Orthogonal polynomials and Painlevé equations

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Orthogonal polynomial with a given measure

Let $d«(x)$ be a positive Borel measure on $\mathbb{R}$. Find orthonormal polynomials

$$p_n(x) = a_{nn}x^n + \ldots + a_{n0}, \quad a_{nn} > 0$$

$$\int p_n(x)\,p_m(x)\,d«(x) = ¢_{nm}, \quad n,m = 0,1,\ldots$$

Moments $\pm_j = \int x^j d«(x)$ Hankel determinant $D_n = \left|\pm_{i+j}\right|_{i,j=0}^{n}$

$$p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}} \begin{vmatrix} \pm_0 & \cdots & \pm_{n-1} & 1 \\ \pm_1 & \cdots & \pm_n & x \\ \vdots & \ddots & \vdots & \vdots \\ \pm_n & \cdots & \pm_{2n-1} & x^n \end{vmatrix}$$

Recurrence

(equiv. to orthonormality, Favard's theorem)

$$xp_n(x) = \sim_n p_{n+1}(x) + {}_n p_n(x) + \sim_{n-1} p_{n-1}(x),$$

$$\sim_n = \frac{a_{nn}}{a_{n+1,n+1}}, \qquad {}_n = \int xp_n^2(x)\,d«(x)$$

Examples

Jacobi $\quad P_n^{\sim,}(x), \quad d\text{«} = (1-x)^{\sim}(1-x)\ \ dx, \quad \operatorname{supp} d\text{«} = [-1,1]$

Laguerre $\quad L_n^{(\sim)}(x), \quad d\text{«} = x^{\sim} e^{-x}, \quad \operatorname{supp} d\text{«} = [0,\infty]$

Hermite $\quad H_n(x), \quad d\text{«} = e^{-x^2}, \quad \operatorname{supp} d\text{«} = [-\infty,\infty]$

Differential equations

$$xy'' + (\sim +1-x)y' + ny = 0, \quad y = L_n^{(\sim)}(x)$$

$$y'' + (2n+1-x^2)y = 0, \quad y = e^{-x^2} H_n(x)$$

Explicit formulas

$$L_n^{(\sim)}(x) = x^{-\sim} e^{x} \frac{1}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\sim}\right)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

G. Szegö: Orthogonal Polynomials, AMS, 1959.

Lioville-Steklov method

How to find asymptotics as $n \to \infty$?

$$e^{-x^2} H_n(x) = \frac{\cos\left(x\sqrt{2n+1} - ® n/2\right)}{\sqrt{2n+1}} H_n(0) + $$
$$+ \frac{1}{\sqrt{2n+1}} \int_0^x \sin\left[\sqrt{2n+1}\left(x-t\right)\right] t^2 e^{-t^2} H_n(t)\,dt$$

Green’s formula for the ODE

$$y'' + (2n+1-x^2)y = 0,$$
$$y = e^{-x^2} H_n(x)$$

Plancherel-Rotach formula

$$e^{-x^2/2} H_n(x) = \frac{2^{\frac{n}{2}+\frac{1}{4}}\sqrt{n!}}{(® n)^{1/4}\sqrt{\sin ¨}} \left\{ \sin\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2¨ - 2¨) + \frac{3®}{4}\right] + O(n^{-1}) \right\},$$

where $\quad x = \sqrt{2n+1}\cos ¨, \quad £ \le ¨ \le ® - £, \quad n \to \infty$

All zeroes x_k of $H_n(x)$ are real-valued and $|x_k| < O\left(\sqrt{2n}\right)$

G. Szegö: Orthogonal Polynomials, AMS, 1959.

Riemann-Hilbert approach

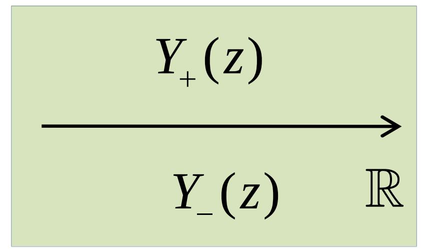
Let $d« = w(z)dz$ and $x^j w(x) \in L^\infty(\mathbb{R}), \quad j \geq 0$

