# On Algebro-Geometric Approach to Diagonal Semi–Hamiltonian Systems of Hydrodynamic Type

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# Outline

# Systems of Hydrodynamic Type

## Definition

A system of the form

$$u^i_t = \sum_{j=1}^{n} v^i_j(u) u^j_x, \quad i = 1, \dots, n$$

on the functions $u^i(x, t)$ is called a system of hydrodynamic type.

# Hamiltonian Systems of Hydrodynamic Type

## Definition

A Hamiltonian of hydrodynamic type is a functional $H[u] = \int h(u)dx$ with the density $h(u)$ independent of the derivatives $u_x,\ u_{xx},\ u_{xxx}, \ldots$

## Definition

The system

$$u^i_t = \{u^i(x), H\} = \sum_{j=1}^{n} A^{ij} \frac{\delta H}{\delta u^j(x)}, \quad A^{ij} = g^{ij}(u)\frac{d}{dx} + \sum_{k=1}^{n} b^{ij}_k u^k_x$$

is called a Hamiltonian system of hydrodynamic type with Dubrovin–Novikov type Poisson bracket.

# Corresponding Flat Metrics

$$u^i_t = \{u^i(x), H\} = \sum_{j=1}^{n} A^{ij} \frac{\delta H}{\delta u^j(x)}, \quad A^{ij} = g^{ij}(u) \frac{d}{dx} + \sum_{k=1}^{n} b^{ij}_k u^k_x.$$

## Theorem (Dubrovin, Novikov, 1983)

*If* $\det g^{ij} \neq 0$ *then*

- $g^{ij}$ *is symmetric;*
- *under local changes* $u^i = u^i(w^1, \ldots, w^n)$ *the coefficient* $g^{ij}$ *is transformed as tensor with two upper indices;*
- *the metric* $g$ *is flat, i.e. the curvature tensor vanishes;*
- $b^{ij}_k = -\sum_{s=1}^{n} g^{ik} \Gamma^j_{ks}$, *where* $\Gamma^j_{ks}$ *is Christoffel symbol of torsion free differential geometric connection compatible with the metric* $g$.

# Diagonal Hamiltonian Systems

$$u^i_t = \lambda_i(u) \cdot u^i_x, \quad i = 1, \dots, n;$$

## Theorem (Tsarev, 1990)

*If all the coefficients $\lambda_i(u)$ are distinct in some domain then*

- *the corresponding metric is diagonal;*
- *variables $u^i$ form a curvilinear orthogonal coordinate system;*
- *the following relations hold:*

$$\frac{\partial_i \lambda_k}{\lambda_i - \lambda_k} = \partial_i \ln \sqrt{g_{kk}} = \Gamma^k_{ki}, \quad i \neq k.$$

# Semi–Hamiltonian Systems

## Definition

A system of the form

$$u_t^i = \lambda_i(u) \cdot u_x^i$$

is called Semi–Hamiltonian if all the coefficients $\lambda_i(u)$ are distinct in some domain and the following relation holds for distinct indices:

$$\partial_j \left( \frac{\partial_i \lambda_k}{\lambda_i - \lambda_k} \right) = \partial_i \left( \frac{\partial_j \lambda_k}{\lambda_j - \lambda_k} \right).$$

# Lamé Equations

$$\sum_{\alpha=1}^{n} \partial_i x^\alpha(u) \cdot \partial_j x^\alpha(u) = 0, \quad \sum_{\alpha=1}^{n} \partial_i x^\alpha(u) \cdot \partial_i x^\alpha(u) = H_i^2(u).$$

## Lamé equations

$$\partial_j \partial_k H_i = \frac{1}{H_j} \cdot \partial_k H_j \cdot \partial_j H_i + \frac{1}{H_k} \cdot \partial_j H_k \cdot \partial_k H_i;$$

$$\partial_j \left( \frac{\partial_j H_i}{H_j} \right) + \partial_i \left( \frac{\partial_i H_j}{H_i} \right) + \sum_{k \neq i, k \neq j} \frac{1}{H_k^2} \cdot \partial_k H_i \cdot \partial_k H_j = 0.$$

# Rotational Coefficients

## Definition

The functions $\beta_{ij}(u)$ defined for distinct $i$, $j$ by the formula

$$\beta_{ij} = \frac{\partial_i H_j}{H_i}$$

are called rotational coefficients of a diagonal metric $g_{ii}(u) = H_i^2(u)$.

## Lamé equations in terms of rotational coefficients

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj};$$

$$\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k\neq i, k\neq j} \beta_{ki}\beta_{kj} = 0.$$

# Zakharov Method

## Dressing Method (Zakharov, 1998)

$n$-wave System

$\downarrow$

Differential Reduction

$\downarrow$

Rotational Coefficients

$\downarrow$

Lamé Coefficients

$\downarrow$

Embedding Functions

# $n$-point Baker–Akhiezer Function

## Spectral data

$$\{\Gamma,\, g;\ \{P_j,\, k_j^{-1}\}_{j=1}^{n};\ R_1, \ldots, R_{l+N};\ \gamma_1, \ldots, \gamma_{g+l+N-1}\}$$

## Definition

The function $\psi(u^1, \ldots, u^n;\ Q|d)$, $Q \in \Gamma$, $d = (d_1, \ldots, d_{l+N})$ with following properties

- in the neighborhood of every point $P_j$

$$\psi(u; Q) = e^{k_j u^j} \left( \xi_0^j(u) + \xi_1^j(u) \cdot k_j^{-1} + \ldots \right);$$

- every point $\gamma_j$ is a simple pole of $\psi(u; Q)$;
- $\psi(u; R_\alpha) = d_\alpha \in \mathbb{C}$.

is called $n$-point Baker–Akhiezer function.

