Integrability of homogeneous Hamiltonian systems in curved spaces.

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Dynamics in Siberia, Novosibirsk, February 29 – March 5, 2016

Integrability of homogeneous Hamiltonian equations

Natural Hamiltonian in flat space

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\mathbf{q}), \qquad (\mathbf{q},\mathbf{p}) \in \mathbb{C}^{2n},$$

with V — homogeneous of degree $k \in \mathbb{Z}$

$$V(\lambda q_1, \ldots, \lambda q_n) = \lambda^k V(q_1, \ldots, q_n)$$

- how to check integrability of such class of potentials in a wide class of functions (without restrictions on degree of a first integral with respect to momenta)?
- what is a counterpart of this class in curved spaces?

Variational equations

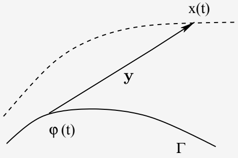
Main Idea

A non-linear system leaves fingerprints of their properties in variational equations.

In a system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots x_n)^T,$$

with known non-stationary particular solution $\boldsymbol{\varphi}(t)$ the substitution $\mathbf{x} = \boldsymbol{\varphi}(t) + \mathbf{y}$ is made



Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = A(t)\mathbf{y}, \qquad A(t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\boldsymbol{\varphi}(t)).$$

Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ*. Then the identity component of the differential Galois group of the variational equations along* Γ *is Abelian.*

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.
- Audin, M., *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés 8, Collection SMF, SMF et EDP Sciences, Paris, 2001.

Particular solutions and Variational equations

Definition

Darboux point $\mathbf{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\mathbf{d}) = \mathbf{d}$$

Particular solution

$$\mathbf{q}(t) = \varphi(t)\mathbf{d}, \ \mathbf{p}(t) = \dot{\varphi}(t)\mathbf{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

$$\ddot{\mathbf{x}} = -\varphi(t)^{k-2} V''(\mathbf{d})\mathbf{x},$$

where $V''(\mathbf{d})$ is the Hessian of V calculated at $\mathbf{d}$. If $V''(\mathbf{d})$ is diagonalisable

$$\ddot{\eta}_i = \lambda_i \varphi(t)^{k-2} \eta_i, \qquad i = 1, \ldots, n,$$

where λ_i for $i = 1, \ldots, n$ are eigenvalues of V''.

By homogeneity of V, $\lambda_n = k - 1$.

Transformation into hypergeometric equation

The Yoshida transformation

$$t \longrightarrow z := \frac{1}{\varepsilon}\varphi(t)^k.$$

Variational equations after transformations:

$$z(1-z)\eta_i'' + \left(\frac{k-1}{k} - \frac{3k-2}{2k}z\right)\eta_i' + \frac{\lambda_i}{2k}\eta_i = 0,$$

where $i = 1, \ldots, n$, have the form of a direct product of hypergeometric differential equation for which the differences of exponents at $z = 0$, $z = 1$ and $z = \infty$ are

$$\rho = \frac{1}{k}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{1}{2k}\sqrt{(k-1)^2 + 8k\lambda_i}.$$

Solvability of Riemann P equation. Kimura theorem

Theorem

The identity component of the differential Galois group of the Riemann P equation is solvable iff

A. *at least one of the four numbers $\rho+\sigma+\tau$, $-\rho+\sigma+\tau$, $\rho-\sigma+\tau$, $\rho+\sigma-\tau$ is an odd integer, or*

B. *the numbers ρ or $-\rho$ and σ or σ and τ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz's table fifteen families*

1	$1/2+l$	$1/2+s$	arbitrary complex number	
2	$1/2+l$	$1/3+s$	$1/3+q$	
3	$2/3+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
4	$1/2+l$	$1/3+s$	$1/4+q$	
5	$2/3+l$	$1/4+s$	$1/4+q$	$l+s+q$ even
6	$1/2+l$	$1/3+s$	$1/5+q$	
7	$2/5+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
8	$2/3+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
9	$1/2+l$	$2/5+s$	$1/5+q$	
10	$3/5+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
11	$2/5+l$	$2/5+s$	$2/5+q$	$l+s+q$ even
12	$2/3+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
13	$4/5+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
14	$1/2+l$	$2/5+s$	$1/3+q$	
15	$3/5+l$	$2/5+s$	$1/3+q$	$l+s+q$ even

where $l, s, q \in \mathbb{Z}$.

Morales-Ramis Theorem

- find all non-zero solutions of

$$V'(\mathbf{d}) = \mathbf{d}$$

- calculate eigenvalues $\lambda_1, \ldots, \lambda_n$ of $V''(\mathbf{d})$

Theorem

If the Hamiltonian system with homogeneous potential is meromorphically integrable, then each (k, λ_i) belong to the following list:

Morales-Ramis table

case	k	λ	
1.	± 2	λ	
2.	k	$p+\frac{k}{2}p(p-1)$	
3.	k	$\frac{1}{2}\left(\frac{k-1}{k}+p(p+1)k\right)$	
4.	3	$-\frac{1}{24}+\frac{1}{6}\left(1+3p\right)^2,$	$-\frac{1}{24}+\frac{3}{32}\left(1+4p\right)^2$
		$-\frac{1}{24}+\frac{3}{50}\left(1+5p\right)^2,$	$-\frac{1}{24}+\frac{3}{50}\left(2+5p\right)^2$
5.	4	$-\frac{1}{8}+\frac{2}{9}\left(1+3p\right)^2$	