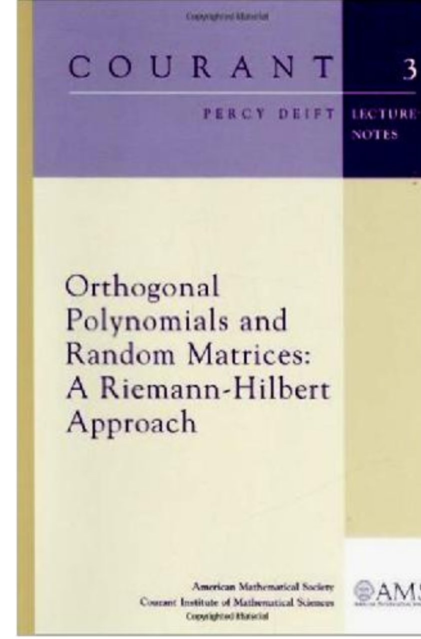
$Y(z)$ is 2×2 - matrix analytic in $\mathbb{C} / \mathbb{R}$

$$Y_+(z) = Y_-(z)\begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

$$Y(z) = \left(I + O(z^{-1})\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty$$

Riemann-Hilbert problem





If $Y(z) = I$ as $z \to \infty$ the solution is explicit

$$Y(z) = \begin{pmatrix} 1 & \dfrac{1}{2® i}\displaystyle\int_{\mathbb{R}} \dfrac{w(s)}{z-s} ds \\ 0 & 1 \end{pmatrix}, \quad \operatorname{Im} z \neq 0$$

by Sokhotskii – Plemelj formula

A.S. Fokas, A.R. Its and A.V. Kitaev, Comm. Math. Phys., 147(1992),395–430.

Riemann-Hilbert approach (continued)

Explicit solution of the RH problem

$$Y(z) = \begin{pmatrix} ®_n(z) & \frac{1}{2®i}\int_{\mathbb{R}} \frac{®_n(s)w(s)}{s-z}ds \\ ¥_{n-1}®_{n-1}(z) & \frac{¥_{n-1}}{2®i}\int_{\mathbb{R}} \frac{®_{n-1}(s)w(s)}{s-z}ds \end{pmatrix}, \quad \operatorname{Im} z \neq 0,$$

where $®_n(z)$ are monic polynomials orthogonal by measure $d« = w(z)dz$

$$®_n(z) = z^n + a_{n,n-1}z^{n-1} + \dots a_{n,0}$$

$$\int_{\mathbb{R}} ®_n(x)®_m(x)w(x)dx = 0, \quad n \neq m$$

$$\int_{\mathbb{R}} ®_n^2(x)w(x)dx = ©_n^2, \quad p_n(z) = ©_n®_n(z),$$

$$¥_n = -2®i©_n^2 = -2®i\frac{D_{n-1}}{D_n}$$

Painlevé equations

PII $$u_{zz} = 2u^3 + zu + n,$$

PIV $$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - n + m\right)u - \frac{2(m+n+1)^2}{u}, \qquad n, m \in \mathbb{Z}$$

