New results on Algebraic Birkhoff Conjecture

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`based on joint work with A.E. Mironov`

Introduction

Anatole Katok colloquium title: Billiard table as a mathematicians playground.

1. Birkhoff billiards

2. Definitions and known results:

1). Birkhoff Billiard flow g^t and the map $\mathcal{B}$.

2). Invariant curves, Phase portraits.

3. Lazutkin,KAM type result existence of Caustics.

4. Integrable billiards. Circular versus Elliptic billiards. Integrals for the (A)Circle and (B)Ellipse:

In case A :

$$F(x, v) = xv_y - yv_x,$$

is the momentum of (x, v) with respect to 0.

In case B:

$$F(x, v) = ((x - c)v_y - yv_x)((x + c)v_y - yv_x)$$

is the product of two momenta with respect to the foci $(\pm c, 0))$ of the ellipse.

5. Birkhoff-Poritsky conjecture: *The only integrable billiards with smooth convex boundaries are Circles and Ellipses.*

6. Algebraic version: *The only billiards admitting polynomial in v integrals are Circles and Ellipses*

Approaches (in alphabetic order)

1) Baryshnikov,Zharnitsky : Birkhoff distributions.

2) Bialy: Total integrability of convex billiards.

3) Bolotin: Polynomial integrals, requires smoothness of certain complex algebraic curve.

4) Delshams, Ramirez-Ros: Small perturbation of ellipse $\Rightarrow$ splitting of separatrices.

5) Glutsyuk: complexification of the reflection law.

6) Kaloshin and Sorrentino: If an integrable billiard is C^2 conjugate to an ellipse (resp. a circle) in a neighborhood of the boundary, then it is an ellipse (resp. a circle).

7) Treschev: Formal power series for local integrability.

8) Our contribution with A. E. Mironov on Algebraic Birkhoff conjecture extends 3).

Results

Notation: $\gamma \to \tilde{\gamma} \subset \mathbb{C}P^2$; $\Gamma \to \tilde{\Gamma} \subset \mathbb{C}P^2$. Absolute:

$$\Lambda \subset \mathbb{C}P^2 : \Lambda = \{x^2 + y^2 + Kz^2 = 0, K = 0, \pm 1\}$$

Theorem.(Bolotin 1990,1992) *Let γ be a smooth non-geodesic arc of the boundary curve of the domain $\Omega \subset \Sigma$. Suppose that Birkhoff billiard inside $\Omega \subset \Sigma$ admits a non-constant polynomial integral Φ on the energy level $\{|v| = 1\}$. It then follows that γ is necessarily algebraic curve. Moreover, let $\tilde{\gamma}$ in $\mathbb{C}P^2$ be the irreducible component of γ.*
If $\tilde{\gamma}$ is a smooth curve, then $\tilde{\gamma}$ is of degree 2.
(For $K = \pm 1$ it is assumed in addition that at least one intersection point of $\tilde{\gamma}$ with the absolute Λ is transversal).

Remark: Smoothness of $\tilde{\gamma}$ is a severe restriction.

Open problem: *How to relax the assumption of smoothnes of $\tilde{\gamma}$?*

Main Theorem

Theorem. (M.B. and A.E.Mironov, 2015-2016) *Let Γ be the dual curve to γ. Denote $\tilde{\Gamma}$ the irreducible component in $\mathbb{C}P^2$ of Γ. Suppose that Birkhoff billiard admits a polynomial integral. Then, either $\tilde{\Gamma}$ has degree 2, or $\tilde{\Gamma}$ necessarily contains singular points. Moreover, all singular and inflection points of $\tilde{\Gamma}$ in $\mathbb{C}P^2$ belong to the Absolute Λ.*

Corollaries and Examples

Corollary 1. *If the Birkhoff billiard inside γ is integrable with an integral which is polynomial in v, then $\tilde{\gamma}$ does not have two real algebraic ovals having a common tangent line.*

Indeed, if there were two such ovals γ and γ_1 (see Fig. 1), then the point (x, y) on Γ dual to the common tangent line τ would be a real singular point of Γ different from O, and hence $x^2 + y^2 \neq 0$. It then follows from Theorem 1 that the Birkhoff billiard inside γ does not admit polynomial integral on the energy level $|v| = 1$.

