Two-Dimensional Geodesic Flows with Polynomial and non-Polynomial Integrals

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  - The protagonist: Puiseux-type integral

$$F = \left(\frac{p_1}{g}\right)^{\epsilon} \prod_{k=1}^{N} (q - b^k(x,t))^{\epsilon_k}, \qquad (1)$$

for the metric

$$ds^2 = dx^2 + g^2(x,t)dt^2; \qquad (2)$$

$$q = \frac{g(x,t)p_2}{p_1}, \quad g(x,t) = -\sum_{m=1}^{N} \epsilon_m b^m(x,t), \quad \epsilon = \sum_{m=1}^{N} \epsilon_m. \qquad (3)$$

(parameters $\epsilon_k \neq 0$)

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- and a more general integral

$$F = p_1^{2K} g^{-\epsilon} \prod_{m=1}^{N} (q - b^m(x,t))^{\epsilon_m}.$$

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$$C_t^k = (kC^{k-1} + (k+2)C^{k+1})C_x^0 - C^0 C_x^{k+1}, \quad k = 0, 1, 2, \ldots,$$

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- Simplest examples

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$$C^k_t = (kC^{k-1} + (k+2)C^{k+1})C^0_x - C^0 C^{k+1}_x, \quad k = 0, 1, 2, \dots,$$

    - a better tool to conjecture a "correct" non-polynomial form of the additional first integral,
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- Simplest examples
- Literature review & discussion
  (to be added by the experts in the audience :)

Integrable Geodesic Flows

- In this talk we consider "geodesic flows", i.e. well-known geodesic equations

$$\ddot{q}^i + \Gamma^i_{ms}\dot{q}^m\dot{q}^s = 0,$$

which can be written in the Hamiltonian form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, ..., n,$$

with the Hamiltonian function

$$H = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}(\mathbf{q})p_i p_j$$

and the corresponding nondegenerate symmetric metric

$$ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(\mathbf{q})dq^i dq^j.$$

Liouville Integrability

- A mechanical system

$$\dot{q}^i = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial q^i}, \quad i = 1, ..., n,$$

where $\mathbf{q} = (q^1, \ldots, q^n), \quad \mathbf{p} = (p_1, \ldots, p_n)$,

- is called *Liouville integrable*, if it possesses $n-1$ functions $F_i(\mathbf{q}, \mathbf{p})$ such that

$$\{F_i, H\}_{\mathbf{p},\mathbf{q}} = 0, \qquad \{F_i, F_j\}_{\mathbf{p},\mathbf{q}} = 0,$$

where the canonical Poisson bracket is

$$\{F, H\}_{\mathbf{p},\mathbf{q}} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} \right).$$

2-dim Integrable Geodesic Flows

- In this talk we restrict our consideration to the *local two dimensional case* only and assume the metric reduced to the following form (for semi-geodesic coordinates $q_1 = x, q_2 = t$)

$$ds^2 = dx^2 + g^2(x,t)dt^2 \tag{4}$$

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- **The simplest example:** polynomial integral $F(x,t,p_1,p_2) = \sum_{k=0}^{N} \frac{(-1)^k a^k(x,t)}{g^{N-k}} p_1^{N-k} p_2^k$ has coefficients $a^k(x,t)$ which satisfy the N component hydrodynamic type system

$$a_t^0 = a^1 a_x^{N-1},\ a_t^k = a^{N-1} a_x^{k-1} + [(k+1)a^{k+1} - (N+1-k)a^{k-1}] a_x^{N-1}, \tag{5}$$

where $k = 1, \ldots, N-1$ and $a^N \equiv 1$, $a^{N-1} \equiv g(x,t)$.

- This system is **semi-Hamiltonian** (M.Bialy, A.Mironov, 2009) so has infinitely many conservation laws and commuting flows of the same hydrodynamic type (so, integrable).

Polynomial *vs* non-polynomial integrals

- (V.V. Kozlov, 1989): "It has long been remarked that all the known first integrals of classical mechanical systems are polynomial w.r.t. velocities (or functions of such polynomials). This observation has no complete explanation yet. For this reason the analytical and geometrical nature of polynomial integrals is of big interest."