Morales-Ramis table

case	k	λ
6.	5	$-\frac{9}{40}+\frac{5}{18}(1+3p)^2, \quad -\frac{9}{40}+\frac{2}{5}(1+5p)^2$
7.	-3	$\frac{25}{24}-\frac{1}{6}(1+3p)^2, \quad \frac{25}{24}-\frac{3}{32}(1+4p)^2$
		$\frac{25}{24}-\frac{3}{50}(1+5p)^2, \quad \frac{25}{24}-\frac{6}{25}(1+5p)^2$
8.	-4	$\frac{9}{8}-\frac{2}{9}(1+3p)^2$
9.	-5	$\frac{49}{40}-\frac{5}{18}(1+3p)^2, \quad \frac{49}{40}-\frac{2}{5}(1+5p)^2$

where p is an integer and λ an arbitrary complex number.

Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

Hénon-Heiles potentials

$$V = \frac{e}{3}x^3 + xy^2$$

Darboux points

$$\mathbf{d}_1 = \left\{e^{-1}, 0\right\}, \quad \mathbf{d}_{2/3} = \left\{\frac{1}{2}, \pm\frac{1}{2}\sqrt{2-e}\right\}$$

Eigenvalues of Hessian $V''(\mathbf{d}_i)$

$$\operatorname{spectrum}(V''(\mathbf{d}_1)) = \left\{\frac{2}{e}, 2\right\}, \quad \operatorname{spectrum}(V''(\mathbf{d}_{2/3})) = \left\{e-1, 2\right\}$$

obstructions on non-trivial eigenvalues $\lambda_1 = \frac{2}{e}$ and $\lambda_2 = e - 1$

$$\begin{aligned}\lambda_1, \lambda_2 \in &\left\{p + \frac{3}{2}p(p-1)\right\} \cup \left\{\frac{1}{2}\left(\frac{2}{3} + 3p(p+1)\right)\right\} \\ &\cup \left\{-\frac{1}{24} + \frac{1}{6}(1+3p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{32}(1+4p)^2\right\} \\ &\cup \left\{-\frac{1}{24} + \frac{3}{50}(1+5p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{50}(2+5p)^2\right\}.\end{aligned}$$

Hénon-Heiles potentials

Theorem (Morales)

Hénon-Heiles system is non-integrable for $e \in \mathbb{C} \setminus \{1, 2, 6, 16\}$.

integrable cases

- $e = 1$
- $e = 6$
- $e = 16$
- to prove non-integrability of case with $e = 2$ higher order variational equations are necessary

$e = -1$

$x = 0$

section plane $(y, \dot{y})$

http://www.maia.ub.es/dsg/hidra/henon.html

Relations between eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$

$\lambda_i(\mathbf{d}) := \Lambda_i(\mathbf{d}) + 1$, are the non-trivial eigenvalues of $V''(\mathbf{d})$.
τ_i is the elementary symmetric polynomial of degree i in $(n-1)$ variables

Theorem

For a generic homogeneous $V \in \mathbb{C}[\mathbf{q}]$ of degree $k > 2$ we have

$$\sum_{[\mathbf{d}] \in \mathcal{D}^\star(V)} \frac{\tau_1(\boldsymbol{\Lambda}(\mathbf{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\mathbf{d}))} = (-1)^n (n+k-2)^r, \quad 0 \leq r \leq n-1,$$

or, alternatively

$$\sum_{[\mathbf{d}] \in \mathcal{D}^\star(V)} \frac{\tau_r(\boldsymbol{\Lambda}(\mathbf{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\mathbf{d}))} = (-1)^{n-r-1} \sum_{i=0}^{r} \binom{n-r-1}{r-i} (k-1)^i.$$

Classification program

- relations give only finite choices of $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n = k-1)$, from these λs we reconstruct potentials
- no new apart already known integrable potentials for $n = 2$ and $2 < k \leq 6$
- for $n = k = 3$ it was found 10 integrable potentials e.g.

$$V_{10} = \frac{4\sqrt{2}q_1^3}{3} + \frac{5q_1q_2^2}{2\sqrt{2}} + q_2^2q_3 + \frac{1}{3}q_3^3$$

with additional first integrals of degree 4 and 6 in momenta.

Maciejewski, A. J. and M. Przybylska, (2004): All meromorphically integrable 2D Hamiltonian systems with homogeneous potentials of degree 3, *Phys. Lett. A* 327(5-6):461–473.

Maciejewski, A. J. and M. Przybylska, (2005): Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential, *J. Math. Phys.* 46(6):062901, 33 pages.

Przybylska M., (2009): Darboux points and integrability of homogenous Hamiltonian systems with three and more degrees of freedom, *Regul. Chaotic Dyn.*14(2):263–311 and *Regul. Chaotic Dyn.*14(3):349–388.

What is analogue of homogeneous systems in curved spaces?

no obvious answer

Our proposition

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right), \qquad V = r^m U(\varphi),$$

where m and k are integers, and $k \neq 0$.

