Naturally graded Lie algebras of slow growth

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$\mathbb{N}$-graded Lie algebras

Definition

A Lie algebra $\mathfrak{g}$ is called $\mathbb{N}$-graded if there is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}_i, \ [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \text{ for all } i, j \in \mathbb{N}.$$

Example

The Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, \dots, e_n, \dots$ with the commutation relations:

$$[e_1, e_i] = e_{i+1}, \ \forall \ i \geq 2.$$

$\mathfrak{m}_2$ and positive part W^+ of the Witt algebra

The Lie algebra $\mathfrak{m}_2$ is defined by its infinite basis $e_1, e_2, \dots, e_n, \dots$ and

$$[e_1, e_i] = e_{i+1}, \quad \forall\ i \geq 2; \qquad [e_2, e_j] = e_{j+2}, \quad \forall\ j \geq 3.$$

The Lie algebra W^+ can be defined by its basis $\{e_i, i \in \mathbb{N}\}$ and

$$[e_i, e_j] = (j-i)e_{i+j},\ \forall i, j \in \mathbb{N}.$$

Narrow graded Lie algebras after Zelmanov and Shalev

Definition

A $\mathbb{N}$-graded Lie algebra $\mathfrak{g}=\bigoplus_{i\in\mathbb{N}}\mathfrak{g}_i$ is called of width d if there exists (minimal) $d \in \mathbb{N}$ such that

$$\dim \mathfrak{g}_i \leq d, \forall i \in \mathbb{N}.$$

The Lie algebras $\mathfrak{m}_0, \mathfrak{m}_2, W^+$ considered above are examples of narrowest graded Lie algebras (with width $d = 1$).

Fialowski in 1983 classified $\mathbb{N}$-graded Lie algebras of width 1. Besides $\mathfrak{m}_0, \mathfrak{m}_2, W^+$, there are other interesting $\mathbb{N}$-graded Lie algebras in her list.

The Lie algebra $\mathfrak{n}_1$

Polynomial matrices defined for $k \in \mathbb{N}$ by

$$e_{3k+1}=\frac{1}{2}\begin{pmatrix}0 & t^{2k+1}\\ 0 & 0\end{pmatrix}, e_{3k+2}=\begin{pmatrix}0 & 0\\ t^{2k+1} & 0\end{pmatrix}, e_{3k+3}=\frac{1}{2}\begin{pmatrix}t^{2k+2} & 0\\ 0 & -t^{2k+2}\end{pmatrix}$$

The linear span $\langle e_1, e_2, e_3, \dots, e_n, \dots \rangle$ is a positively graded subalgebra $\mathfrak{n}_1$ in the loop Lie algebra $\mathfrak{s}l(2,\mathbb{K}) \otimes \mathbb{K}[t]$. It is $\mathbb{N}$-graded with one-dimensional homogeneous components:

$$\mathfrak{n}_1 = \oplus_{i=1}^{+\infty} \langle e_i \rangle \subset \mathfrak{s}l(2,\mathbb{K}) \otimes \mathbb{K}[t],$$

with the Lie bracket

$$[e_i, e_j] = c_{i,j} e_{i+j},\ \ c_{i,j} = \begin{cases} 1, \text{if } j-i \equiv 1 \bmod 3; \\ 0, \text{if } j-i \equiv 0 \bmod 3; \\ -1, \text{if } j-i \equiv -1 \bmod 3. \end{cases}$$