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- (В.В.Козлов, О рациональных интегралах геодезических потоков 2014): "Несколько преувеличенное внимание к полиномиальным по импульсам интегралам связано со следующим простым наблюдением. Пусть нам известен аналитический (или даже формально аналитический) по импульсам первый интеграл F ... Разложим F в (формальный) ряд по однородным формам относительно импульсов p_1 и p_2 : $F = \sum F_m$, $\deg F_m = m$. Легко проверить, что каждая однородная форма F_m будет полиномиальным интегралом. Автором настоящей заметки в связи с этим высказывалась точка зрения, что все первые интегралы задачи о геодезических (и даже более общих обратимых систем из механики) являются либо полиномами, либо функциями от полиномов. Во всяком случае все известные нам конкретные примеры интегрируемых обратимых систем укладываются в эту схему."

Integrable classical mechanical systems and integrable hydrodynamic type systems

- 1.5-dim integrable systems $H = p^2/2 + V(x,t)$, with polynomial integrals (V.V. Kozlov, 1989; M. V. Deryabin, 1997), polynomial & non-polynomial integrals (M.V. Pavlov & S.P. Tsarev, 2013):

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- 2-dim integrable geodesic flows with polynomial integrals (M.Bialy & A.Mironov, 2009; M.V. Pavlov & S.P. Tsarev, J. Phys.A, 2016: a regular procedure for construction of conservation laws, commuting flows $\Rightarrow$ coefficients of the integrals via GHM)

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- **Now:** The universal infinite hydrodynamic chain for arbitrary integrals of 2-dim integrable geodesic flows, polynomial & non-polynomial first integrals as N-component hydrodynamic reductions (M.V. Pavlov & S.P. Tsarev, 2015).

Effective 1.5-dim Hamiltonian and Momentum

- Take a first integral $F(x, t, p_1, p_2)$ of the 2-dim geodesic flow for the metric $ds^2 = dx^2 + g^2(x, t)dt^2$;

Effective 1.5-dim Hamiltonian and Momentum

- Take a first integral $F(x, t, p_1, p_2)$ of the 2-dim geodesic flow for the metric $ds^2 = dx^2 + g^2(x, t)dt^2$;
- introduce $s = p_2/p_1$. then $p(x, t, s) = (1 + g^{-2}s^2)^{-1/2} \Rightarrow s(x, t, p) = g\sqrt{1 - p^2}/p$

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- introduce the new 1.5-dim *effective* Hamiltonian $\tilde{H}(x, t) = g(x, t)\sqrt{1 - p^2}$, with the Hamilton's equations

$$x' = \frac{\partial \tilde{H}}{\partial p}, \quad p' = -\frac{\partial \tilde{H}}{\partial x}, \tag{6}$$

and the reduced first integral $\lambda(x, t, p)$:
$F(x, t, p_1, p_2) = \phi(H)\lambda(x, t, p(x, t, s))$

- Instead of the effective momentum p we introduce $q = \frac{g(x,t)p_2}{p_1} = -\frac{p}{\sqrt{1-p^2}}$, and consider the Liouville equation $\{F, H\} = 0$ for $\lambda(x, t, q)$:

$$\lambda_t = gq\lambda_x + (1+q^2)\lambda_q g_x. \tag{7}$$

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- Expand $\mu(x, t, q) = -\ln\lambda$ w.r.t. the parameter $q \to \infty$:

$$\mu(x, t, q) = -\ln\lambda = \frac{C^0(x, t)}{q} + \frac{C^1(x, t)}{q^2} + \frac{C^2(x, t)}{q^3} + \frac{C^3(x, t)}{q^4} + \dots$$

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- Then (7) is equivalent to the hydrodynamic chain ($g = -C^0$)

$$C_t^k = (kC^{k-1} + (k+2)C^{k+1})C_x^0 - C^0 C_x^{k+1}, \quad k = 0, 1, 2, \dots, \tag{8}$$