Rational solutions

$$u_{PII}(z) = \frac{d}{dz}\log\frac{Q_{n-1}(z)}{Q_n(z)}$$

$Q_n(z)$ - Vorobiev-Yablonskii polynomials

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\left[Q_n Q_n'' - (Q_n')^2\right], \qquad Q_0 = 1, \quad Q_1 = z$$

recurrence

$$u_{PIV}(z) = -2z + \frac{d}{dz}\log\frac{H_{m,n+1}(z)}{H_{m+1,n}(z)}$$

$H_{m,n}(z)$ - Generalized Hermite polynomials

$$\begin{cases} 2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH^2_{m,n}, \\ 2nH_{m,n+1}H_{m,n-1} = -H_{m,n}H''_{m,n} + \left(H'_{m,n}\right)^2 + 2nH^2_{m,n}, \end{cases}$$

$$H_{0,0} = H_{0,1} = H_{1,0} = 1, \qquad H_{1,1} = 2z.$$

Generalized Hermite polynomials

Main properties of $H_{m,n}$

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$$

classic Hermite poly

- $H_{m,0}(z) = H_{0,m}(z) = H_m(z)$
- $H_{m,n}(z) = c_{m,n} W\left(H_m(z), H_{m+1}(z), \ldots, H_{m+n}(z)\right),$
- Let $p_n(x)$ are orthogonal with weight $w(x,z) = (x-z)^m e^{-x^2}$ then

$$x p_n(x) = p_{n+1}(x) + a_n(z,m) p_n(x) + b_n(z,m) p_{n-1}(x),$$

$$a_n(z,m) = -\frac{1}{2}\frac{d}{dz} \ln \frac{H_{n+1,m}}{H_{n,m}}, \quad b_n(z,m) = \frac{n H_{n+1,m} H_{n-1,m}}{2 H_{n,m}^2}.$$

- GUE matrix model with m- fold degeneracy of $n+1$- st eigenvalue $a_{n+1} = z$

$$D_n(z) = \frac{1}{n!} \int_{\infty}^{\infty} \ldots \int_{\infty}^{\infty} \prod_{1 \le i < j \le n} (a_i - a_j)^2 \prod_{k=1}^{n} (a_k - z)^m e^{-a_k^2} d a_k.$$ - partition function

$$D_n(z) = A_{m,n} H_{m,n}(cz), \quad c = i\sqrt{2/3}, \quad A_{m,n} = \text{const}.$$

Riemann-Hilbert problem for PIV

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2z^2u - \frac{2(2n+1)^2}{u},$$

$$u_{PIV}(z) = -2z + \frac{d}{dz}\log\frac{H_{n,n+1}(z)}{H_{n+1,n}(z)}$$

Scaling as $n \to \infty$

$$z = x\sqrt{n}, \quad \text{a} = ¶\sqrt{n}, \quad \Theta = \frac{1}{2}(¶^2 - x^2)$$

$Y(¶, x)$ is 2×2 - matrix analytic in $¶ \in \mathbb{C}/\mathbb{R}$

$$Y_+(¶, x) = Y_-(¶, x)\begin{pmatrix} 1 & 2®ie^{-2n\Theta(¶,x)} \\ 0 & 1 \end{pmatrix}, \quad ¶ \in \mathbb{R}$$

$$Y_\pm(¶, x) = \left(I + O(¶^{-1})\right)\begin{pmatrix} ¶^{2n} & 0 \\ 0 & ¶^{-2n} \end{pmatrix}, \quad ¶ \to \infty$$