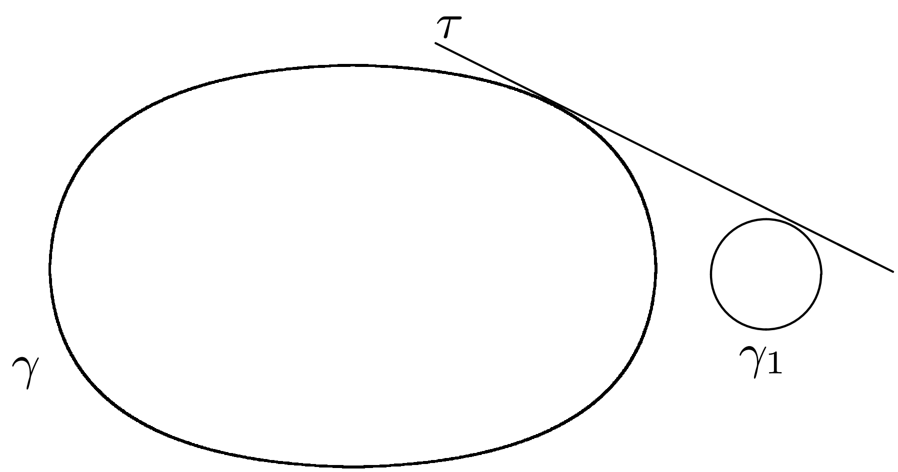

Fig. 1 Non-integrable Birkhoff billiard inside γ.

Example: Consider real algebraic curve

$$y^2 = F(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)f(x), \quad x_j \in \mathbb{R}, \quad x_1 < x_2 < x_3 < x_4,$$

where $f(x)$ is a real polynomial such that $F(x) > 0$ for $x \in (x_1, x_2)$ and $x \in (x_3, x_4)$. Then Corollary 1 applies with

$$\gamma = \{(x, \pm\sqrt{F(x)}),\ x \in [x_1, x_2]\},\ \gamma_1 = \{(x, \pm\sqrt{F(x)}),\ x \in [x_3, x_4]\}.$$

Moreover, since the algebraic curve $\tilde{\gamma} \subset \mathbb{C}P^2$ is always singular, then Theorem by Bolotin does not apply.

Corollary 2. *Assume that $\tilde{\Gamma}$ is a non-singular curve (of degree > 2) in $\mathbb{C}P^2$ and has a smooth real oval Γ (for example, $\tilde{\Gamma}$ is a nonsingular cubic). Then the dual curve γ is also an oval and Birkhoff billiard inside γ is not integrable by Theorem 1. Notice, that in this case Bolotin's theorem does not apply, since $\tilde{\gamma}$ is necessarily singular in this case. The inflection points of $\tilde{\Gamma}$ correspond to singular points of $\tilde{\gamma}$.*

Corollary3. *Let* γ *be the dual of Fermat oval* $\Gamma = \{x^{2n} + y^{2n} = 1,\ n > 1\}$. *Notice that* $\tilde{\Gamma}$ *is irreducible, non-singular curve and so by Theorem 1 the Birkhoff billiard inside* γ *is not integrable. One can easily compute that in this case the oval* γ *can be written as follows:*

$$\gamma = \{x^{\frac{2n}{2n-1}} + y^{\frac{2n}{2n-1}} = 1\}.$$

Therefore (the algebraic curve) γ *is a strictly convex* C^1 *curve in the plane which has 4 singular points* $(\pm 1, 0), (0, \pm 1)$ *corresponding to 4 inflection points* $(\pm 1, 0), (0, \pm 1)$ *of* Γ. *So Bolotin's theorem does not apply in this case also.*

