Lefschetz trace formulas for flows on foliated manifolds

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The setting

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a foliated flow (i.e., ϕ^t takes each leaf to a leaf).

A Lefschetz number of the flow ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

H^j is some cohomology theory associated to $\mathcal{F}$, Tr is some trace.

The corresponding Lefschetz trace formula:

$$L(\phi) = \text{a contribution of closed orbits and fixed points of the flow.}$$

Simple flows

Definition

A closed orbit c of period l (not necessarily minimal) of the flow ϕ is called simple, if

$$\det(\mathrm{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$

Definition

A fixed point x of the flow ϕ is called simple if

$$\det(\mathrm{id} - \phi^t_* : T_xM \to T_xM) \neq 0, \quad t \neq 0.$$

Simple flows

- $\mathrm{Fix}(\phi)$ the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of $\mathrm{Fix}(\phi)$ (the union of leaves with fixed points).
  Observe that M^0 is ϕ-invariant.
- $M^1 = M \setminus M^0$ the transitive point set.

Definition

The foliated flow ϕ is simple, i.e.:

- all of its fixed points and closed orbits are simple,
- its orbits in M^1 are transverse to the leaves:

$$T_x M = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M^1,$$

  where Z is the infinitesimal generator of ϕ (a vector field on M).

Guiilemin-Sternberg formula

There is a canonical expression for the right-hand side of the Lefschetz trace formula, which follows from the Guiilemin-Sternberg formula.

In $\mathcal{D}'(\mathbb{R}^+)$,

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)} + \sum_p \varepsilon_p |1 - e^{\varkappa_p t}|^{-1},$$

c runs over all closed orbits and p over all fixed points of ϕ:

- $l(c)$ the minimal period of c,
- $\varepsilon_l(c) :=$ sign det $\left(\mathrm{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}\right), x \in c$.
- $\varepsilon_p :=$ sign det $\left(\mathrm{id} - \phi^t_* : T_p\mathcal{F} \to T_p\mathcal{F}\right), t > 0$.
- $\varkappa_p \neq 0$ is a real number such that

$$\bar{\phi}^t_* : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\varkappa_p t}x.$$

Problems

Problem

To define a Lefschetz number of the flow ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

- H^j is some cohomology theory associated with $\mathcal{F}$,
- Tr is a trace,

in such a way that the above Guillemin-Sternberg formula holds.

Motivation:

Deninger's program to study zeta- and L-functions for algebraic schemes over the integers, in particular, the Riemann zeta-function (Berlin, ICM, 1998).

Nonsingular flows

ASSUMPTIONS:

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a simple foliated flow.
- ϕ has no fixed points:
  - all the closed orbits are simple,
  - all the orbits in M are transverse to the leaves.

Leafwise de Rham complex

$(\Omega(\mathcal{F}), d_{\mathcal{F}})$ the leafwise de Rham complex of $\mathcal{F}$:

- $\Omega^{\cdot}(\mathcal{F}) = C^{\infty}(M, \Lambda^{\cdot} T^{*}\mathcal{F})$ smooth leafwise differential forms;
- $d_{\mathcal{F}} : \Omega^{\cdot}(\mathcal{F}) \to \Omega^{\cdot+1}(\mathcal{F})$ the leafwise de Rham differential.

In a foliated chart with coordinates $(x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that leaves are given by $y = c$,
a p-form $\omega \in \Omega^{p}(\mathcal{F})$ is written as

$$\omega = \sum_{\alpha_1<\alpha_2<\ldots<\alpha_p} a_{\alpha}(x, y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

and $d_{\mathcal{F}}\omega \in \Omega^{p+1}(\mathcal{F})$ is given by

$$d_{\mathcal{F}}\omega = \sum_{j=1}^{n-1} \sum_{\alpha_1<\alpha_2<\ldots<\alpha_p} \frac{\partial a_{\alpha}}{\partial x_j}(x, y) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

Leafwise de Rham cohomology

- The reduced leafwise de Rham cohomology of $\mathcal{F}$:

$$\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \overline{\operatorname{im} d_{\mathcal{F}}},$$

  the closure is in C^∞-topology.
- ϕ is a foliated flow $\Longrightarrow d_{\mathcal{F}} \circ \phi^t = \phi^t \circ d_{\mathcal{F}}$.
  The induced action:

$$\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F}).$$

Question

The trace of $\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$?