RH problem for weight function

$$w(¶, x) = e^{-n(¶^2 - x^2)}$$

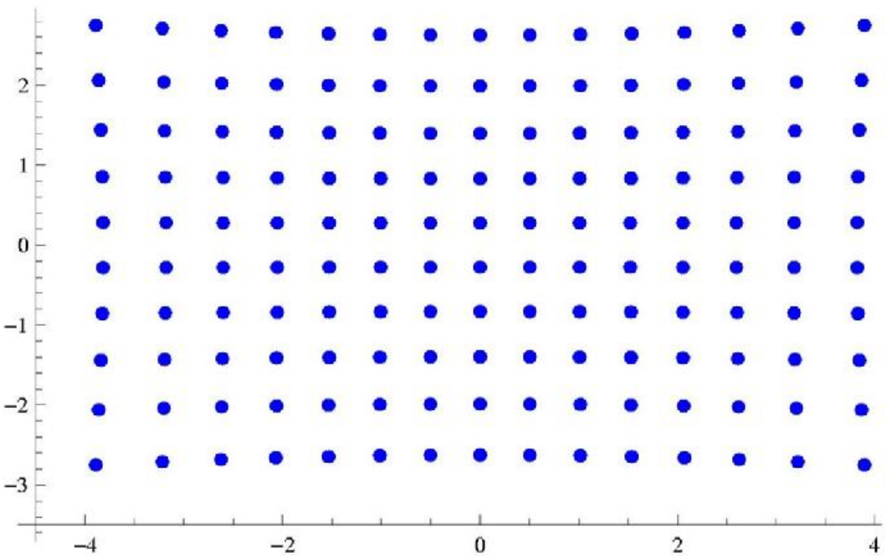
As $x = ¶$ we have

$$Y(¶, x) = \begin{pmatrix} H_{n,n}(¶) & \dots \\ ¥_{n-1}H_{n,n-1}(¶) & \dots \end{pmatrix}$$

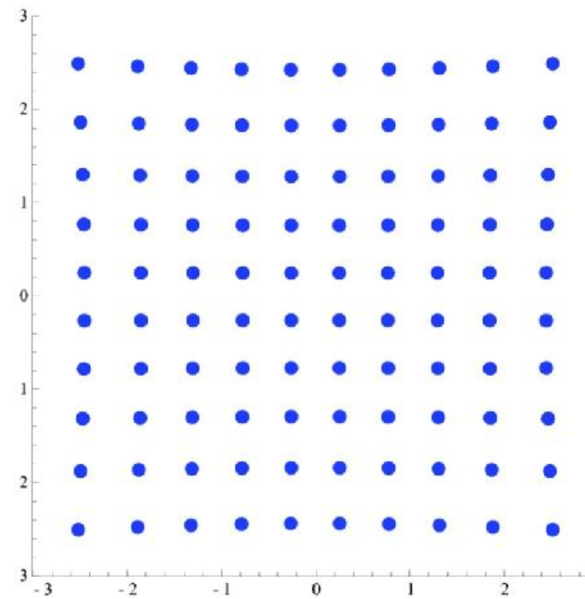
$$u_{PIV}(z) = -2x\sqrt{n} - \lim_{¶\to\infty}\left(¶\,\partial_x Y_+(¶, x)\right)_{12}$$

Distribution of zeroes

Zeroes in the complex plane of $H_{m,n}(z)$



$m = 15, \quad n = 10$



$m = 10, \quad n = 10$

What is asymptotics as $m, n \to \infty$?

(under scaling $m = n$ and $z = x\sqrt{n}, \quad x = O(1)$)

Asymptotic "undressing" of the RH problem

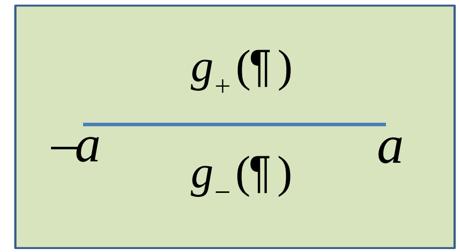
$$W(\P, x) \equiv \begin{pmatrix} e^{n\ell} & 0 \\ 0 & e^{-n\ell} \end{pmatrix} Y(\P, x) \begin{pmatrix} e^{-ng(\P)-n\ell} & 0 \\ 0 & e^{ng(\P)+n\ell} \end{pmatrix}, \quad \P \in \mathbb{C}/\mathbb{R}.$$

$$g(\P) = \int_{-a}^{a} \ln(\P - s)\,^\circ(s)ds, \quad a \in \mathbb{R}, \quad a = O(1)$$