# Restrictions on Spectral Data

## Restriction on the curve $\Gamma$

There is an involution $\sigma : \Gamma \to \Gamma$ with following properties

- $\sigma^2 = id$, $\sigma(k_i) = -k_i$;
- $\sigma$ has only $2(n+N)$ fixed points: $P_1, \ldots, P_n$, $R_{l+1}, \ldots, R_{l+N}$ $Q_1, \ldots, Q_{n+N}$;
- $\sigma(R_1 + \cdots + R_l) = R_1 + \cdots + R_l$, $\sigma(\gamma_1 + \cdots + \gamma_{g+l+N-1}) = \gamma_1 + \cdots + \gamma_{g+l+N-1}$.

# Algebro–Geometric Diagonal Metrics

## Theorem

*For spectral data with some restrictions functions*

$$x^k(u) = \psi(u; Q_k), \quad k = 1, \dots, n + N,$$

*are embedding functions for n-dimensional submanifold of* $\mathbb{R}^{n+N}$ *with orthogonal coordinates* $(u^1, \dots, u^n)$*, and the Lamé coefficients of induced metric are*

$$H_i(u) = \epsilon_i \cdot \xi_0^i(u),$$

*where* $\epsilon_i \in \mathbb{C}$ *are defined by the spectral data.*

# Vector Space of Baker–Akhiezer Functions

## Linear Properties of Algebro–Geometric Diagonal Metrics

$$\psi(u; Q|d^{(x)}) + \psi(u; Q|d^{(y)}) = \psi(u; Q|d^{(x)} + d^{(y)})$$

$$\psi(u; Q|d^{(x)}) \to x(u);\ \psi(u; Q|d^{(y)}) \to y(u);$$

$$\psi(u; Q|d^{(x)} + d^{(y)}) \to x(u) + y(u);$$

$$H_i^{(x)}(u) + H_i^{(y)}(u) = H_i^{(x+y)}(u).$$

# Vector Space of Baker–Akhiezer Functions

## Lemma

*Embedding functions* $x(u) = (x^1(u), \dots, x^{n+N}(u))$ *and* $y(u) = (y^1(u), \dots, y^{n+N}(u))$ *defined by Baker–Akhiezer functions with same spectral data and distinct norming vectors have the following property:*

$$\left(\partial_i x^1(u), \dots, \partial_i x^{n+N}(u)\right) = \lambda_i(u) \cdot \left(\partial_i y^1(u), \dots, \partial_i y^{n+N}(u)\right), \; i = 1, \dots,$$

## Lemma

*For any choice of spectral data and two norming vectors* $d^{(x)}$, $d^{(y)}$ *there are* $n$ *functions* $\lambda_1(u), \dots, \lambda_n(u)$ *such that*

$$\partial_i \psi(u; Q|d^{(x)}) = \lambda_i(u) \cdot \partial_i \psi(u; Q|d^{(y)}), \quad i = 1, \dots n.$$

# Algebro–Geometric Diagonal Systems

## Theorem

*If the functions*

$$\lambda_i(u) = \frac{\partial_i \psi(u; Q|d^{(x)})}{\partial_i \psi(u; Q|d^{(y)})}$$

*are distinct in some domain then they are coefficients of a diagonal Semi–Hamiltonian system of hydrodynamic type with the corresponding metric defined by Baker–Akhiezer function* $\psi(u; Q|d^{(y)})$, *i.e.*

$$\frac{\partial_i \lambda_k}{\lambda_i - \lambda_k} = \partial_i \ln H_k^{(y)}$$

# Hydrodynamic Integrals

## Definition

A hydrodynamic integral is a functional $I[u] = \int P(u)dx$ with the density $P(u)$ independent of the derivatives $u_x, u_{xx}, u_{xxx}, \dots$, that satisfies

$$I_t \equiv 0.$$

## Corollary

*For any point $Q_0 \in \Gamma$ the function $P_0(u) = \psi(u; Q_0|d^{(y)})$ is a density of a hydrodynamic integral of a system with coefficients $\lambda_1, \dots, \lambda_n$.*

# Hydrodynamic Commuting Flows

## Corollary

*Two flows generated by systems with the following coefficients commute*

$$\lambda_i(u) = \frac{\partial_i \psi(u; Q|d^{(x)})}{\partial_i \psi(u; Q|d^{(y)})}; \quad \mu_i(u) = \frac{\partial_i \psi(u; Q|d^{(\widehat{x})})}{\partial_i \psi(u; Q|d^{(y)})}.$$

# Combescure Equivalence

## Definition

Two diagonal metrics are called Combescure equivalent if their rotational coefficients are the same.

## Theorem

*Metrics* $H_i^{(x)}(u)$ *and* $H_i^{(y)}(u)$ *are Combescure equivalent.*

## Corollary

*Any metric defined by Baker–Akhiezer function is a metric of hypersurface.*