Nakagawa and Yoshida

$$H = T(\mathbf{p}) + V(\mathbf{q}).$$

where T and V are homogeneous functions of integer degrees.
To find a straight line particular solution one must solve overdetermined system of nonlinear equations

$$T'(\mathbf{c}) = \mathbf{c}, \qquad V'(\mathbf{c}) = \mathbf{c},$$

Main integrability theorem. Auxiliary sets

$$\mathcal{I}_0(k,m) := \left\{ \frac{1}{k}(mp+1)(2mp+k) \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_1(k,m) := \left\{ \frac{1}{2k}(mp-2)(mp-k) \;\middle|\; p = 2r+1, r \in \mathbb{Z} \right\},$$

$$\mathcal{I}_2(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{2}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_3(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{3}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_4(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{4}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_5(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_6(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{2}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_a(k,m) := \mathcal{I}_0(k,m) \cup \mathcal{I}_1(k,m) \cup \mathcal{I}_2(k,m).$$

Main integrability theorem. Main theorem

Theorem

Assume that $U(\varphi)$ is a complex meromorphic function and there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. If the Hamiltonian system defined by Hamiltonian

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right), \qquad V = r^m U(\varphi),$$

is integrable in the Liouville sense, then number

$$\lambda := 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)},$$

belongs to set $\mathcal{I}(k, m)$ which is defined by the following table

Main integrability theorem. Integrability table

No.	k	m	$\mathcal{I}(k,m)$
1	$k=-2(mp+1)$	m	$\mathbb{C}$
2	$k\in\mathbb{Z}\setminus\{0\}$	m	$\mathcal{I}_{\mathrm{a}}(k,m)$
3	$k=2(mp-1)\pm\frac{1}{3}m$	$3q$	$\bigcup_{i=0}^{6}\mathcal{I}_i(k,m)$
4	$k=2(mp-1)\pm\frac{1}{2}m$	$2q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_4(k,m)$
5	$k=2(mp-1)\pm\frac{3}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_6(k,m)$
6	$k=2(mp-1)\pm\frac{1}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_5(k,m)$

Table : Integrability table. Here $k,m,p,q\in\mathbb{Z}$ and $k\neq 0$.

Proof. Particular solution

$$\dot{r} = \frac{\partial H}{\partial p_r} = r^{m-k} p_r,$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = r^{m-k-3} p_\varphi^2 - \frac{1}{2}(m-k) r^{m-k-1} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - m r^{m-1} U(\varphi),$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = r^{m-k-2} p_\varphi,$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -r^m U'(\varphi).$$

If $U'(\varphi_0) = 0$ for a certain $\varphi_0 \in \mathbb{C}$, then system has invariant manifold

$$\mathcal{N} = \left\{ (r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 | \varphi = \varphi_0, \ p_\varphi = 0 \right\}.$$

particular solution lying on $\mathcal{N}$

$$\dot{r} = r^{m-k} p_r, \qquad \dot{p}_r = -\frac{1}{2}(m-k) r^{m-k-1} p_r^2 - m r^{m-1} U(\varphi_0)$$

Proof. Variational equations

$[R, P_R, \Phi, P_\Phi]^T$ are variations of $[r, p_r, \varphi, p_\varphi]^T$.
Variational equations along the particular solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix} = \mathbf{C}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix},$$

with

$$\mathbf{C} = \begin{bmatrix} lr^{l-1}p_r & r^l & 0 & 0 \\ -\frac{1}{2}(l-1)lr^{l-2}p_r^2 - (m-1)mr^{m-2}U(\varphi_0) & -lr^{l-1}p_r & 0 & \\ 0 & 0 & 0 & r^{l-2} \\ 0 & 0 & -r^m U''(\varphi_0) & 0 \end{bmatrix}$$

where auxiliary parameter $l = m - k$.

Proof. Normal variational equations (NVEs)

Equations for Φ and P_Φ form a closed subsystem NVEs. that give one second-order differential equation

$$\ddot{\Phi}+P\dot{\Phi}+Q\Phi=0,\quad P=(k-m+2)r^{m-k-1}p_r,\quad Q=r^{2m-k-2}U''(\phi_0).$$

Rationalization

$$t\longrightarrow z=\frac{U(\varphi_0)}{E}r^m(t),$$

for $E\neq 0$, that gives immediately

$$\dot{z}^2=-2Em^2r^{m-k-2}z^2(z-1),\ \ddot{z}=Emr^{m-k-2}z\left[(k-4m+2)z+3m-k-2\right].$$

NVEs after such a change of independent variable takes the form

$$z(z-1)\Phi''(z)+\left[\frac{2m+k+2}{2m}z-\frac{k+m+2}{2m}\right]\Phi'(z)+\frac{k(1-\lambda)}{2m^2}\Phi(z)=0,$$

where prime denotes derivative with respect to z and

$$\lambda=1+\frac{U''(\varphi_0)}{kU(\varphi_0)}.$$

Proof. Gauss hypergeometric differential equation

Form of Gauss hypergeometric differential equation

$$z(z-1)\Phi''(z) + \left[(\alpha+\beta+1)z-\gamma\right]\Phi'(z) + \alpha\beta\Phi(z) = 0,$$

with parameters

$$\alpha = \frac{k+2-\Delta}{4m}, \qquad \beta = \frac{k+2+\Delta}{4m}, \qquad \gamma = \frac{k+2+m}{2m},$$

where

$$\Delta = \sqrt{(k-2)^2 + 8k\lambda}.$$

The differences of exponents at singularities $z=0$, $z=1$ and at $z=\infty$

$$\rho = 1-\gamma, \quad \sigma = \gamma-\alpha-\beta = \frac{1}{2}, \quad \tau = \beta-\alpha$$

and for our equation

$$\rho = \frac{m-k-2}{2m}, \qquad \sigma = \frac{1}{2}, \qquad \tau = \frac{\Delta}{2m}.$$