$$e^{ng(\P)} = \P^{n}\left(1 + O(\P^{-1})\right), \quad \P \to \infty$$

$$\ell = 2\int_{-a}^{a} \ln|\P - s|\,^\circ(s)ds - \P^{2}, \quad -a < \P < a.$$

Deift-Zhou parametrix



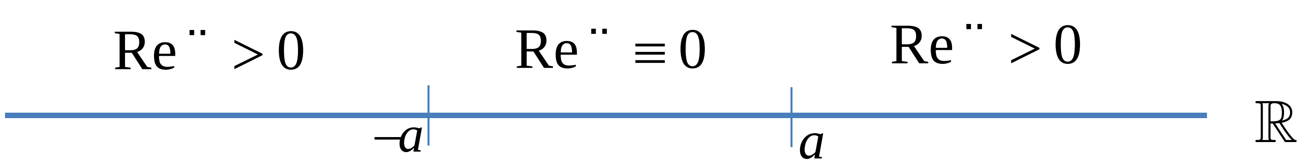
$W(\P, x)$ is 2×2 - matrix analytic in $\P \in \mathbb{C}/\mathbb{R}$

$$W_+(\P, x) = W_+(\P, x)\begin{pmatrix} e^{n(\text{¨}_+ - \text{¨}_-)} & e^{-n(\text{¨}_+ + \text{¨}_-)} \\ 0 & e^{-n(\text{¨}_+ - \text{¨}_-)} \end{pmatrix}, \quad \P \in \mathbb{R}$$

$$W_\pm(\P, x) = I + O(\P^{-1})$$

$$\text{¨}(\P) \equiv \P^{2} - 2g(\P) + \ell,$$

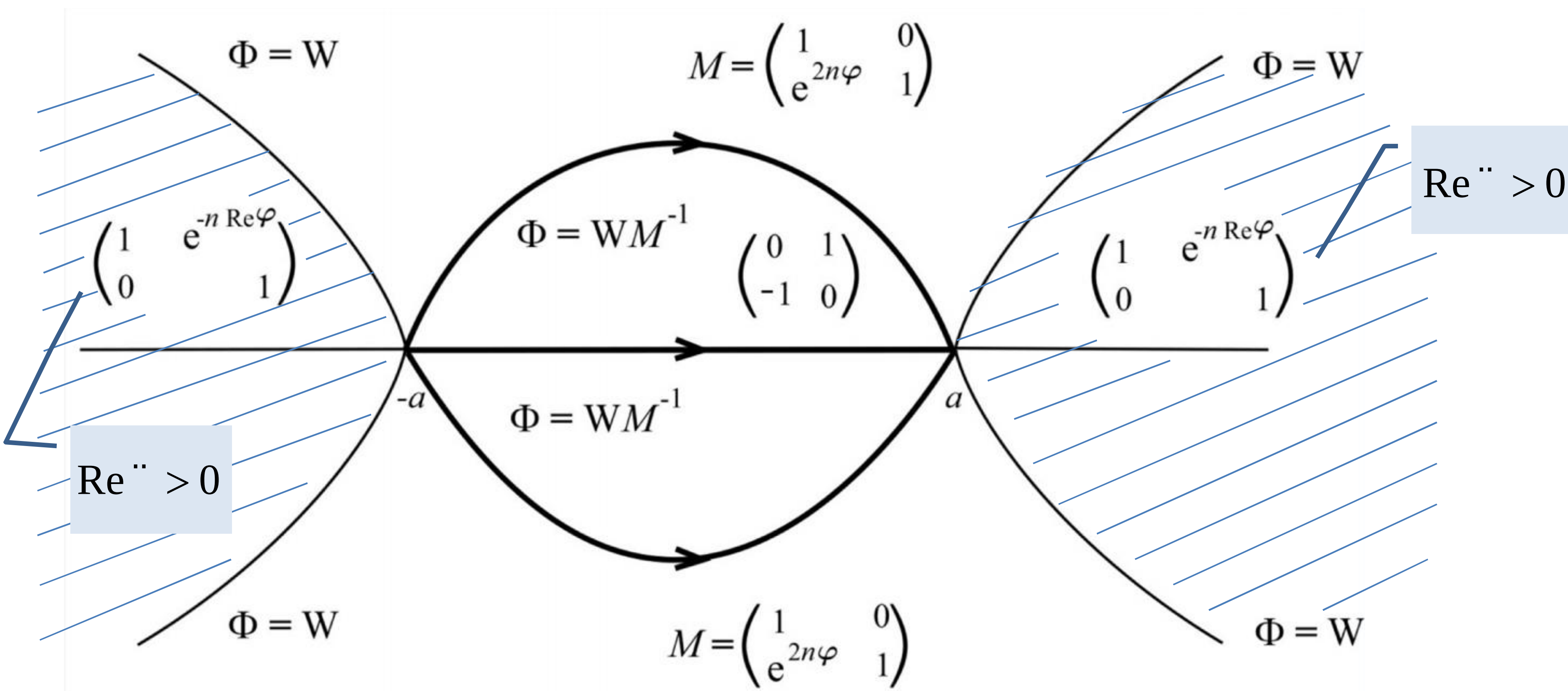
$$\operatorname{Re}\text{¨} = \frac{\text{¨}_+ + \text{¨}_-}{2}, \quad \P \in \mathbb{R}$$