The leafwise Hodge decomposition

- g the Riemannian metric on M such that the infinitesimal generator Z of the flow ϕ is of length one and is orthogonal to the leaves — a bundle-like metric (so $\mathcal{F}$ is a Riemannian foliation.).
- $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplacian on $\Omega(\mathcal{F})$ (a second order tangentially elliptic differential operator on M).
- $\mathcal{H}(\mathcal{F})$ the space of leafwise harmonic forms on M:

$$\mathcal{H}(\mathcal{F}) = \{\omega \in \Omega(\mathcal{F}) : \Delta_{\mathcal{F}}\omega = 0\}.$$

Theorem (Alvarez Lopez - Yu. K)

The Hodge isomorphism

$$\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F}).$$

The Lefschetz distribution

For any $f \in C_c^\infty(\mathbb{R})$, define

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F}),$$

where $\Pi : L^2\Omega(\mathcal{F}) \to L^2\mathcal{H}(\mathcal{F})$ is the orthogonal projection.

A_f is a smoothing operator:

The Schwartz kernel $K_{A_f} = K_{A_f}(x,y)|dy|$ of A_f is smooth:

$$A_f u(x) = \int_M K_{A_f}(x,y) u(y) |dy|.$$

In particular, A_f is of trace class and

$$\operatorname{Tr} A_f = \int_M \operatorname{tr} K_{A_f}(x,x)|dx|.$$

The Lefschetz distribution

For any $f \in C^\infty_c(\mathbb{R})$,

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\,dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F}),$$

where $\Pi : L^2\Omega(\mathcal{F}) \to L^2\mathcal{H}(\mathcal{F})$ is the orthogonal projection.

The Lefschetz distribution $L(\phi) \in \mathcal{D}'(\mathbb{R})$:

$$< L(\phi), f >= \operatorname{Tr}^s A_f := \sum_{j=1}^{n-1} (-1)^j \operatorname{Tr} A_f^{(i)}, \quad f \in C^\infty_c(\mathbb{R}),$$

where $A_f^{(i)}$ is the restriction of A_f to $\Omega^i(\mathcal{F})$.

The Lefschetz formula

Theorem (Alvarez Lopez - Y.K.)

Assume that ϕ is simple and has no fixed points.

- *On $\mathbb{R}\setminus\{0\}$*

$$L(\phi)=\sum_c l(c)\sum_{k\neq 0}\varepsilon_{kl(c)}(c)\delta_{kl(c)},$$

when c runs over all closed orbits of ϕ and $l(c)$ denotes the minimal period of c.

- *In some neighborhood of 0 in $\mathbb{R}$:*

$$L(\phi)=\chi_\Lambda(\mathcal{F})\cdot\delta_0.$$

$\chi_\Lambda(\mathcal{F})$ the Λ-Euler characteristic of $\mathcal{F}$ given by the holonomy invariant transverse measure Λ (Connes, 1979).

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- $M^1 = M \setminus M^0$ the transitive point set.

Definition

The foliated flow ϕ is simple, i.e.:

- all of its fixed points and closed orbits are simple,
- its orbits in M^1 are transverse to the leaves.

Difficulties

$\mathcal{F}$ is a foliation almost without holonomy:

If ϕ is simple, then:

- M^0 is a finite union of compact leaves,
- only the leaves in M^0 may have non-trivial holonomy groups.

In particular, $\mathcal{F}$ is not a Riemannian foliation.

- The leafwise Laplacian $\Delta_{\mathcal{F}}$ is transversally elliptic only on the transitive point set M^1, not on M^0.
- As a consequence, the operator

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F})$$

is not a smoothing operator. Its Schwartz kernel is smooth on $M^1 \times M^1$ and singular near $M^0 \times M^0$.
So its trace is not well-defined.

The transitive point set and its blow-up

- M_l^1, $l = 1, \ldots, r$, the connected components of $M^1 (= M \setminus M^0)$:
$$(M^1, \mathcal{F}^1) = \bigsqcup_l (M_l^1, \mathcal{F}_l^1).$$
- M^l is the closure of M_l^1:
$$M^l = \overline{M_l^1}.$$
Thus, M_l is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}_l$ tangent to the boundary.
- Put
$$M^{\mathrm{c}} := \bigsqcup_l M_l, \quad \mathcal{F}^{\mathrm{c}} := \bigsqcup_l \mathcal{F}_l.$$
- The flow lifts to a simple foliated flow $\phi^{\mathrm{c},t}$ of $\mathcal{F}^{\mathrm{c}}$ tangent to ∂M^{c}.

Riemannian metric on the transitive point set

There exists a Riemannian metric g^1 on M^1:

- M^1_l equipped with $g_l := g^1|_{M^1_l}$ is a manifold of bounded geometry;
- g^1 is bundle-like for $\mathcal{F}^1$;
- $\mathcal{F}^1_l$ a Riemannian foliation of bounded geometry;
- ϕ^t_l a flow of bounded geometry.

Remarks:

- g^1 is singular at M^0.
- Each (M^1_l, g^1_l) is a Riemannian manifold with cylindrical ends.

