Differentiation of Hyperelliptic Functions: Genus 3 Case

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Abelian function

An *Abelian function* is a meromorphic function in $\mathbb{C}^g$ with a lattice of periods $\Gamma \subset \mathbb{C}^g$ of rank $2g$.

We say that an Abelian function is a meromorphic function on the complex torus $T^g = \mathbb{C}^g / \Gamma$.

Denote the coordinates in $\mathbb{C}^g$ by $u = (u_1, u_3, u_5, \ldots, u_{2g-1})$.

If $f(u)$ is Abelian, then $\frac{\partial}{\partial u_k} f(u)$ is Abelian for the same periods.

Hyperelliptic functions

We consider the model for a plane hyperelliptic curve of genus g

$$\begin{aligned}\mathcal{V}_\lambda = & \{(x, y) \in \mathbb{C}^2 : \\ & y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \ldots + \lambda_{4g} x + \lambda_{4g+2}\}.\end{aligned}$$

Denote by $\mathcal{B} \subset \mathbb{C}^{2g}$ the subspace of parameters such that $\mathcal{V}_\lambda$ is non-singular for $\lambda \in \mathcal{B}$.

$$\begin{array}{c} \mathcal{U} \\ \Big\downarrow{\scriptstyle \mathbb{C}^g/\Gamma} \\ \mathcal{B} \end{array}$$

A *hyperelliptic function of genus* g is a smooth function defined on an open dense subset of $\mathbb{C}^g \times \mathcal{B}$, such that for each $\lambda \in \mathcal{B}$ it's restriction on $\mathbb{C}^g \times \lambda$ is Abelian with T^g the Jacobian $\mathcal{J}_\lambda$ of $\mathcal{V}_\lambda$.

Genus 1

Plane elliptic curve of genus 1

$$\mathcal{V}_\lambda = \{(x, y) \in \mathbb{C}^2 \colon y^2 = x^3 + \lambda_4 x + \lambda_6\}.$$

$$\updownarrow$$

Weierstrass form: $\qquad y^2 = 4x^3 - g_2 x - g_3$

$$\mathcal{B} = \{(g_2, g_3) \in \mathbb{C}^{2g} \colon g_2^3 \neq 27 g_3^2\}$$

$$\begin{array}{c} \mathcal{U} \\ \Big\downarrow {\scriptstyle \mathbb{C}/\Gamma} \\ \mathcal{B} \end{array}$$

Γ is generated by (ω_1, ω_2),

$$g_2 = 60 \sum (2k\omega_1 + 2m\omega_2)^{-4},$$

$$g_3 = 140 \sum (2k\omega_1 + 2m\omega_2)^{-6}.$$

$$\wp(u; g_2, g_3) = \frac{1}{u^2} + \sum_{(k,m)\neq(0,0)} \left(\frac{1}{(u - 2k\omega_1 - 2m\omega_2)^2} - \frac{1}{(2k\omega_1 + 2m\omega_2)^2} \right).$$

Genus 1

F. G. Frobenius, L. Stickelberger, *Über die Differentiation der elliptischen Functionen nach den Perioden und Invarianten*, J. Reine Angew. Math., 92 (1882), 311–337.

$$\mathcal{L}_0 = L_0 - u\partial_u, \qquad \mathcal{L}_1 = \partial_u,$$
$$\mathcal{L}_2 = L_2 - \zeta(u; -4\lambda_4, -4\lambda_6)\partial_u.$$

Here $\wp(u; g_2, g_3)$ and $\zeta(u; g_2, g_3)$ are Weierstrass functions, and the fields L_k on $\mathcal{B}$ are

$$L_0 = 4\lambda_4\partial_{\lambda_4} + 6\lambda_6\partial_{\lambda_6}, \qquad L_2 = 6\lambda_6\partial_{\lambda_4} - \frac{4}{3}\lambda_4^2\partial_{\lambda_6}.$$

Lie algebra

$$[\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1, \qquad [\mathcal{L}_0, \mathcal{L}_2] = 2\mathcal{L}_2,$$
$$[\mathcal{L}_1, \mathcal{L}_2] = \wp(u; -4\lambda_4, -4\lambda_6)\mathcal{L}_1.$$

Problem of Differentiation of Hyperelliptic Functions

A *hyperelliptic function of genus* g is a smooth function defined on an open dense subset of $\mathbb{C}^g \times \mathcal{B}$, such that for each $\lambda \in \mathcal{B}$ it's restriction on $\mathbb{C}^g \times \lambda$ is Abelian with T^g the Jacobian $\mathcal{J}_\lambda$ of $\mathcal{V}_\lambda$.