“Lens” domain in RH problem

$$\begin{pmatrix} e^{n(\varphi_+-\varphi_-)} & e^{-n(\varphi_++\varphi_-)} \\ 0 & e^{-n(\varphi_+-\varphi_-)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_-} & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & e^{-n(\varphi_++\varphi_-)} \\ -e^{n(\varphi_++\varphi_-)} & 0 \end{pmatrix}}_{= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_+} & 1 \end{pmatrix}.$$

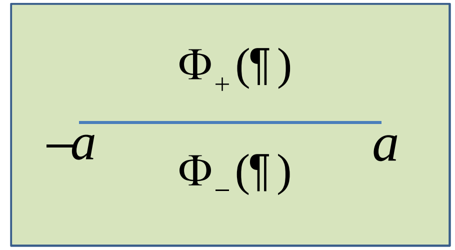
$$\Phi_\pm(\P, x) \equiv W_\pm(\P, x) \begin{pmatrix} 1 & 0 \\ \mp e^{2n\varphi} & 1 \end{pmatrix}, \quad \mathrm{Im}\,\P \lessgtr 0.$$



Model RH problem

$$\Phi_+ = \Phi_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \P \in [-a, a],$$

$$\Phi_\pm(\P) \to I, \quad \P \to \infty.$$



Explicit solution

$$\Phi(\P) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} \sqrt[4]{\dfrac{\P - a}{\P + a}} & 0 \\ 0 & \sqrt[4]{\dfrac{\P + a}{\P - a}} \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.$$

P.Deift et al., Comm. Pure Appl. Math. 1999. V. 52. No. 11. P. 1335–1425

Explicit Deift-Zhou parametrix

$$g(\P) = \int_{-a}^{a} \ln(\P - s)\,{}^{\circ}(s)\,ds, \quad a \in \mathbb{R}, \quad a = O(1)$$

$${}^{\circ}(\P) = \frac{1}{\circledR}\sqrt{a^2 - \P^2} + \frac{x}{\circledR}\sqrt{\frac{a-\P}{a+\P}}, \qquad a = \sqrt{2},$$

$$g(\P) = -\ln\left(\P - \sqrt{\P^2 - 2}\right) + \frac{1}{4}\left(\P - \sqrt{\P^2 - 2}\right)^2$$

Outside of “lens” as $\mathrm{Re}\,\text{¨} < 0$

$$Y(\P, z) = e^{n\ell \pm_3}\Phi(\P)e^{n(2g-\ell)\pm_3}\left(I + O(n^{-1})\right), \quad n \to \infty,$$