Twisted loop algebra $\mathfrak{n}_2 = \oplus_{i=1}^{+\infty} \langle f_i \rangle \subset \mathfrak{sl}(3, \mathbb{K}) \otimes \mathbb{K}[t]$,

$$f_{8k+1} = \begin{pmatrix} 0 & t^{2k} & 0 \\ 0 & 0 & t^{2k} \\ 0 & 0 & 0 \end{pmatrix}, f_{8k+2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^{2k+1} & 0 & 0 \end{pmatrix}, f_{8k+3} = \begin{pmatrix} 0 & 0 \\ t^{2k+1} & 0 \\ 0 & -t^{2k+1} \end{pmatrix}$$

$$f_{8k+4} = \begin{pmatrix} t^{2k+1} & 0 & 0 \\ 0 & -2t^{2k+1} & 0 \\ 0 & 0 & t^{2k+1} \end{pmatrix}, f_{8k+5} = \begin{pmatrix} 0 & t^{2k+1} & 0 \\ 0 & 0 & -t^{2k+1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_{8k+6} = \begin{pmatrix} 0 & 0 & t^{2k+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_{8k+7} = \begin{pmatrix} 0 & 0 & 0 \\ t^{2k+2} & 0 & 0 \\ 0 & t^{2k+2} & 0 \end{pmatrix}, f_{8k+8} = \begin{pmatrix} t^{2k+2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[f_q, f_l] = d_{q,l} f_{q+l}, \; q, l \in \mathbb{N}.$$

Descending central series and natural grading

Let $\mathfrak{g}$ be a Lie algebra and its descending central series is

$$\mathfrak{g}^1=\mathfrak{g} \supset \mathfrak{g}^2=[\mathfrak{g},\mathfrak{g}] \supset \dots \supset \mathfrak{g}^k=[\mathfrak{g},\mathfrak{g}^{k-1}] \supset \dots$$

$\mathfrak{g}$ is called nilpotent if there exists s such that $\mathfrak{g}^s \neq 0, \mathfrak{g}^{s+1}=0$.

One can consider its associated graded Lie algebra

$$\mathrm{gr}_C\mathfrak{g} = \oplus_{i=1}^{+\infty}\left(\mathfrak{g}^i/\mathfrak{g}^{i+1}\right)$$

with the Lie bracket:

$$\left[x+\mathfrak{g}^{i+1}, y+\mathfrak{g}^{j+1}\right] = [x,y]+\mathfrak{g}^{i+j+1}, x \in \mathfrak{g}^i, y \in \mathfrak{g}^j.$$

Definition

A Lie algebra $\mathfrak{g}$ is called naturally graded if it is isomorphic to its associated graded $\mathrm{gr}_C\mathfrak{g}$.

Some remarks

The Lie algebra $\mathfrak{m}_0$ is naturally graded:

$$\mathfrak{m}_0 \cong \mathrm{gr}_C \mathfrak{m}_0 = \oplus_{i=1}^{+\infty} \mathfrak{m}_{0i}.$$

But its first homogeneous component is two-dimensional now:

$$\mathfrak{m}_{01} = \langle e_1, e_2 \rangle, \mathfrak{m}_{02} = \langle e_3 \rangle, \dots, \mathfrak{m}_{0i} = \langle e_{i+1} \rangle, i \geq 2.$$

However the positive part W^+ of the Witt algebra and $\mathfrak{m}_2$ are not naturally graded.

$$\mathrm{gr}_C \mathfrak{m}_2 \cong \mathrm{gr}_C W^+ \cong \mathrm{gr}_C \mathfrak{m}_0 \cong \mathfrak{m}_0.$$

Finite-dimensional case: Carnot Lie algebras

The natural grading has very important property

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i \in \mathbb{N}. \quad (1)$$

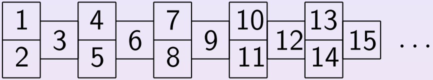
In particular it means that $\mathfrak{g}$ is generated by its first homogeneous component $\mathfrak{g}_1$.
If a naturally graded Lie algebra $\mathfrak{g}$ is finite-dimensional, i.e. it means that exists N such that $\mathfrak{g}_i = 0, i > N$, then $\mathfrak{g}$ is nilpotent.

Definition

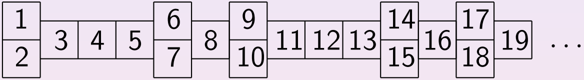
A finite-dimensional $\mathbb{N}$-graded Lie algebra is called Carnot Lie algebra if it satisfies (1).

The Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are naturally graded and they have the width $d = 2$ (as naturally graded Lie algebras).

$\mathfrak{n}_1$:



$\mathfrak{n}_2$:



Theorem (M. Vergne, 1970)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be a naturally graded of maximal class, i.e.*

$$\dim \mathfrak{g}_1 = 2, \dim \mathfrak{g}_i = 1, i \geq 2.$$

then $\mathfrak{g} \cong \mathfrak{m}_0$

Growth of Lie algebras

Suppose that an infinite-dimensional Lie algebra $\mathfrak{g}$ is generated by a finite-dimensional subspace V_1. For $n > 1$, let V^n denote the $\mathbb{K}$-linear span of all products in elements of V_1 of length at most n with arbitrary arrangements of brackets. Clearly $V_1 \subset V_2 \subset \dots \subset V_n \subset \dots$ is an ascending chain of finite-dimensional subspaces of $\mathfrak{g}$ and $\cup_{i=1}^{+\infty} V_i = \mathfrak{g}$. The Gelfand-Kirillov dimension of $\mathfrak{g}$ is

$$GKdim \mathfrak{g} = \limsup_{n \to +\infty} \frac{\log \dim V_n}{\log n}.$$

A finite Gelfand-Kirillov dimension means that there exists a polynomial $P(x)$ such that $\dim V_n < P(n)$ for all $n > 1$.

The growth function is $F(n) = \dim V_n = \dim(\mathfrak{g}/\mathfrak{g}^{n+1})$. For the Lie algebras $\mathfrak{m}_0, \mathfrak{m}_2$ and W^+ (maximal class or filiform) we have

$$\dim V_n = n+1$$

and it is the slowest possible growth. An arbitrary naturally graded Lie algebra $\mathfrak{g}$ of width 2 grows not faster than $2n$. For instance if $\mathfrak{g} = \mathfrak{n}_1$ we have

$$\dim V_n = \left[\frac{3n+1}{2}\right].$$

All these Lie algebras have $GKdim\mathfrak{g} = 1$.

Central extensions of Lie algebras

A central extension of a Lie algebra $\mathfrak{g}$ is an exact sequence

$$0 \to V \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras and their homomorphisms, in which the image of $V \to \tilde{\mathfrak{g}}$ is contained in the centre $Z(\tilde{\mathfrak{g}})$.
At the level of vector spaces we have

$$0 \to V \to V \oplus \mathfrak{g} \to \mathfrak{g} \to 0$$

where the Lie bracket $[,]_{V\oplus\mathfrak{g}}$ in the vector space $V \oplus \mathfrak{g}$ is defined by the formula

$$[(v,g),(w,h)]_{V\oplus\mathfrak{g}} = (c(g,h),[g,h]_{\mathfrak{g}}).$$

The Jacobi identity for this Lie bracket is equivalent to $c : \mathfrak{g} \times \mathfrak{g} \to V$ being cocycle.

Naturally graded Lie algebras as central extensions

Let $\tilde{\mathfrak{g}} = \oplus_{i=1}^{k} \mathfrak{g}_i$ be a naturally graded Lie algebra.
Obviously $\mathfrak{g}_k \subset Z(\tilde{\mathfrak{g}})$.
One can consider the central extension

$$0 \to \mathfrak{g}_k \to \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/\mathfrak{g}_k \to 0.$$

Let fix $e_1^k, \dots, e_{m_k}^k$ a basis of $\mathfrak{g}_k$, then we can write our two-cocycle in the coordinates

$$c(\cdot,\cdot) = c_1(\cdot,\cdot)e_1^k + \dots + c_{m_k}(\cdot,\cdot)e_{m_k}^k,$$

Naturally graded Lie algebras as central extensions

We have

$$c_1(\cdot,\cdot), \dots, c_{m_k}(\cdot,\cdot) \in H^2_{(k)}(\tilde{\mathfrak{g}}/\mathfrak{g}_k, \mathbb{K}).$$