Denote the field of hyperelliptic functions by $\mathcal{F}$.

- Find the generators of the $\mathcal{F}$-module $\operatorname{Der} \mathcal{F}$ of derivations of the field $\mathcal{F}$ and their action on $\mathcal{F}$.
- Describe the structure of Lie algebra $\operatorname{Der} \mathcal{F}$ (i.e. find the commutation relations).

V. M. Buchstaber, D. V. Leikin, *Solution of the Problem of Differentiation of Abelian Functions over Parameters for Families of* (n, s)*-Curves*, Funct. Anal. Appl., 42:4, (2008).

Genus 2

V. M. Buchstaber,
Polynomial dynamical systems and Korteweg–de Vries equation,
Proc. Steklov Inst. Math., 294 (2016), 176–200.

Set $\zeta_j = \frac{\partial}{\partial_{u_j}} \ln \sigma(u;\lambda)$, $j = 1, 3$, where
$\sigma(u;\lambda)$ is the two-dimensional hyperelliptic sigma-function.

$$\mathcal{L}_0 = L_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3}, \qquad \mathcal{L}_1 = \partial_{u_1},$$

$$\mathcal{L}_2 = L_2 + (-\zeta_1 + \frac{4}{5}\lambda_4 u_3)\partial_{u_1} - u_1\partial_{u_3}, \qquad \mathcal{L}_3 = \partial_{u_3},$$

$$\mathcal{L}_4 = L_4 + (-\zeta_3 + \frac{6}{5}\lambda_6 u_3)\partial_{u_1} - (\zeta_1 + \lambda_4 u_3)\partial_{u_3},$$

$$\mathcal{L}_6 = L_6 + \frac{3}{5}\lambda_8 u_3\partial_{u_1} - \zeta_3\partial_{u_3}.$$

Polynomial bundle

$$\begin{tikzcd} \mathcal{U} \arrow[r, dashed, "\varphi"] \arrow[d, "\pi"] & \mathbb{C}^{3g} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, hook] & \mathbb{C}^{2g} \end{tikzcd}$$

$$\begin{matrix} \mathcal{L}_1 & \mathcal{L}_3 & \dots & \mathcal{L}_{2g-1} \\ \mathcal{L}_0 & \mathcal{L}_2 & \dots & \mathcal{L}_{4g-2} \end{matrix}$$

$$\varphi : (u, \lambda) \mapsto \begin{pmatrix} \wp_{1;1} & \wp_{1;3} & \cdots & \wp_{1;2g-1} \\ \wp_{2;1} & \wp_{2;3} & \cdots & \wp_{2;2g-1} \\ \wp_{3;1} & \wp_{3;3} & \cdots & \wp_{3;2g-1} \end{pmatrix}, \qquad \wp_{i;j} = -\frac{\partial^{i+1}}{\partial_{u_1}^{i} \partial_{u_j}} \ln \sigma(u; \lambda).$$

V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin,
Kleinian functions, hyperelliptic Jacobians and applications,
Reviews in Mathematics and Math. Physics, 10:2, (1997).

For $i, k \in \{1, 3, \ldots, 2g-1\}$ we have the relations

$$\begin{aligned}
\wp_{3;i} =& 6\wp_2\wp_{1;i} + 6\wp_{1;i+2} - 2\wp_{0,3;i} + 2\lambda_4\delta_{i,1}, \\
\wp_{2;i}\wp_{2;k} =& 4\left(\wp_2\wp_{1;i}\wp_{1;k} + \wp_{1;k}\wp_{1;i+2} + \wp_{1;i}\wp_{1;k+2} + \wp_{;k+2,i+2}\right) - \\
& - 2(\wp_{1;i}\wp_{;3,k} + \wp_{1;k}\wp_{;3,i} + \wp_{;k,i+4} + \wp_{;i,k+4}) + \\
& + 2\lambda_4(\delta_{i,1}\wp_{1;k} + \delta_{k,1}\wp_{1;i}) + 2\lambda_{i+k+4}(2\delta_{i,k} + \delta_{k,i-2} + \delta_{i,k-2}),
\end{aligned}$$

where $\wp_{i;j} = -\dfrac{\partial^{i+1}}{\partial_{u_1}^i \partial_{u_j}} \ln \sigma(u;\lambda), \quad \wp_{i;j,k} = -\dfrac{\partial^{i+2}}{\partial_{u_1}^i \partial_{u_j}\partial_{u_k}} \ln \sigma(u;\lambda).$

Corollary

Consider the map $\varphi : \mathcal{U} \dashrightarrow \mathbb{C}^{\frac{g(g+9)}{2}}$, with coordinates (x, y, λ),

$\varphi : (u, \lambda) \mapsto (x, y, \lambda) = (x_{i,k}, y_{k,j}, \lambda_s) = (\wp_{i;k}(u, \lambda), \wp_{;k,j}(u, \lambda), \lambda_s)$.