The simplest reduction: polynomial integrals

- The N-parametric reduction

$$\lambda(q,\mathbf{a}) = (1+q^2)^{-N/2}\left(q^N + \sum_{k=0}^{N-1} q^k a^k(x,t)\right)$$
$$= (1+q^2)^{-N/2}\prod_{k=1}^{N}(q-b^k(x,t)).$$

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- $$b_t^k = (1+(b^k)^2)\sum_{m=1}^{N} b_x^m - \left(\sum_{n=1}^{N} b^n\right) b^k b_x^k. \qquad (9)$$

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- The Riemann surface

$$\lambda(b^k, q) = (1+q^2)^{-N/2}\prod_{k=1}^{N}(q - b^k). \qquad (10)$$

provides infinite series of conservation laws and commuting flows for (9).

The polynomial integrals: finding their coefficients & the metric

- We can find conservation laws of the system for b^k (resp. a^k) expanding the Riemann surface at infinity (the Kruskal series) or at the branching points (N principal series of conservation laws).

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- We can find the commuting flows using the generating function expansions.
- We can find solutions of the N-component hydrodynamic type system (here, the equations for b^k and $g(x,t) = -\sum_{k=1}^{N} b^k$) using the Generalized Hodograph Method.

Conservation laws

- Expand $q = q(\lambda, b^k)$ w.r.t. λ at the vicinity of each root b^k of the Riemann surface $\lambda(b^k, q) = (1+q^2)^{-N/2}\prod_{k=1}^{N}(q-b^k)$:

$$q^{(k)}(\lambda) = b^k + \lambda q_1^{(k)} + \lambda^2 q_2^{(k)} + \dots, \qquad \lambda \to 0. \tag{11}$$

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$$q^{(k)}(\lambda) = b^k + \lambda q_1^{(k)} + \lambda^2 q_2^{(k)} + \dots, \qquad \lambda \to 0. \tag{11}$$

- Substitute $q^{(k)}(\lambda)$ into $p = -\frac{q}{\sqrt{1+q^2}}$ obtaining

$$p^{(k)}(\lambda) = -\frac{q^{(k)}(\lambda)}{\sqrt{1 + (q^{(k)}(\lambda))^2}} = h^k + \lambda p_1^{(k)} + \lambda^2 p_2^{(k)} + \dots, \tag{12}$$

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- Substitute $p^{(k)}(\lambda)$ into the generating function p of conservation laws

$$(p^{(k)}(\lambda))_t + \left(\sqrt{1-(p^{(k)}(\lambda))^2}\sum_{n=1}^{N} b^n\right)_x = 0$$

(cont.)

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  and expand both sides w.r.t. the parameter $\lambda \to 0$. Matching the coefficients of the same powers of λ one obtains N infinite series of conservation laws.

- Alternatively: take expansions at infinity as $q \to \infty$ and $\lambda \to 1$. Set $\mu = -\ln\lambda$, so $\mu \to 0$:

$$p(\mu) = -1 + \frac{\mu^2}{2(C^0)^2} - \mu^3\frac{C^1}{(C^0)^4} + \mu^4\left(-\frac{C^2}{(C^0)^5} + \frac{5(C^1)^2}{2(C^0)^6} - \frac{3}{8(C^0)^4}\right) \qquad (14)$$

  Substitute into (13), note that $a^{N-1} = -C^0$ and equate the coefficients at equal powers of μ obtaining the Kruskal series of conservation laws.

Commuting flows

- The conservation laws for the generating function of all higher commuting flows can be represented formally as

$$\partial_{\tau(\eta)} p(\mu) = \partial_x G(p(\mu), p(\eta)), \tag{15}$$

where $\mu = -\ln\lambda$, $p(\eta)$ is obtained by formally replacing the parameter μ by the parameter η and

$$G(p(\mu), p(\eta)) = \frac{\sqrt{1-p^2(\mu)}}{\sqrt{1-p^2(\eta)}} + \frac{1}{2}p(\mu)\ln\frac{p(\eta)+1}{p(\eta)-1} + \tag{16}$$

$$+\ln\frac{p(\mu)-p(\eta)}{\sqrt{1-p^2(\mu)}+\sqrt{1-p^2(\eta)}}.$$

The so called “vertex” operator $\partial_{\tau(\eta)}$ is not yet determined and should be specified separately for different cases below.