Angular billiard

1. Riemannian billiard.
2. Outer billiard (Neumann, Moser, Tabachnikov)
3. Our model: **Angular billiard**

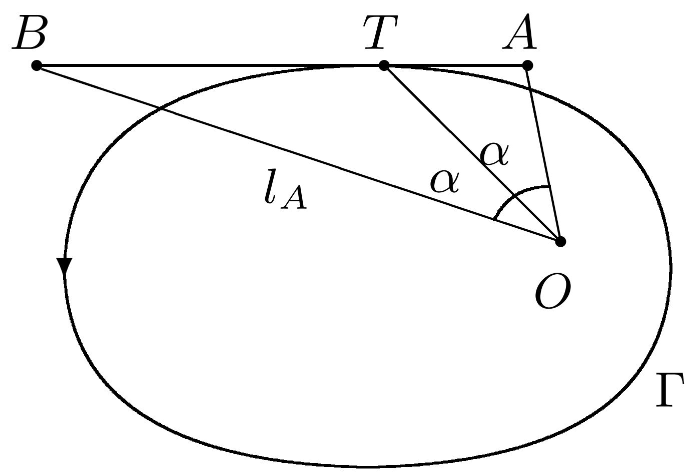

Fig.2 $\angle AOT = \angle TOB, \quad \mathcal{A}(A) = B.$

If OA is not orthogonal to OT, then there is unique line l_A (different from OA) such that the angle between OA and OT is equal to the angle between lines OT and l_A. If l_A is not parallel to AT, then there is the intersection point B of the tangent line and l_A.

Let $\mathcal{S} \subset U$ be a curve defined by the condition

$$\mathcal{S} = \{A : l_A \| AT\}.$$

We introduce the mapping

$$\mathcal{A} : U \backslash \mathcal{S} \to U, \qquad \mathcal{A}(A) = B.$$

On the curve $\mathcal{S}$ the mapping $\mathcal{A}$ can not be defined. We call the mapping $\mathcal{A}$ the *Angular Billiard map of* Γ.

KAM type-Theorem. *If* Γ *is sufficiently smooth, then there exist invariant curves of* $\mathcal{A}$ *arbitrary close to* Γ. *Thus, there is a neighborhood of* Γ *where* $\mathcal{A}$ *is well-defined.* .

The global dynamics of $\mathcal{A}$ can be complicated, for example if OA is orthogonal to OT (i.e. $OA = l_A$), then $\mathcal{A}(A) = A$, this happens when $\angle ATO < \frac{\pi}{2}$, denote such a point P, $OP \perp OT$. Points P form a curve $\mathcal{P}$ (see Fig. 3).

For arbitrary point A from the interval $(-\infty; P)$ there is a unique point B from the interval $(P; S)$ such that the angle between lines AO and OT is equal to the angle between lines OT and OB or since $OP \perp OT$ it is equivalent to $\angle AOP = \angle BOP$. Hence,

$$\mathcal{A}(A) = B, \quad \mathcal{A}(B) = A, \quad \mathcal{A}(P) = P,$$

and $\mathcal{A}$ maps the half line $(-\infty; P)$ on the interval $(S; P)$.