Kimura theorem. Condition A

The condition A of Kimura theorem is fulfilled if at least one of the following numbers

$$\begin{aligned} \rho+\sigma+\tau &= \frac{2m-k-2+\Delta}{2m}, \\ -\rho+\sigma+\tau &= \frac{k+2+\Delta}{2m}, \\ \rho-\sigma+\tau &= \frac{-k-2+\Delta}{2m}, \\ \rho+\sigma-\tau &= \frac{2m-k-2-\Delta}{2m} \end{aligned}$$

is an odd integer.

- If it is the first one, then $\lambda \in \mathcal{I}_0(k, m)$,
- if it is the second one, then $\lambda \in \mathcal{I}_1(k, m)$,
- if the third or fourth of the above numbers is an odd integer, then $\lambda \in \mathcal{I}_0(k, m) \cup \mathcal{I}_1(k, m)$.

Kimura theorem. Condition B

In this case the quantities ρ or $-\rho$, σ or $-\sigma$ and τ or $-\tau$ must belong to Schwarz's table. As $\sigma = \frac{1}{2}$ only items 1, 2, 4, 6, 9, or 14 of the Schwarz table are allowed.

Case 1.

- $\pm\rho = 1/2 + s$, for a certain $s \in \mathbb{Z}$, then $k = -2(mp + 1)$ for a certain $p \in \mathbb{Z}$. In this case τ is an arbitrary number, so λ is arbitrary.
- $\pm\tau = 1/2 + p$, for a certain $p \in \mathbb{Z}$, then $\lambda \in \mathcal{I}_2(k, m)$. In this case ρ-arbitrary, and thus k can be arbitrary.

Similar analysis for items 2, 4, 6, 9, or 14 of the Schwarz table

Example

$$H = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^m \cos\varphi. \tag{1}$$

It corresponds to $U(\varphi) = -\cos\varphi$. As $U'(\varphi) = \sin\varphi$, we take $\varphi_0 = 0$.
Since $U''(0)/U(0) = -1$, we have $\lambda = (k-1)/k$.
Comparing this value with forms of λ in sets $\mathcal{I}_j(k,m)$ for $j = 0, \ldots, 6$:

- if $\lambda \in \mathcal{I}_0(k,m)$, then $2m^2p^2 + (k+2)mp + 1 = 0$, and this implies
$$[4mp + k + 2]^2 = k^2 + 4k - 4,$$
- if $\lambda \in \mathcal{I}_1(k,m)$, then $m^2p^2 - (k+2)mp + 2 = 0$, and this implies
$$[2(mp-1) - k]^2 = k^2 + 4k - 4,$$
- if $\lambda \in \mathcal{I}_2(k,m)$, then
$$[m(2p+1)]^2 = k^2 + 4k - 4,$$

Example

- if $\lambda \in \mathcal{I}_3(k, m)$, then
$$[2m(3p+1)]^2 = 9(k^2+4k-4),$$
- if $\lambda \in \mathcal{I}_4(k, m)$, then
$$[m(4p+1)]^2 = 4(k^2+4k-4),$$
- if $\lambda \in \mathcal{I}_5(k, m)$, then
$$[2m(5p+1)]^2 = 25(k^2+4k-4),$$
- if $\lambda \in \mathcal{I}_6(k, m)$, then
$$[2m(5p+2)]^2 = 25(k^2+4k-4).$$

If one of the above conditions is fulfilled, then we have equality

$$k^2+4k-4 = q^2, \quad \text{for a certain } q \in \mathbb{Z}$$

that can be rewritten as

$$(k+2+q)(k+2-q) = 8$$

Example

$$(k+2+q)(k+2-q)=8$$

- Considering all decompositions of $8=(\pm1)\cdot(\pm8)=(\pm2)\cdot(\pm4)=(\pm4)\cdot(\pm2)=(\pm8)\cdot(\pm1)$, we obtain that $k\in\{-5,1\}$.
- With these values of k one can easily find that $\lambda=(k-1)/k\in\mathcal{I}_0(k,m)$ iff $m\in\{-1,1\}$.
- Hence, we have the following four cases with m, k and $l=m-k$:

$$\begin{array}{llll} 1. & m=1, & k=-5, & l=6,\\ 2. & m=-1, & k=1, & l=-2,\\ 3. & m=1, & k=1, & l=0,\\ 4. & m=-1, & k=-5, & l=4, \end{array} \tag{2}$$

Example

- Similarly, if $\lambda \in (k-1)/k \in \mathcal{I}_1$ with $k \in \{-5, 1\}$, then $m \in \{-2, -1, 1, 2\}$.
- Besides the above cases we have additionally

$$\begin{array}{llll} 5. & m=2, & k=1, & l=1, \\ 6. & m=-2, & k=1, & l=-3, \\ 7. & m=2, & k=-5, & l=7, \\ 8. & m=-2, & k=-5, & l=3. \end{array} \tag{3}$$

- No other cases when the necessary conditions for the integrability given by our Theorem.
- Surprisingly all cases (2) are integrable and in fact superintegrable.

