. Syst.* , 34(11):4589–4615.
2. Duval G. and Maciejewski A. J., 2015, Integrability of potentials of degree $k \neq \pm 2$. Second order variational equations between Kolchin solvability and abelianity, *Discrete Contin. Dyn. Syst.* , 35(5):1969–2009.

Rationalization

$$\left.\begin{aligned} x'' &= r_\alpha(z)x, \\ y'' &= r_\gamma(z)y + \omega(z)x^2, \end{aligned}\right\} \qquad (\mathrm{VE}^{\gamma}_{2,\alpha})$$

where

$$\omega(z) = z^{-\frac{3}{2}-\frac{1}{2k}}(1-z)^{-\frac{5}{4}}$$

In our case $k = -3$, $\lambda_\alpha = -1$, $\lambda_\gamma = -4$. Solutions of VE_1:

$$x_1 = z^{1/3}(1-z)^{3/4}, \quad I_\alpha = \int \frac{1}{x_1^2} = 2\frac{z^{1/3}}{\sqrt{1-z}} + \frac{1}{3}B(z, 1/3, 1/2)$$

$$y_1 = z^{2/3}(1-z)^{3/4}, \quad I_\gamma = \frac{5z-3}{z^{1/3}\sqrt{1-z}} - \frac{5}{6}B(z, 2/3, 1/2)$$

Integrals

$$\Phi = \int \omega(z) y_1 x_1^2 = z - \frac{1}{2} z^2$$

but

$$\Phi_\gamma := \int \omega(z) y_2 x_1^2$$

is very complicated.
However we can consider also

$$\Psi_\gamma := \int \Phi I'_\gamma = \frac{z^{2/3}}{\sqrt{1-z}} + \frac{1}{3} B(z, 2/3, 1/2)$$

Key point

Necessary condition for integrability: there exists Ostrowski relation

$$d_\alpha I_\alpha + d_\gamma I_\gamma \in K = \mathbb{C}(z)[\omega]$$

with $(d_\alpha, d_\gamma) \in \mathbb{C}^2 \setminus \{(0,0)\}$. But it does not exist!. (Theorem 5.3 in ref. 2)