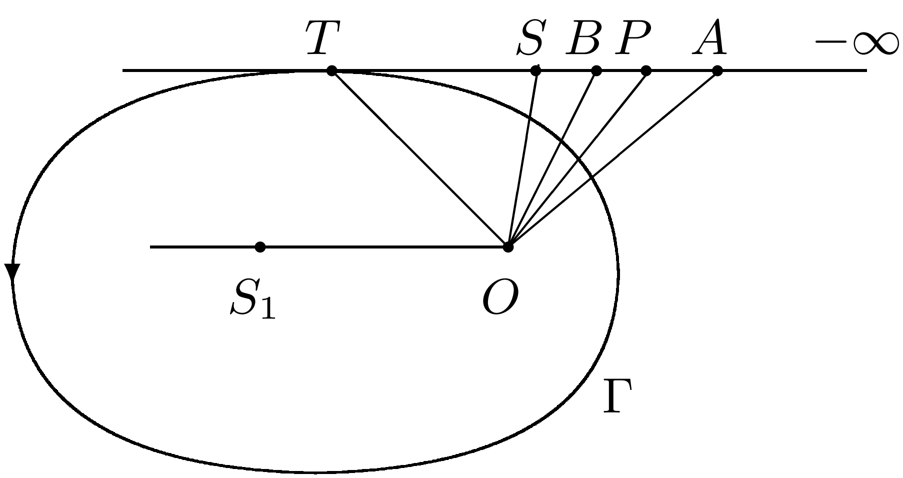

Fig. 3 $\angle ATO < \frac{\pi}{2}$, $\mathcal{A}(A) = B$, $\mathcal{A}(B) = A$, $\mathcal{A}(-\infty; P) = (P; S)$.

Polar and Projective Duality

Teorem. *In a neidhborhood of the boundary curve* Γ *the Angular billiard is Dual to Birkhoff billiard inside* γ*. More precisely, if a line* a *is transformed to* b *by* $\mathcal{B}$*, then the for the dual points one has:* $\mathcal{A}(A) = B.$

Correspondence of polar duality with respect to the unite circle: It acts on *non-oriented* lines. Given a line l not passing through O we write l in the form $< n, x >= p,\ p > 0$, where n is an outward unite normal to l then the dual point corresponding to l is by definition $L = n/p$. In other words, the corresponding point L lies on the normal radius to the line at the distance equal $1/p$:

$$l = \{< n, x >= p,\ p > 0\} \leftrightarrow L = n/p.$$

Polar duality extends to the usual projective duality:

$$l = \{kx + ly = mz\} \ \leftrightarrow \ (k : l : m).$$

By small letters we denote the lines and by capital letters the corresponding dual points.
We denote by Γ the dual curve to γ consisting of points which are dual to the tangent lines of γ.

Remarkably, the duality preserves the incidence relation and dual to Γ is γ again. More precisely, if l is tangent to γ at Q then the dual line q is tangent to Γ at L (see Fig. 4).

Furthermore, if a, b are two oriented positive lines in the plane so that the Birkhoff billiard map $\mathcal{B}$ transforms a to b. Let $Q \in \gamma$ be the point of reflection and let l be the tangent line to γ at the reflection point Q. Then the dual points A, B lie on the line q which is tangent to Γ at L. Moreover, AOL and BOL are equal, so Angular billiard rule holds:

$$\mathcal{A}(A) = B.$$

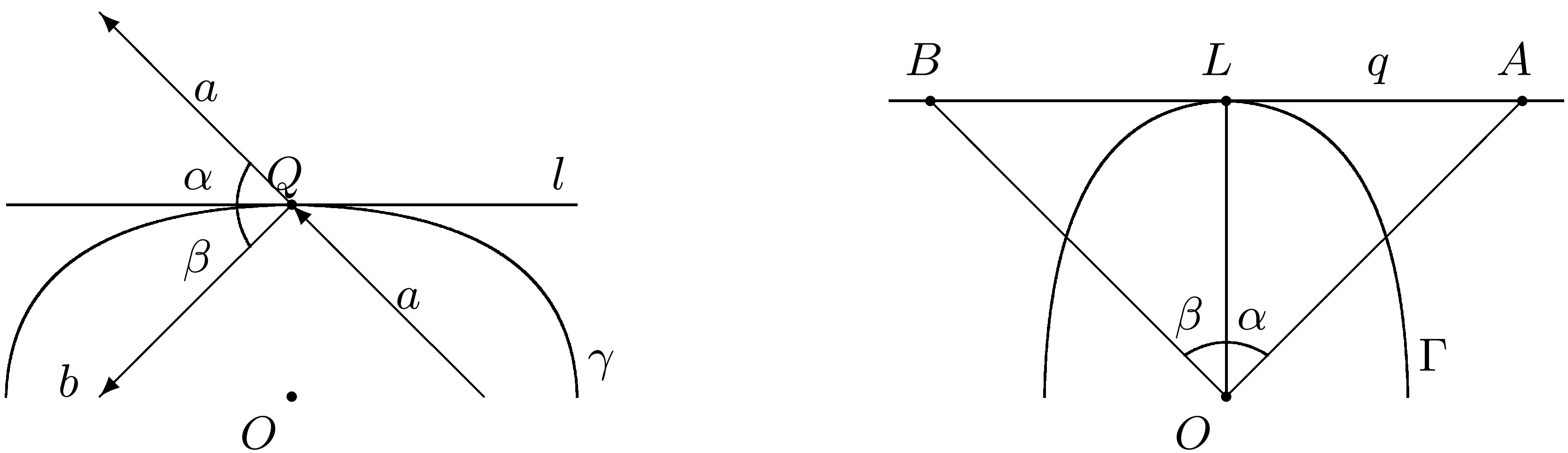

Fig. 4 Polar duality; $\beta = \alpha$.

Indeed, the angle α between the lines a and l equals the angle $\angle AOL$ because OA is normal to a and OL is normal to l, by definition of polar duality. Analogously, the angle β between the lines l and b equals the angle $\angle LOB$ (see Fig. 4). Moreover, α and β are equal by the reflection law of Birkhoff billiard.

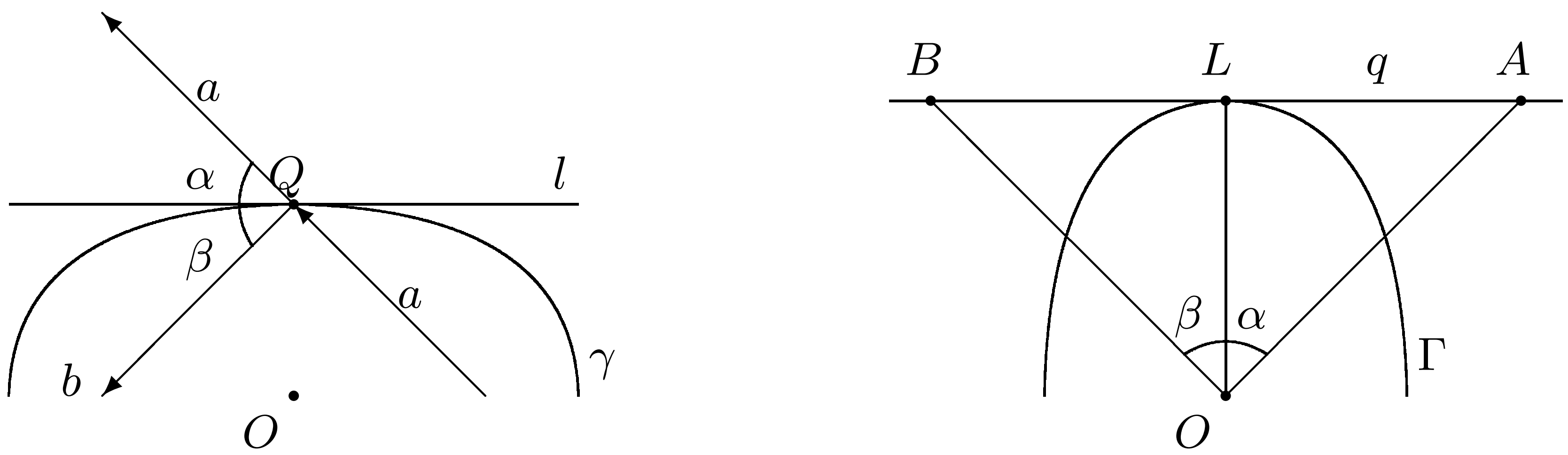

Fig. 4 Polar duality; $\beta = \alpha$.