Local model for g^1 near a compact leaf

Take a compact leaf $L \subset M^0$. Then, by the local stability theorem,

- a tubular nbhd V of L in M is diffeomorphic to a tubular nbhd V_L of L in the suspension foliated manifold $(M_L = \widetilde{L} \times_\Gamma \mathbb{R}, \mathcal{F}_L)$:

$$V \subset M \equiv V_L \subset M_L = \widetilde{L} \times_\Gamma \mathbb{R},$$

- the flow ϕ^t on $V \equiv V_L$:

$$\phi^t([\tilde{y}, x]) = [\phi^t_x(\tilde{y}), e^{\varkappa_L t} x], \quad [\tilde{y}, x] \in V_L \subset M_L = \widetilde{L} \times_\Gamma \mathbb{R},$$

- the Riemannian metric g^1 on $M^1 \equiv M_L \setminus L = \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\})$:

$$g^1 = g_{\mathcal{F}_L} + \frac{dx^2}{x^2}, \quad [\tilde{y}, x] \in \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\}),$$

where $g_{\mathcal{F}_L}$ is a leafwise Riemanian metric on $(M_L, \mathcal{F}_L)$.

Differential operators on the blow-up

- The blow up of the transitive point set M^1:

$$M^c = \bigsqcup_l M_l, \quad \mathcal{F}^c = \bigsqcup_l \mathcal{F}_l,$$

  M_l a connected compact manifold with boundary,
  $\mathcal{F}_l$ a smooth foliation tangent to the boundary:

$$\mathring{M}_l \equiv M_l^1, \quad \mathring{\mathcal{F}}_l \equiv \mathcal{F}_l^1.$$

- We transfer the Riemannian metric g^1 to $\mathring{M}_l$. Then $\mathring{M}_l$ is a manifold of bounded geometry and $\mathring{\mathcal{F}}_l$ is a Riemannian foliation of bounded geometry.
- $d_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham differential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $\delta_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham codifferential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $D_{\mathring{\mathcal{F}}_l} = d_{\mathring{\mathcal{F}}_l} + \delta_{\mathring{\mathcal{F}}_l}$.

Smoothing operators

Definition

Let $\mathcal{A}$ be the Fréchet algebra of functions $\psi : \mathbb{R} \to \mathbb{C}$ such that the Fourier transform $\hat{\psi}$ satisfies that, for every $k \in \mathbb{N}$, there is some $A_k > 0$ such that, for all $\xi \in \mathbb{R}$,

$$|\hat{\psi}(\xi)| \leq A_k e^{-k|\xi|} .$$

$\mathcal{A}$ contains all functions with compactly supported Fourier transform, as well as the Gaussians $x \mapsto e^{-tx^2}$ with $t > 0$.

Definition

For any $\psi \in \mathcal{A}$, $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$ and l, the operator

$$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$$

is a smoothing operator on $\mathring{M}_l$, but its kernel is singular near $\partial\mathring{M}_l$.

b-calculus (R. Melrose)

Theorem (Alvarez Lopez, Yu.K., Leichtnam)

$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$ gives rise to $P_l \in \Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)$.

- The Schwartz kernel K_{P_l} is smooth in the interior $\mathring{M}_l \times \mathring{M}_l$.
- K_{P_l} has a C^∞ extension to $M_l \times M_l \setminus \partial M_l \times \partial M_l$ that vanishes to all orders at $(\partial M_l \times M_l) \cup (M_l \times \partial M_l)$.
- Consider a tubular neighborhood of $L \subset \pi_0(\partial M_l)$ with coordinates $(\rho, y), \rho \in (0,\infty), y \in L$.
  Then $K_{P_l} = K_{P_l}(\rho, y, \rho', y')u(\rho', y')|d\rho'||dy'|$ has the form

$$K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'} \kappa_{P_l}\left(\rho, y, \frac{\rho'}{\rho}, y'\right),$$

  where $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

b-trace

In a tubular neighborhood of L with coordinates $\rho \in (0, \epsilon_0), y \in L$,

$$P_l u(\rho, y) = \int K_{P_l}(\rho, y, \rho', y') u(\rho', y') |d\rho'||dy'|,$$

$$K_{P_l}(\rho, y, \rho', y') = \frac{1}{\rho'} \kappa_{P_l}(\rho, y, \frac{\rho'}{\rho}, y'),$$

and $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

Definition

$$ {}^{\mathrm{b}}\mathrm{Tr}\,(P_l) = \lim_{\epsilon \to 0} \left(\int_{\rho > \epsilon} K_{P_l}(\rho, y, \rho, y) |d\rho||dy| + \ln \epsilon \int \kappa_{P_l}(0, y, 1, y) |dy| \right).$$

Key fact

The functional ${}^{\mathrm{b}}\mathrm{Tr}$ doesn't have trace propertry, but ${}^{\mathrm{b}}\mathrm{Tr}\,[P, P']$ is expressed in terms of traces of some explicit integral operators on ∂M_l.