Example. Superintegrable cases

Case 1.

$$H = \frac{1}{2}r^6\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,$$

$$F_1 := r^2 p_\varphi^2 \cos(2\varphi) - r^3 p_r p_\varphi \sin(2\varphi) + r^{-1}\sin\varphi\sin(2\varphi),$$

$$F_2 := r^2 p_\varphi^2 \sin(2\varphi) + r^3 p_r p_\varphi \cos(2\varphi) - r^{-1}\sin\varphi\cos(2\varphi).$$

Case 2.

$$H = \frac{1}{2}r^{-2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,$$

$$F_1 := r^{-2} p_\varphi^2 \cos(2\varphi) + r^{-1} p_r p_\varphi \sin(2\varphi) + r\sin\varphi\sin(2\varphi),$$

$$F_2 := -r^{-2} p_\varphi^2 \sin(2\varphi) + r^{-1} p_r p_\varphi \cos(2\varphi) + r\sin\varphi\cos(2\varphi).$$

Example. Superintegrable cases

Case 3.

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,$$

$$F_1 := r^{-1}p_\varphi^2\cos\varphi + p_r p_\varphi \sin\varphi + \frac{1}{2}r^2\sin^2\varphi,$$

$$F_2 := \left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi + r^{-1}p_r p_\varphi\cos(2\varphi) - r\sin\varphi.$$

Case 4.

$$H = \frac{1}{2}r^4\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,$$

$$F_1 := rp_\varphi^2\cos\varphi - r^2 p_r p_\varphi\sin\varphi + \frac{1}{2}r^{-2}\sin^2\varphi,$$

$$F_2 := r^4\left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi - r^3 p_r p_\varphi\cos(2\varphi) - r^{-1}\sin\varphi.$$

Example. Integrable cases

- In cases with parameters given in (3) we have integrable as well as non-integrable systems.
- Namely cases 5 and 8 are integrable.

Case 5.

$$H = \frac{1}{2} r \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^2 \cos\varphi,$$

$$F := r^{-1}(p_\varphi^2 - r^2 p_r^2)\cos\varphi + r^2(1+\cos^2\varphi) + 2p_r p_\varphi \sin\varphi.$$

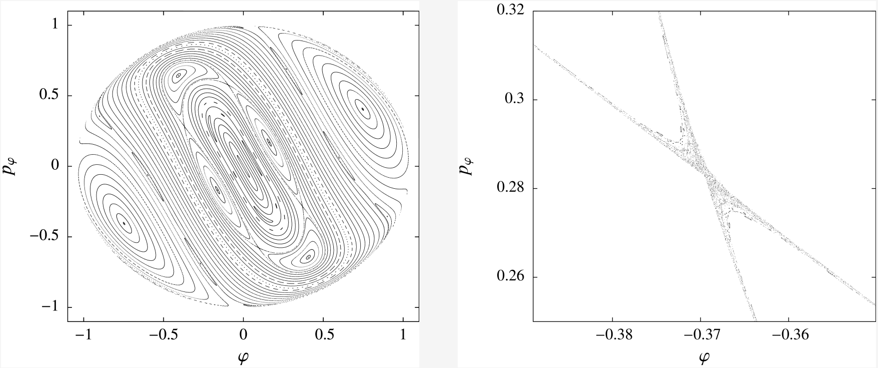
Case 8.

$$H = \frac{1}{2} r^3 \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-2} \cos\varphi,$$

$$F := r(p_\varphi^2 - r^2 p_r^2)\cos\varphi + r^{-2}(1+\cos^2\varphi) - 2r^2 p_r p_\varphi \sin\varphi.$$

- Poincaré sections for Hamiltonian systems with parameters given in cases 6 and 7 in (3) show chaotic area.

Example. Non-integrable case 6

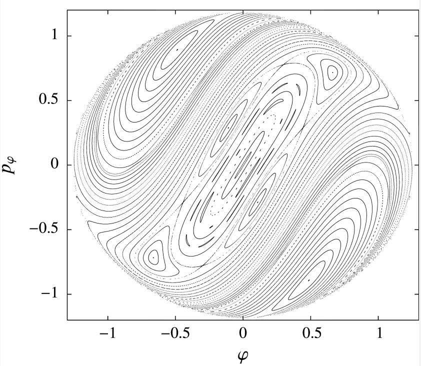


(a) section plane $r = 1$ with coordinates (φ, p_φ)

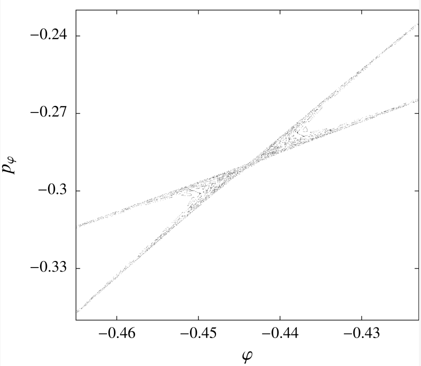
(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = -2, k = 1$ corresponding to case 6

