The Manin reduction and the Calogero Gold Fish

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The Manin Reduction

- The Benney hydrodynamic chain

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0,\ \ k = 0, 1, ...$$

contains infinitely many moments $A^k(x,t)$ and infinitely many corresponding equations. A very important task is to find and extract finite component reductions.

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contains infinitely many moments $A^k(x,t)$ and infinitely many corresponding equations. A very important task is to find and extract finite component reductions.
- In comparison with that approach, Yu. Manin observed (1981, Physica D) that the Benney hydrodynamic chain also admits infinitely many *non-hydrodynamic* reductions selected by the ansatz
$$p = \lambda - \sum_{k=1}^{N} \frac{b_k(x,t)}{\lambda - a_k(t)},$$
where p is a generating function of conservation law densities for the Benney hydrodynamic chain and the generating equation of conservation laws is
$$p_t = \left(\frac{p^2}{2} + A^0\right)_x.$$

Computations

- Substitution this ansatz

$$p = \lambda - \sum_{k=1}^{N} \frac{b_k(x,t)}{\lambda - a_k(t)}$$

into

$$p_t = \left(\frac{p^2}{2} + A^0\right)_x$$

leads to the relationship (at zeroth order with respect to parameter λ)

$$A^0 = \sum_{k=1}^{N} b_k$$

and to $2N$ equations (first and second orders with respect of parameter λ, respectively), i.e.

$$b_{k,t} - a_k b_{k,x} + \sum_{m \neq k} \frac{(b_k b_m)_x}{a_k - a_m} = 0, \quad b_{k,x} + a'_k = 0.$$

Computations

- Last N equations of this system

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can be integrated immediately, i.e.

$$b_k(x,t) = c_k(t) - x a_k'(t),$$

where $c_k(t)$ are arbitrary functions.

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- Taking into account these dependencies first N equations reduce to the form

$$x \left(a''_k - 2 \sum_{m \neq k} \frac{a'_k a'_m}{a_k - a_m} \right) = c'_k + a_k a'_k - \sum_{m \neq k} \frac{a'_k c_m + c_k a'_m}{a_k - a_m}.$$

- Thus we obtain $2N$ component system

$$a_k'' = 2\sum_{m\neq k}\frac{a_k' a_m'}{a_k - a_m}, \quad c_k' + a_k a_k' = \sum_{m\neq k}\frac{a_k' c_m + c_k a_m'}{a_k - a_m}.$$

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- First N equations

$$a_k'' = 2\sum_{m\neq k} \frac{a_k' a_m'}{a_k - a_m}$$

are nothing but the system derived by F. Calogero (1978). We shall call this system the Calogero Gold Fish.

- Taking into account

$$b_k(x,t) = c_k(t) - x a_k'(t),$$

the rational ansatz

$$p = \lambda - \sum_{k=1}^{N} \frac{b_k(x,t)}{\lambda - a_k(t)}$$

becomes

$$p = \lambda - \sum_{m=1}^{N} \frac{c_m}{\lambda - a_m} + x \sum_{m=1}^{N} \frac{a_m'}{\lambda - a_m}.$$

The Integration

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- This means that $p(x,t,\lambda)$ depends on x linearly, i.e.

$$p = \lambda + (g(t,\lambda) - x) f(t,\lambda),$$

where

$$f(t,\lambda) = \sum_{m=1}^{N} \frac{a_m'}{a_m - \lambda}, \quad g(t,\lambda) f(t,\lambda) = \sum_{m=1}^{N} \frac{c_m}{a_m - \lambda}.$$

The Integration

- Substitution the linear ansatz

$$p = \lambda + (g(t,\lambda) - x) f(t,\lambda),$$

into

$$p_t = \left(\frac{p^2}{2} + A^0\right)_x$$

yields two equations

$$f_t = -f^2, \quad g_t = -\lambda - \frac{1}{f} \sum_{m=1}^{N} a'_m,$$

where we took into account that

$$A^0 = \sum_{m=1}^{N} c_m - x \sum_{m=1}^{N} a'_m.$$

The Integration

- Integration of the equation in

$$f_t = -f^2,$$

yields

$$f = \frac{1}{t + C(\lambda)},$$

where $C(\lambda)$ is not yet determined function of the parameter λ.

- Taking into account that this equation is equivalent to the equation

$$f(t, \lambda) = \sum_{m=1}^{N} \frac{a'_m}{a_m - \lambda},$$

The Integration

- Integration of the equation in
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- Taking into account that this equation is equivalent to the equation
$$f(t,\lambda) = \sum_{m=1}^{N} \frac{a'_m}{a_m - \lambda},$$
we obtain the relationship
$$t + C(\lambda) = D(\lambda) \prod_{m=1}^{N} (a_m - \lambda),$$
where $D(\lambda)$ is not yet determined function of the parameter λ.

- Since the l.h.s. of this expression

$$t + C(\lambda) = D(\lambda) \prod_{m=1}^{N} (a_m - \lambda),$$

depends linearly on t, then r.h.s. of this expression also must linearly depend on t. Thus all coefficients (symmetric polynomials) of the polynomial

$$\prod_{m=1}^{N} (\lambda - a_m)$$

depend linearly on t, i.e. roots $a_k(t)$ of the polynomial

$$\lambda^N - (A_1 t + B_1)\lambda^{N-1} + (A_2 t + B_2)\lambda^{N-2} - ... + (-1)^N (A_N t + B_N) = 0$$

satisfy the Calogero Gold Fish system.

- Thus we proved
  Theorem: *N component Calogero Gold Fish system*

$$a_k'' = 2 \sum_{m \neq k} \frac{a_k' a_m'}{a_k - a_m}$$

has a general solution parameterised by $2N$ *arbitrary constants, where functions* $a_k(t)$ *are roots of polynomial*

$$\lambda^N - (A_1 t + B_1)\lambda^{N-1} + (A_2 t + B_2)\lambda^{N-2} - ... + (-1)^N (A_N t + B_N) = 0,$$

whose coefficients depend linearly on t.