Integrable Angular Billiards

We shall call the Angular billiard *integrable* if there is a function

$$G : U\backslash\mathcal{S} \to \mathbb{R}$$

such that

$$G(A) = G(\mathcal{A}(A)), \qquad \forall A \in U\backslash\mathcal{S}.$$

Example 1 *Let Γ be an ellipse defined by the equation*

$$F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,\; G(x,y) = \frac{F(x,y)}{(x-x_0)^2 + (y-y_0)^2}.$$

Here $O(x_0, y_0)$ is arbitrary point inside Γ,G is the integral.

Suppose that the Birkhoff billiard flow g^t admits a polynomial integral Φ on the energy level $\{|v| = 1\}$. One can assume that $\Phi(q, v)$ is a *homogeneous* polynomial of a certain *even* degree n in $\sigma(v) = xv_y - yv_x,\; v_x, v_y$:

$$\Phi = \Phi(\sigma, v_x, v_y).$$

And moreover, Φ vanishes on tangent vectors to γ.

Theorem. *Let γ be a convex closed curve and $\Phi(\sigma, v_x, v_y)$ be a homogeneous polynomial integral of even degree n, vanishing on the tangent vectors to the boundary γ. Then the Angular billiard corresponding to Γ is also integrable with the integral of the form*

$$G_1(x,y) = \frac{F_1(x,y)}{(\sqrt{x^2+y^2})^n}, \qquad F_1(x,y) = \Phi(1,-y,x),$$

where F_1 is a (non-homogeneous) polynomial of degree n. Moreover F_1 vanishes on Γ.

Let v be a unite tangent vector along the line a with positive momentum $\sigma(v) = p > 0$. We have:

$$\Phi(\sigma(v), v) = \sum_{k+l+m=n} f_{klm} (v_x)^k (v_y)^l p^m.$$

The dual point A of the line a has the coordinates:

$$x = \frac{n_1}{p} = \frac{\cos\varphi}{p} = \frac{v_y}{p}, \quad y = \frac{n_2}{p} = \frac{\sin\varphi}{p} = \frac{-v_x}{p} \Rightarrow p = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}.$$

$$G_1(x,y) = p^n \left(\sum_{k+l+m=n} f_{klm} (-y)^k (x)^l \right) = p^n F_1(x,y) = \frac{F_1(x,y)}{(x^2+y^2)^{n/2}},$$

where the polynomial in brackets is denoted by F_1. Since G_1 vanishes on Γ then also F_1 does.

Let Γ be defined by the equation $f = 0$, where f is an irreducible polynomial in $\mathbb{C}[x, y]$ of degree d. Since $F_1 = 0$ on Γ one can write F_1 in the form:

$$F_1(x, y) = f^k(x, y) g_1(x, y), k \in \mathbb{Z}_+.$$

It is important, that f, g_1 can be assumed to be *real* polynomials. Next we replace G_1 by $G := G_1^{\frac{1}{k}}$:

$$G(x, y) = \frac{(F_1(x, y))^{\frac{1}{k}}}{(x^2 + y^2)^{\frac{n}{2k}}} := \frac{F}{(x^2 + y^2)^m}, F := (F_1(x, y))^{\frac{1}{k}} = fg, \qquad m = \frac{n}{2k}.$$

Then G is also an integral, which also vanishes on Γ, but F, g are not necessarily polynomials anymore.

Lemma. For the integral F of Angular billiard for Γ, for all small ε, and $(x, y) \in \Gamma$ we have:

(1)
$$F(x + \varepsilon F_y, y - \varepsilon F_x) \left(-\frac{\mu}{\varepsilon}\right)^{2m} = F(x + \mu F_y, y - \mu F_x),$$

$$\mu = -\frac{(x^2 + y^2)\varepsilon}{x^2 + y^2 + 2\varepsilon(xF_y - yF_x)}.$$

Remarkable Identity

For any function f we define affine Hessian:

$$H(f) := f_y(f_{xx}f_y - f_{xy}f_x) + f_x(f_{yy}f_x - f_{xy}f_y) = f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2.$$

Theorem. *The following formula holds true*

(2) $$g^3(x,y)H(f(x,y)) = c_1(x^2+y^2)^{3m-3},$$

where c_1 is a constant.