Example. Non-integrable case 7



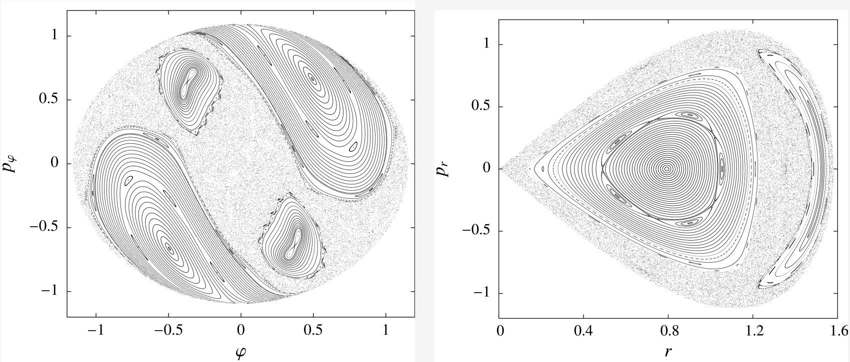
(a) section plane $r = 1$ with coordinates (φ, p_φ)



(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.3$ for Hamiltonian system given by (1) with $m = 2, k = -5$ corresponding to case 7

Example 2. Non-integrable cases for family $k = -2(mp+1)$

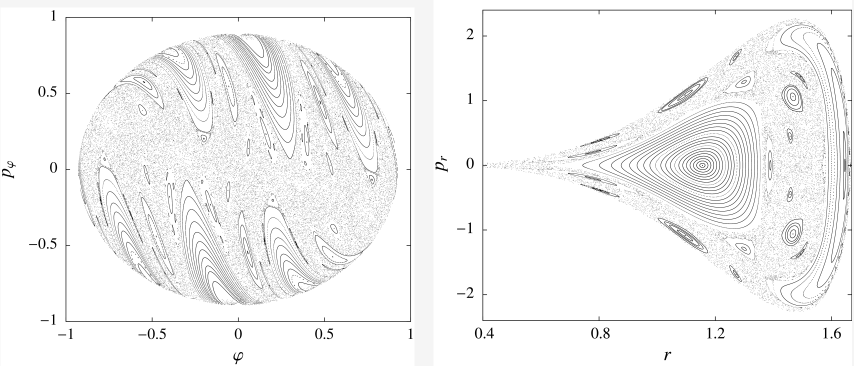


(a) section plane $r = 1$ with coordinates (φ, p_φ)

(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = -2, k = 2$

Example. Non-integrable cases for family $k = -2(mp+1)$

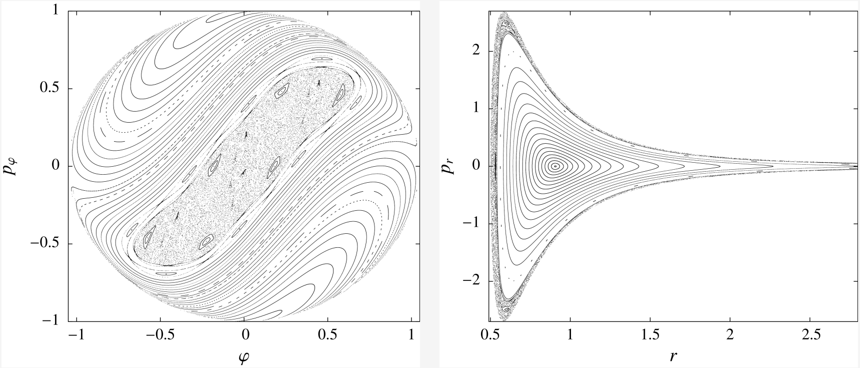


(a) section plane $r = 1$ with coordinates (φ, p_φ)

(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.6$ for Hamiltonian system given by (1) with $m = -1, k = 8$

Example. Non-integrable cases for family $k = -2(mp + 1)$



(a) section plane $r = 1$ with coordinates (φ, p_φ)

(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = 1, k = -6$

Another analogue in curved spaces

n dimensional constant curvature spaces $S^n_{[\kappa]}$: the sphere $\mathbb{S}^n$ for $\kappa > 0$, Euclidean space $\mathbb{E}^n$ for $\kappa = 0$ and and hyperbolic space $\mathbb{H}^n$ for $\kappa < 0$

$$\mathrm{C}_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ 1 & \text{for} \quad \kappa = 0, \\ \cosh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0, \end{cases}$$

$$\mathrm{S}_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ x & \text{for} \quad \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0. \end{cases}$$

These functions satisfy the following identities

$$\mathrm{C}^2_\kappa(x) + \kappa\,\mathrm{S}^2_\kappa(x) = 1, \quad \mathrm{S}'_\kappa(x) = \mathrm{C}_\kappa(x), \quad \mathrm{C}'_\kappa(x) = -\kappa\,\mathrm{S}_\kappa(x).$$