Operators on the transitive point set

Since $M^{\mathrm{c}} = \bigsqcup_l M_l, \mathcal{F}^{\mathrm{c}} = \bigsqcup_l \mathcal{F}_l$, we get the operator

$$P \equiv \bigoplus_l P_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathcal{F}^{\mathrm{c}}})$$

$$\in \Psi_{\mathrm{b}}^{-\infty}(M^{\mathrm{c}}; \textstyle\bigwedge T\mathcal{F}^{\mathrm{c}*}) \equiv \displaystyle\bigoplus_l \Psi_{\mathrm{b}}^{-\infty}(M_l; \textstyle\bigwedge T\mathcal{F}_l^*) \,.$$

In particular, its b-trace ${}^{\mathrm{b}}\mathrm{Tr}\,(P)$ is well-defined.
The b-supertrace of P:

$${}^{\mathrm{b}}\mathrm{Tr}\,^{\mathrm{s}}(P) = \sum_{j=1}^{n-1} (-1)^j \, {}^{\mathrm{b}}\mathrm{Tr}\,(P^{(j)}),$$

where $P^{(j)}$ is the restriction to j-forms.

Derivative of the b-supertrace

We follow the heat kernel approach to index theory:

Fix an even $\psi \in \mathcal{A}$ and $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$.
For $u > 0$, let

$$P_{\psi_u,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(uD_{\mathcal{F}^{\mathrm{c}}})$$

Since the b-trace is not a trace, $\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) \neq 0$.

Theorem

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma})\,.$$

Notation

Theorem

$$\frac{d}{du} {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma}) ,$$

- $\widetilde{L}$ the universal covering of L, $\Gamma_L := \pi_1 \widetilde{L}$.
- T^*_γ the induced action of $\gamma \in \Gamma_L$ on Γ_L-invariant operators on $\widetilde{L}$.
- Tr_{Γ_L} the Γ_L-trace on Γ_L-invariant operators on $\widetilde{L}$.
- $\widetilde{R}_{\widetilde{L},u,t} = u\tilde{\eta} \wedge \tilde{\phi}^{t*}_L \psi'(uD_{\widetilde{L}})$.
- $\tilde{\eta}$ a closed one-form on $\widetilde{L}$, the lift of a closed one-form η on L. If we consider η as a closed leafwise 1-form on the suspension manifold $M_L = \widetilde{L} \times_\Gamma \mathbb{R}$, then there exists a 1-form ω on M_L satisfying $T\mathcal{F}_L = \ker\omega$ such that $d\omega = \eta \wedge \omega$.
- $\phi^t_L : L \to L$ the restriction of the flow to L.

More notation

Theorem

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma}) \,,$$

- $\varkappa_L \neq 0$ a real number such that, for $p \in L$,

$$\bar{\phi}^t_* : N_p\mathcal{F} \to N_p\mathcal{F}, \quad x \to e^{\varkappa_L t}x.$$

- $t_{L,\gamma} = -\varkappa_L^{-1} \log a_{L,\gamma}$ relative periods, where a homomorphism $\gamma \in \Gamma_L \mapsto a_{L,\gamma} \in \mathbb{R}^+$ is given by the holonomy homomorphism

$$\gamma \in \Gamma_L \mapsto \bar{h}_{L,\gamma} \in \mathrm{Diffeo}_+(\mathbb{R},0), \quad \bar{h}_{L,\gamma}(x) = a_{L,\gamma}x.$$

Variation of the b-supertrace and Lefschetz distribution

For $u, v > 0$,

$$ {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_v,f}) - {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}) \,, $$

$$ \widetilde{S}_{\widetilde{L},u,v,t} = \int_u^v \widetilde{R}_{\widetilde{L},w,t}\, dw = \widetilde{\eta}\wedge \tilde{\phi}_L^{t*} \frac{\psi(vD_{\widetilde{L}}) - \psi(uD_{\widetilde{L}})}{D_{\widetilde{L}}} \,. $$

Definition

The Lefschetz distribution

$$ \langle L(\phi), f\rangle = {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_v,f}) - \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}). $$

Here the right-hand side is independent of v.

Trace formula

Theorem

$L(\phi)$ is a well-defined distribution on $\mathbb{R}_+$ given by

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot \delta_{kl(c)}$$

on $\mathbb{R}_+$, where c runs over all closed orbits of ϕ^t, $l(c)$ denotes the minimal period of c, and x is an arbitrary point of c.

Concluding remarks

Remark

The next problem is to give a cohomological interpretation of the limit as $v \to +\infty$ of

$$ {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_v,f}) - \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}). $$

Remark

Contribution of fixed points as in the Guillemin-Sternberg formula.