Inside of “lens” as $\mathrm{Re}\,\text{¨} > 0$

$$Y(\P, z) = e^{n\ell \pm_3}\Phi(\P)\begin{pmatrix} 1 & 0 \\ e^{2n\text{¤}} & 1 \end{pmatrix}e^{n(2g-\ell)\pm_3}\left(I + O(n^{-1})\right), \quad n \to \infty,$$

Asymptotics of generalized Hermite's

Inversion formula

$$H_{n,n}(z) = (Y)_{11}(z,x) = \Phi_{11}(\text{¶})e^{2ng(\text{¶})}\left(I + O(n^{-1}\right), \quad \text{¶} = x, \quad z = x\sqrt{n},$$

Outside the circle $|x| > \sqrt{2}$

$$H_{n,n}(x\sqrt{n}) \approx \frac{1}{2}\left[\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\frac{1}{4}} + \left(\frac{x+\sqrt{2}}{x-\sqrt{2}}\right)^{\frac{1}{4}}\right]\frac{e^{\frac{n}{4}(x-\sqrt{x^2-2})^2}}{(x-\sqrt{x^2-2})^n}, \quad n \to \infty$$

Inside the circle $|x| < \sqrt{2}$

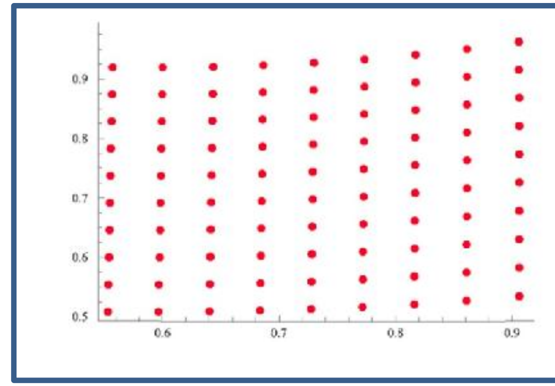
$$H_{n,n}(x\sqrt{n}) \approx \frac{1}{2}\left[\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)^{\frac{1}{4}} \cos\left(n\text{®}\int_x^{\sqrt{2}} \text{°}(s)ds + \frac{\text{®}}{4}\right) + \right.$$

$$\left. + \left(\frac{x+\sqrt{2}}{x-\sqrt{2}}\right)^{\frac{1}{4}} \cos\left(n\text{®}\int_x^{\sqrt{2}} \text{°}(s)ds - \frac{\text{®}}{4}\right)\right] \exp\left\{n\int_{-\sqrt{2}}^{\sqrt{2}} \ln|x-s|\,\text{°}(s)ds\right\}.$$

Generalized Hermite, numeric check

$$\circledR \int_x^{\sqrt{2}} {}^\circ(s)\,ds =$$

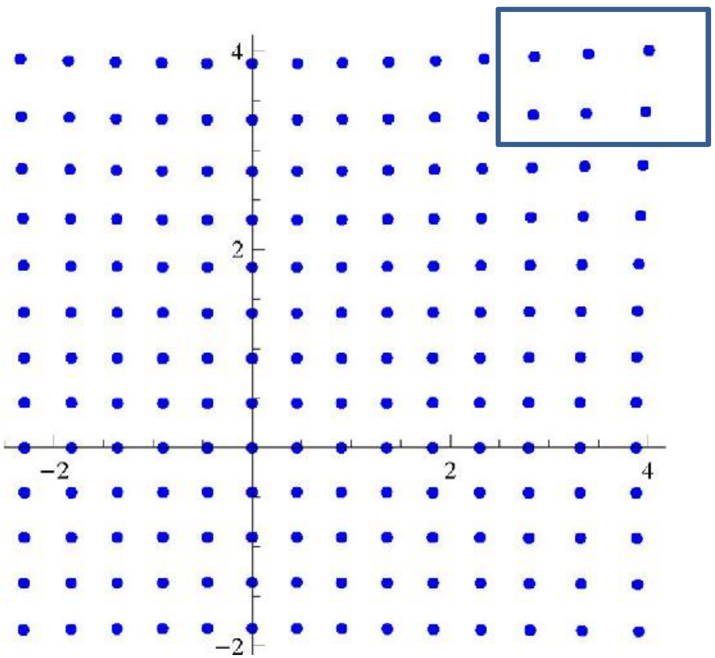
$$= \frac{3}{2} x\sqrt{2-x^2} + (x\sqrt{2}-1)\,\mathrm{Arctg}\,\frac{x}{\sqrt{2-x^2}}.$$