Another analogue in curved spaces

Point $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in S^n_{[\kappa]}$ can be characterised by variables $(r, \varphi_2, \ldots, \varphi_n)$ defined in the following way

$$x_0 = \mathrm{C}_\kappa(r), \quad x_1 = \mathrm{S}_\kappa(r)\cos\varphi_2,$$

$$x_i = \mathrm{S}_\kappa(r)\prod_{j=2}^{i}\sin\varphi_j\cos\varphi_{j+1} \quad \text{for} \quad j = 2, \ldots, n-1, \; x_n = \mathrm{S}_\kappa(r)\prod_{j=2}^{n}\sin\varphi_j.$$

Metrics

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{S}^2_\kappa(r)\left(\mathrm{d}\varphi_2^2 + \sum_{i=3}^{n}\left(\prod_{j=2}^{i-1}\sin^2\varphi_j\right)\mathrm{d}\varphi_i^2\right)$$

and corresponding kinetic energy T as

$$T = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\mathrm{S}^2_\kappa(r)\left(\dot{\varphi}_2^2 + \sum_{i=3}^{n}\left(\prod_{j=2}^{i-1}\sin^2\varphi_j\right)\dot{\varphi}_i^2\right).$$

Another analogue in curved spaces

If we introduce canonical momenta $p_1 = \partial T/\partial \dot{r}$ and $p_i = \partial T/\partial \dot{\varphi}_i$ for $i = 2, \ldots, n$, then

$$T = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{\mathrm{S}_\kappa^2(r)} + \sum_{i=3}^{n} \frac{p_i^2}{\mathrm{S}_\kappa^2(r)\prod_{j=2}^{i-1}\sin^2\varphi_j}\right),$$

As the equivalent of homogeneous potential of degree $k \in \mathbb{Z}$ in flat space we will consider

$$V = \mathrm{S}_\kappa^k(r)U(\varphi_2, \ldots, \varphi_n).$$

Final form of the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{\mathrm{S}_\kappa^2(r)} + \sum_{i=3}^{n} \frac{p_i^2}{\mathrm{S}_\kappa^2(r)\prod_{j=2}^{i-1}\sin^2\varphi_j}\right) + \mathrm{S}_\kappa^k(r)U(\varphi_2, \ldots, \varphi_n).$$

Another analogue in curved spaces. Case $n = 2$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathrm{S}_\kappa(r)^2}\right) + \mathrm{S}_\kappa^k(r)U(\varphi), \tag{4}$$

with $k \in \mathbb{Z}$. Hamilton equations

$$\dot{r} = p_r, \qquad \dot{p}_r = \frac{p_\varphi^2}{\mathrm{S}_\kappa^3(r)}\,\mathrm{C}_\kappa(r) - k\,\mathrm{S}_\kappa^{k-1}(r)\,\mathrm{C}_\kappa(r),$$

$$\dot{\varphi} = \frac{p_\varphi}{\mathrm{S}_\kappa^2(r)}, \qquad \dot{p}_\varphi = -k\,\mathrm{S}_\kappa^{k-1}(r)U'(\varphi),$$

have an invariant plane given by $\varphi(t) = \varphi_0 = \text{const}$ with $U'(\varphi_0) = 0$ and $p_\varphi = 0$.
We consider a particular solution on the energy level $H = e$

$$H = \frac{1}{2}p_r^2 + \mathrm{S}_\kappa^k(r)U(\varphi_0).$$

Case $n = 2$. Integrability theorem

Theorem

If the Hamiltonian system governed by Hamilton function (4) is meromorphically integrable, then at each Darboux point the pair (k, λ) belongs to one of the following list

case	k	λ
1	k	$-\dfrac{(k-4p)(2p-1)}{k}$
2	k	$\dfrac{2p(k-2+4p)}{k}$
3	k	$-\dfrac{(k+4p)(k-4(1+p))}{8k}$
4	$-2+4p$	*arbitrary*
6	$k=2q-1$	$-\dfrac{(-2+3k+12p)(3k-2(5+6p))}{72k}$

Problems with algebraic Hamiltonians

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right), \qquad V = r^m U(\varphi),$$

What about the case when m or k are rational?

First we consider natural Hamiltonian system $H = T + V$ with an algebraic potential V

Equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{q} = \mathbf{p}, \qquad \frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{p} = -\partial_{\mathbf{q}} V(\mathbf{q}), \tag{H}$$

where

$$\partial_{\mathbf{q}} V = [\partial_1 V, \ldots, \partial_n V]^T = \left[\frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_n} \right]^T$$

$$\mathbf{X}_H = \mathbf{p} \cdot \partial_{\mathbf{q}} - \partial_{\mathbf{q}} V(\mathbf{q}) \cdot \partial_{\mathbf{p}}$$