The Kruskal Series of Commuting flows

- Take

$$p(\eta) = -1 + \frac{\eta^2}{2(C^0)^2} - \eta^3 \frac{C^1}{(C^0)^4} + \eta^4 \left(-\frac{C^2}{(C^0)^5} + \frac{5(C^1)^2}{5(C^0)^6} - \frac{3}{8(C^0)^4} \right)$$

and the following expansion of the vertex operator

$$\partial_{\tau(\eta)} = \ln \eta \partial_{t^0} + \frac{1}{\eta}\partial_{t^1} + \partial_{t^2} + \eta \partial_{t^3} + \eta^2 \partial_{t^4} + \dots$$

to match with the expansion of $G(p(\mu), p(\eta))$, $t^0 \equiv x$ and $t^1 \equiv t$, so

$$(p(\mu))_{t^0} = (p(\mu))_x \,, \quad (p(\mu))_{t^1} = -\left(C^0 \sqrt{1 - p^2(\mu)} \right)_x ,$$

$$(p(\mu))_{t^2} = \left(\frac{C^1}{C^0} \sqrt{1 - p^2(\mu)} - p(\mu) \ln C^0 + \frac{1}{2} \ln \frac{p(\mu)+1}{p(\mu)-1} - p(\mu) \ln 2 \right)$$

$$(p(\mu))_{t^3} = \left[\left(\frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} + \frac{1}{2C^0} \right) \sqrt{1 - p^2(\mu)} - \frac{C^1}{(C^0)^2} p(\mu) - \frac{\ }{C} \right.$$

Generalized Hodograph Method & nonlinear superposition

- Take the algebraic system

$$x + t(2C^1 - C^0 b^i) = \ln\left(\sqrt{1+q^2} - q\right) - q\sqrt{1+q^2} + \tag{17}$$

$$+ C^0 \frac{\sqrt{1+q^2}}{q - b^i}\left(\sum_{m=1}^{N} \frac{1}{q - b^m} - \frac{Nq}{1+q^2}\right)^{-1},$$

where $q(\mathbf{b}, \lambda)$ is the inverse function to the function $\lambda(\mathbf{b}, q)$ of the Riemann surface.

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Theorem

Hydrodynamic type system (9) has infinitely many particular solutions $b^i(x, t)$ in the implicit form given by (17) with a free parameter λ.

-

Generalized Hodograph Method & nonlinear superposition

- Expanding the generating function $p(\mathbf{b}, \lambda)$ at different points on the Riemann surface, one can construct infinite multiparametric series of new solutions $g(\mathbf{b}(x,t))$. Introduce functions

$$W_i(\mathbf{b}, \lambda) = \ln\left(\sqrt{1+q^2} - q\right) - q\sqrt{1+q^2} + \\ + C^0 \frac{\sqrt{1+q^2}}{q - b^i}\left(\sum_{m=1}^{N} \frac{1}{q - b^m} - \frac{Nq}{1+q^2}\right)^{-1},$$

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$$+C^0\frac{\sqrt{1+q^2}}{q-b^i}\left(\sum_{m=1}^{N}\frac{1}{q-b^m}-\frac{Nq}{1+q^2}\right)^{-1},$$

- *Kruskal series*. Asymptotic expansion at infinity gives

$$W_i(\mathbf{b},\mu)=\ln\mu+\frac{1}{\mu}W_i^{(-1)}(\mathbf{b})+W_i^{(0)}(\mathbf{b})+\mu W_i^{(1)}(\mathbf{b})+\mu^2W_i^{(2)}(\mathbf{b})+..$$

where, for instance,

$$W_i^{(-1)}(\mathbf{b})=C^0b^i-2C^1.$$

$$W_i^{(0)}(\mathbf{b})=(b^i)^2-\frac{C^1}{C^0}b^i+\frac{2(C^1)^2-3C^0C^2}{(C^0)^2}-\log C^0-\log 2.$$