For the proof we extract terms of order ε^3 in power series of the equation (1) of Lemma. Then it turns out that these terms form a complete derivative

$$L_v\left(H(g(x,y)f(x,y))(x^2+y^2)^{-3m+3}\right) = 0$$

Notation: For any polynomial $p(x, y)$, we denote by $\tilde{p}(x, y, z)$ the corresponding homogeneous polynomial of the same degree as $p(x, y)$.

Lemma. *The identity*

(3) $$\tilde{g}_1^3(x,y,z)(\mathrm{Hess}(\tilde{f}(x,y,z)))^k + c(x^2+y^2)^{k(3m-3)} = \tilde{f}(x,y,z)\tilde{h}(x,y,z),$$

holds true for all $(x, y, z) \in \mathbb{C}^3$, where c is a constant, $\tilde{h}$ is a homogeneous polynomial, and

$$\mathrm{Hess}(\tilde{f}(x,y,z)) := \det \begin{pmatrix} \tilde{f}_{xx} & \tilde{f}_{xy} & \tilde{f}_{xz} \\ \tilde{f}_{xy} & \tilde{f}_{yy} & \tilde{f}_{yz} \\ \tilde{f}_{xz} & \tilde{f}_{yz} & \tilde{f}_{zz} \end{pmatrix}.$$

Reminder

Given an algebraic curve C of degree d, $p \in C$ a regular point, let T be the tangent line at p.

The *order* r of the inflection point: $I_p(C, T) = r + 2$,

1) $r \geq 1$, and
2) $r + 2 \leq d$, by Bezout theorem.

Theorem.
$I_p(C, Hess(C)) = r$, and hence $\leq d - 2$.

Proof of the Main Theorem

Consider the situation in $\mathbb{C}P^2$. Any intersection point in $\mathbb{C}P^2$ between Hessian curve of $\mathrm{Hess}(\tilde{\Gamma})$ with $\tilde{\Gamma}$ is either singular or inflection point of $\tilde{\Gamma}$. So, if there is a singular or inflection point $(x_0 : y_0 : z_0) \in \tilde{\Gamma}$ such that $x_0^2 + y_0^2 \neq 0$, it then follows from (3) that $c = 0$. Therefore, $\mathrm{Hess}(\tilde{f}) \equiv 0$ since $\tilde{g}_1 \neq 0$ identically on $\tilde{\Gamma}$. This implies that $\tilde{\Gamma}$ is a line, but this is impossible. Let us prove now that $\tilde{\Gamma}$ must have singular points. If on the contrary $\tilde{\Gamma}$ is a smooth curve, then it follows from (3) that all inflection points must belong to two lines L_1 and L_2 defined by the equations

$$L_1 = \{x + iy = 0\}, \qquad L_2 = \{x - iy = 0\}.$$

Recall, d is the degree of $\tilde{\Gamma}$. Then the Hessian curve intersects $\tilde{\Gamma}$ exactly in inflection points, and moreover, it is remarkable fact that the intersection multiplicity of such a point of intersection equals exactly the order of inflection point, and hence does not exceed $(d-2)$. Furthermore, the lines L_1 and L_2 intersect $\tilde{\Gamma}$ maximum in $2d$ points together . Hence, we have altogether counted with multiplicities not more than $2d(d-2)$, but on the other hand the Hessian curve has degree $3(d-2)$ and thus by Bezout theorem the number of intersection points with multiplicities is $3d(d-2)$. This contradiction shows that $\tilde{\Gamma}$ can not be a smooth curve unless $d = 2$.

THANKS!

AND JOIN PLAYING BILLIARDS!