Extension of the system

V is algebraic, so

$$F(\mathbf{q}, V(\mathbf{q})) := f_0(\mathbf{q}) + f_1(\mathbf{q})V(\mathbf{q}) + \cdots + f_m(\mathbf{q})V(\mathbf{q})^m = 0, \qquad (5)$$

where $f_0(\mathbf{q}), \ldots, f_m(\mathbf{q}) \in \mathbb{C}(\mathbf{q})$, $F \in \mathbb{C}(\mathbf{q})[u]$.
We have

$$\partial_i V(\mathbf{q}) := -\frac{1}{\partial_u F(\mathbf{q}, V(\mathbf{q}))}\partial_i F(\mathbf{q}, V(\mathbf{q})), \quad \text{for} \quad i = 1, \ldots, n,$$

where

$$\partial_u F := \sum_{r=0}^{m} r f_r u^{r-1}, \quad \text{and} \quad \partial_i F := \sum_{r=0}^{m} (\partial_i f_r) u^r.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{q}) = L_{\mathbf{X}_H}(V)(\mathbf{q}) = \partial_{\mathbf{q}} V(\mathbf{q}) \cdot \mathbf{p} = -\frac{\mathbf{p} \cdot \partial_{\mathbf{q}} F(\mathbf{q}, V(\mathbf{q}))}{\partial_u F(\mathbf{q}, V(\mathbf{q}))}$$

Extended system

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} &= \mathbf{p},\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} &= \frac{1}{\partial_u F(\mathbf{q}, u)}\partial_{\mathbf{q}} F(\mathbf{q}, u),\\
\frac{\mathrm{d}}{\mathrm{d}t}u &= -\frac{\mathbf{p}\cdot\partial_{\mathbf{q}} F(\mathbf{q}, u)}{\partial_u F(\mathbf{q}, u)}
\end{aligned}\right\} \qquad \text{(HP)}$$

Extended system. Properties.

Lemma

If $(\mathbf{q}(t), \mathbf{p}(t))$ *be a solution of system* (H) *with initial condition* $(\mathbf{q}(0), \mathbf{p}(0)) = (\mathbf{q}_0, \mathbf{p}_0)$, *then* $(\mathbf{q}(t), \mathbf{p}(t), u(t))$ *with* $u(t) := V(\mathbf{q}(t))$ *is a solution of* (HP) *satisfying initial condition* $(\mathbf{q}(0), \mathbf{p}(0), u(0)) = (\mathbf{q}_0, \mathbf{p}_0, V(\mathbf{q}_0))$.

Lemma

System (HP) *has two polynomial first integrals*

$$K = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + u, \quad \text{and} \quad F = F(\mathbf{q}, u).$$

Extended system. Poisson structure

Theorem

Extended system is Hamiltonian with respect to the Poisson structure

$$J(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & \mathbf{E}_n & \mathbf{0} \\ -\mathbf{E}_n & \mathbf{0} & \mathbf{C}(\mathbf{z}) \\ \mathbf{0}^T & -\mathbf{C}(\mathbf{z})^T & 0 \end{bmatrix}, \quad \mathbf{C} = \frac{1}{\partial_u F}\partial_{\mathbf{q}} F.$$

and it has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \{\mathbf{z}, K(\mathbf{z})\} = J(\mathbf{z})\partial_{\mathbf{z}} K(\mathbf{z}), \qquad K(\mathbf{z}) = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} + u.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{E}_n & \mathbf{0} \\ -\mathbf{E}_n & \mathbf{0} & \mathbf{C} \\ \mathbf{0}^T & -\mathbf{C}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{C} \\ -\mathbf{C}^T\mathbf{p} \end{bmatrix}.$$

- non-Hamiltonian version of Morales-Ramis theory *M. Ayoul and N. T. Zung, Galoisian obstructions to non-Hamiltonian integrability,* C. R., Math., Acad. Sci. Paris *348, No. 23-24, 1323-1326 (2010).*

Algebraic Hamiltonian

Now we assume that Hamiltonian is an algebraic function i.e. exists a function $F \in \mathbb{C}(\mathbf{q}, \mathbf{p})[u]$ such that

$$F(\mathbf{q}, \mathbf{p}, H(\mathbf{q}, \mathbf{p})) = 0,$$

where

$$F(q, p, u) = \sum_{i=0}^{m} F_i(\mathbf{q}, \mathbf{p}) u^i, \quad F_i \in \mathbb{C}(\mathbf{q}, \mathbf{p}).$$

If we introduce $\mathbf{y} = (\mathbf{q}, \mathbf{p}) \in \mathbb{C}^{2n}$, then

$$F(\mathbf{y}, H(\mathbf{y})) = 0$$

Differentiation gives

$$\partial_i H := \frac{\partial H}{\partial y_i} = -\frac{\partial_i F}{\partial_u F}$$

Extended system

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial z_{n+i}} = -\frac{\partial_{i+n} F}{\partial_u F},$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial H}{\partial z_i} = \frac{\partial_i F}{\partial_u F},$$
$$\dot{u} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \{\mathbf{z}, K(\mathbf{z})\} = J(\mathbf{z})\partial_{\mathbf{z}} K(\mathbf{z}), \qquad K(\mathbf{z}) = u.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\\ u\end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{E}_n & -\frac{1}{\partial_u F}\partial_{\mathbf{p}} F \\ -\mathbf{E}_n & \mathbf{0} & \frac{1}{\partial_u F}\partial_{\mathbf{q}} F \\ \left(\frac{1}{\partial_u F}\partial_{\mathbf{p}} F\right)^T & -\left(\frac{1}{\partial_u F}\partial_{\mathbf{q}} F\right)^T & 0 \end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ 1\end{bmatrix}.$$

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