- A simplest two-component *rational* case (for a first integral) is

$$F = \frac{q - b^1}{q - b^2} = \frac{(b^1 - b^2)p_2 - b^1 p_1}{(b^1 - b^2)p_2 - b^2 p_1},$$

where the corresponding hydrodynamic type system

$$b^1_t = (1 + b^1 b^2) b^1_x - (1 + (b^1)^2) b^2_x, \quad b^2_t = (1 + (b^2)^2) b^1_x - (1 + b^1 b^2) b^2_x$$

takes the homogeneous form

$$u_t = 2v_x, \quad v_t = -\frac{1}{2}(\ln u)_x,$$

where we introduce the notation

$$u = \frac{1}{(b^1 - b^2)^2}, \quad v = \frac{1}{2} \frac{b^1 + b^2}{b^1 - b^2}.$$

Corresponding first integral becomes

$$F = \frac{p_2 + \left(v + \frac{1}{2}\right) p_1}{p_2 + \left(v - \frac{1}{2}\right) p_1}.$$

The Simplest Rational Case

- This hydrodynamic type system appears in fluid mechanics (barotropic fluid), gas dynamics (polytropic gas), also well known as a dispersionless limit of 2DToda Lattice. In Riemann invariants it has the form
$$r^1_x = (r^2 - r^1) r^1_t, \quad r^2_x = (r^1 - r^2) r^2_t,$$
where $u = -(r^1 - r^2)^2$ and $v = r^1 + r^2$.
- According to the Generalised Hodograph Method, a general solution of this two-component hydrodynamic type system can be written in the following implicit form
$$x = -\frac{1}{2}\frac{\partial_1 h - \partial_2 h}{r^1 - r^2}, \quad t = \frac{1}{2}(\partial_1 h + \partial_2 h),$$
where the function $h(r^1, r^2)$ is a general solution of the Euler–Darboux–Poisson equation
$$\partial_{12} h = -\frac{1}{2}\frac{\partial_1 h - \partial_2 h}{r^1 - r^2}.$$

The Simplest Rational Case

- Taking into account that $g = -C^0$ and $u = g^{-2}$, then $g^2 = -(r^1 - r^2)^{-2}$. Thus

$$ds^2 = dx^2 + g^2(t,x)dt^2 = (dx + igdt)(dx - igdt)$$

$$= \frac{1}{4}\left(d\frac{\partial_1 h - \partial_2 h}{r^1 - r^2} - \frac{1}{r^1 - r^2} d(\partial_1 h + \partial_2 h)\right)\left(d\frac{\partial_1 h - \partial_2 h}{r^1 - r^2} + \frac{1}{r^1 - r^2} d(\partial \ldots\right.$$

$$= \frac{1}{4}\left(-2\frac{d(\partial_2 h)}{r^1 - r^2} - \frac{\partial_1 h - \partial_2 h}{(r^1 - r^2)^2} d(r^1 - r^2)\right)\left(2\frac{d(\partial_1 h)}{r^1 - r^2} - \frac{\partial_1 h - \partial_2 h}{(r^1 - r^2)^2}(dr^1 \ldots\right.$$

$$= \frac{1}{4(r^1 - r^2)^2}\left(-2h_{22} + \frac{\partial_1 h - \partial_2 h}{r^1 - r^2}\right)\left(2h_{11} - \frac{\partial_1 h - \partial_2 h}{r^1 - r^2}\right) dr^1 dr^2$$

$$= -\frac{(h_{11} + h_{12})(h_{22} + h_{12})}{(r^1 - r^2)^2} dr^1 dr^2 = -\frac{\tilde{h}_1 \tilde{h}_2}{(r^1 - r^2)^2} dr^1 dr^2,$$

where $\tilde{h} = h_1 + h_2$ is a new solution of the Euler–Darboux–Poisson equation.

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