x-plane $n = 30$

$21 \le j, k \le 30$

$x = z / \sqrt{n}$



z-plane $H_{n,n}(z)$ $n = 17$

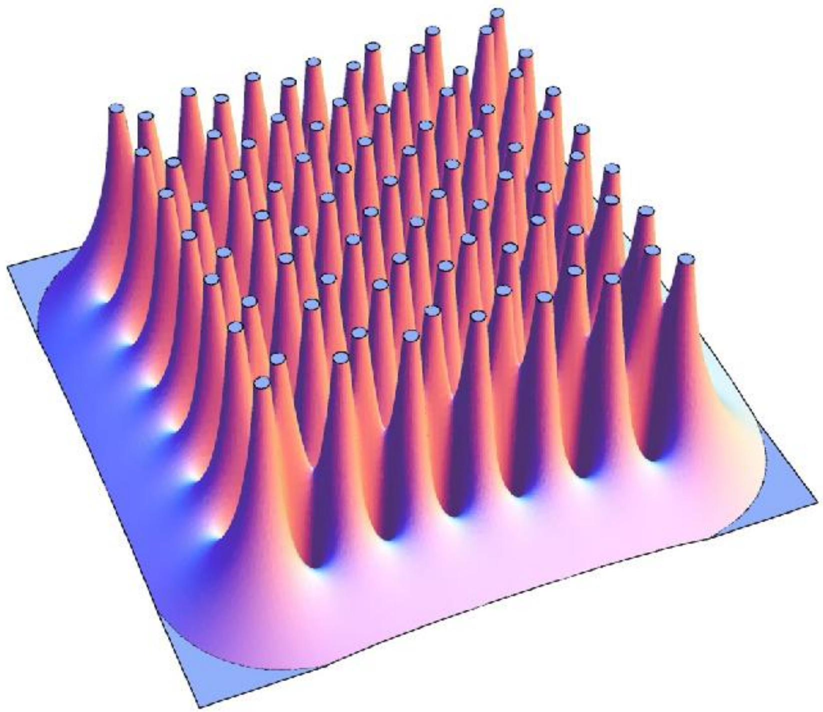
$$\mathrm{Re}\left(\frac{3}{2} x\sqrt{2-x^2} + (x\sqrt{2}-1)\,\mathrm{arctg}\,\frac{x}{\sqrt{2-x^2}}\right) = $$

$$= \frac{1}{n}\left(\mathrm{Re}\,\mathrm{arctg}\,\frac{x-\sqrt{2}}{x+\sqrt{2}} + \circledR\, j\right), \quad j \in \mathbb{Z}$$

$$\mathrm{Im}\left(\frac{3}{2} x\sqrt{2-x^2} + (x\sqrt{2}-1)\,\mathrm{arctg}\,\frac{x}{\sqrt{2-x^2}}\right) = $$

$$= \frac{1}{n}\left(\mathrm{Im}\,\mathrm{arctg}\,\frac{x-\sqrt{2}}{x+\sqrt{2}} + \circledR\, k\right), \quad k \in \mathbb{Z}$$

Thank you for attention!



$$u_{PIV}(z) = -2z + \frac{d}{dz}\log\frac{H_{m,n+1}(z)}{H_{m+1,n}(z)}$$