Set $x_{k+1} = x_{k,1}$. The image of φ lies in $\mathcal{S} \subset \mathbb{C}^{\frac{g(g+9)}{2}}$:

$$\begin{aligned}
x_4 =& 6x_2^2 + 4x_{1,3} + 2\lambda_4,\\
x_{3,k} =& 6x_2 x_{1,k} + 6x_{1,k+2} - 2y_{3,k},\\
x_3^2 =& 4x_2^3 + 4x_2 x_{1,3} + 4y_{3,3} - 4x_{1,5} + 4\lambda_4 x_2 + 4\lambda_6,\\
x_3 x_{2,k} =& \left(4x_2^2 + 2x_{1,3} + 2\lambda_4\right) x_{1,k} + 4x_2 x_{1,k+2} - 2x_{1,k+4} +\\
& + 4y_{3,k+2} - 2y_{5,k} - 2x_2 y_{3,k} + 2\lambda_8 \delta_{3,k},\\
x_{2,i} x_{2,k} =& 4\left(x_2 x_{1,i} x_{1,k} + x_{1,k} x_{1,i+2} + x_{1,i} x_{1,k+2} + y_{k+2,i+2}\right) -\\
& - 2(x_{1,i} y_{3,k} + x_{1,k} y_{3,i} + y_{k,i+4} + y_{i,k+4}) +\\
& + 2\lambda_{i+k+4}(2\delta_{i,k} + \delta_{k,i-2} + \delta_{i,k-2}).
\end{aligned}$$

Theorem

The projection $\pi_1\colon \mathbb{C}^{\frac{g(g+9)}{2}} \to \mathbb{C}^{3g}$ *on the first* $3g$ *coordinates gives the isomorphism* $\mathcal{S} \simeq \mathbb{C}^{3g}$.

Corollary

The projection $\pi_3\colon \mathbb{C}^{\frac{g(g+9)}{2}} \to \mathbb{C}^{2g}$ *on the last* $2g$ *coordinates gives a polynomial map* $p\colon \mathbb{C}^{3g} \to \mathbb{C}^{2g}$.

$$\begin{tikzcd}
 & \mathbb{C}^{\frac{g(g+9)}{2}} \arrow[r, equal] & \mathbb{C}^{3g} \times \mathbb{C}^{\frac{g(g-1)}{2}} \times \mathbb{C}^{2g} \arrow[dl, "\pi_1"] \arrow[ddl, bend left, "\pi_3"] \\
\mathcal{U} \arrow[r, dashed, "\varphi"] \arrow[d, "\pi"] & \mathcal{S} \simeq \mathbb{C}^{3g} \arrow[u, hook] \arrow[d, "p"] & \\
\mathcal{B} \arrow[r, hook] & \mathbb{C}^{2g} &
\end{tikzcd}$$

Generators of the polynomial Lia algebra in $\mathbb{C}^{3g}$

$$\begin{tikzcd} \mathcal{U} \arrow[r, dashed, "\varphi"] \arrow[d, "\pi"] & \mathbb{C}^{3g} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, hook] & \mathbb{C}^{2g} \end{tikzcd}$$

$$\begin{matrix} \mathcal{L}_1 & \mathcal{L}_3 & \dots & \mathcal{L}_{2g-1} \\ \mathcal{L}_0 & \mathcal{L}_2 & \dots & \mathcal{L}_{4g-2} \end{matrix}$$

Denote the ring of polynomials in $\lambda \in \mathbb{C}^{2g}$ by $\mathcal{P}$.

Find $3g$ polynomial vector fields on $\mathbb{C}^{3g}$ such that for any $P \in \mathcal{P}$ we have

$$\mathcal{L}_k(p^*(P)) \in p^*\mathcal{P}.$$

Genus 1

The projection p takes the form

$$\lambda_4 = \frac{1}{2}x_4 - 3x_2^2,$$
$$\lambda_6 = \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 + 2x_2^3.$$

The vector fields are

$$\begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ x_3 & x_4 & 12x_2x_3 \\ \frac{2}{3}x_4 - 2x_2^2 & 3x_2x_3 & 2x_2x_4 + 3x_3^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} \end{pmatrix}$$

The polynomial Lie algebra is

$$[\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1, \quad [\mathcal{L}_0, \mathcal{L}_2] = 2\mathcal{L}_2, \quad [\mathcal{L}_1, \mathcal{L}_2] = x_2\mathcal{L}_1.$$

Genus 2

In the formulas below we write y_4 instead of $x_{1,3}$, y_5 instead of $x_{2,3}$ and y_6 instead of $x_{3,3}$ to shorten down the formulas.

The projection p takes the form

$$\begin{aligned}
\lambda_4 &= -3x_2^2 + \frac{1}{2}x_4 - 2y_4, \\
\lambda_8 &= -\frac{1}{2}(x_4 y_4 - x_3 y_5 + x_2 y_6) + \left(4x_2^2 + y_4\right) y_4, \\
\lambda_6 &= 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2 x_4 - 2x_2 y_4 + \frac{1}{2}y_6, \\
\lambda_{10} &= 2x_2 y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4 y_6.
\end{aligned}$$

Genus 2: The polynomial Lie algebra

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k, \quad [\mathcal{L}_1, \mathcal{L}_2] = x_2\mathcal{L}_1 - \mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_3] = 0,$$

$$[\mathcal{L}_1, \mathcal{L}_4] = y_4\mathcal{L}_1 + x_2\mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_6] = y_4\mathcal{L}_3,$$

$$[\mathcal{L}_2, \mathcal{L}_3] = -\left(y_4 + \frac{4}{5}\lambda_4\right)\mathcal{L}_1, \quad [\mathcal{L}_3, \mathcal{L}_6] = \frac{3}{5}\lambda_8\mathcal{L}_1 + \left(3x_2y_4 - \frac{1}{2}y_6\right)\mathcal{L}_3,$$

$$[\mathcal{L}_2, \mathcal{L}_4] = \frac{8}{5}\lambda_6\mathcal{L}_0 - \frac{1}{2}y_5\mathcal{L}_1 - \frac{8}{5}\lambda_4\mathcal{L}_2 + \frac{1}{2}x_3\mathcal{L}_3 + 2\mathcal{L}_6,$$

$$[\mathcal{L}_3, \mathcal{L}_4] = \left(3x_2y_4 - \frac{1}{2}y_6 + \frac{6}{5}\lambda_6\right)\mathcal{L}_1 + (y_4 - \lambda_4)\mathcal{L}_3,$$

$$[\mathcal{L}_2, \mathcal{L}_6] = \frac{4}{5}\lambda_8\mathcal{L}_0 - \frac{1}{2}(x_3y_4 - x_2y_5)\mathcal{L}_1 + \frac{1}{2}y_5\mathcal{L}_3 - \frac{4}{5}\lambda_4\mathcal{L}_4,$$

$$[\mathcal{L}_4, \mathcal{L}_6] = -2\lambda_{10}\mathcal{L}_0 - \frac{1}{2}\left(x_2^2y_5 - x_2x_3y_4 - \frac{1}{2}x_4y_5 + \frac{1}{2}x_3y_6 + y_4y_5\right)\mathcal{L}_1 +$$

$$+\frac{6}{5}\lambda_8\mathcal{L}_2 + \frac{1}{2}(x_3y_4 - x_2y_5)\mathcal{L}_3 - \frac{6}{5}\lambda_6\mathcal{L}_4 + 2\lambda_4\mathcal{L}_6.$$

Genus 3: projection p

In the formulas below we write y_{k+3} instead of $x_{k,3}$ and z_{k+5} instead of $x_{k,5}$.

$$\begin{array}{c} \mathbb{C}^{3g} \\ \big\downarrow p \\ \mathbb{C}^{2g} \end{array}$$

$$\lambda_4 = \frac{1}{2}x_4 - 3x_2^2 - 2y_4,$$

$$\lambda_6 = \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 + 2x_2^3 - 2x_2y_4 + \frac{1}{2}y_6 - 2z_6,$$

$$\lambda_8 = -\frac{1}{2}(x_4y_4 - x_3y_5 + x_2y_6) + \left(4x_2^2 + y_4\right) y_4 - 2x_2z_6 + \frac{1}{2}z_8,$$

$$\lambda_{10} = \frac{1}{4}y_5^2 + 2x_2y_4^2 - \frac{1}{2}y_4y_6 - \frac{1}{2}(x_4z_6 - x_3z_7 + x_2z_8) + 2z_6y_4 + 4x_2^2z_6,$$

$$\lambda_{12} = 4x_2y_4z_6 + z_6^2 - \frac{1}{2}(y_6z_6 - y_5z_7 + y_4z_8),$$

$$\lambda_{14} = \frac{1}{4}z_7^2 + 2x_2z_6^2 - \frac{1}{2}z_6z_8.$$

Thank you!