Proposition

Let $c_1(\cdot,\cdot), \dots, c_{m_k}(\cdot,\cdot)$ and $\tilde{c}_1(\cdot,\cdot), \dots, \tilde{c}_{m_k}(\cdot,\cdot)$ be two k-sets of cocycles from $H^2_{(k)}(\tilde{\mathfrak{g}}/\mathfrak{g}_k, \mathbb{K})$.
They define isomorphic extensions iff linear spans $\langle c_1(\cdot,\cdot), \dots, c_{m_k}(\cdot,\cdot)\rangle$ and $\langle \tilde{c}_1(\cdot,\cdot), \dots, \tilde{c}_{m_k}(\cdot,\cdot)\rangle$ are in the same orbit of $\mathrm{Aut}_{gr}(\tilde{\mathfrak{g}}/\mathfrak{g}_k)$ in the space of m_k-planes in $H^2_{(k)}(\tilde{\mathfrak{g}}/\mathfrak{g}_k, \mathbb{K})$.

Remark

$$\dim \mathfrak{g}_k \le \dim H^2_{(k)}(\tilde{\mathfrak{g}}/\mathfrak{g}_k, \mathbb{K})$$

$\mathrm{Aut}_{gr}(\mathcal{L}(2,3))$-action on $\mathbb{P}H^2_{(4)}(\mathcal{L}(2,3),\mathbb{R})$

Minimal model for $\mathcal{L}(2,3)$:

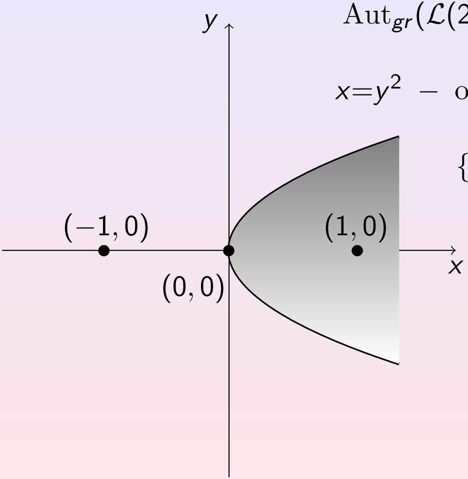
$$\begin{gathered} da^1{=}db^1{=}0, \;\; da^2{=}a^1\wedge b^1, \\ da^3{=}a^1\wedge a^2, \;\; db^3 = b^1\wedge a^2; \end{gathered} \tag{2}$$

$$H^2_{(4)}(\mathcal{L}(2,3),\mathbb{R}) = \langle a^1\wedge a^3,\; a^1\wedge b^3{+}a^3\wedge b^1,\; b^1\wedge b^3\rangle.$$

Graded automorphisms, $\mathrm{Aut}_{gr}(\mathcal{L}(2,3)){=}GL(2,\mathbb{R})$

$$A = \begin{pmatrix} \alpha & \beta \\ \rho & \mu \end{pmatrix} \to \det A \cdot \begin{pmatrix} \alpha^2 & 2\rho\alpha & \rho^2 \\ \alpha\beta & \rho\beta{+}\alpha\mu & \mu\rho \\ \beta^2 & 2\mu\beta & \mu^2 \end{pmatrix}$$

3 orbits of $\mathrm{Aut}_{gr}(\mathcal{L}(2,3))$-action on $\mathbb{P}H^2_{(4)}(\mathcal{L}(2,3),\mathbb{R})$



$\mathrm{Aut}_{gr}(\mathcal{L}(2,3)) = GL(2,\mathbb{R})$

$x = y^2$ – orbit of $(0{:}0{:}1)$

$\{(\rho^2{:}\mu\rho{:}\mu^2)\} \subset \mathbb{R}P^2$

$\mu \neq 0$

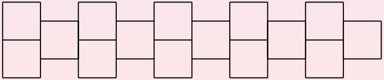
Two real loop algebras

Two real forms $\mathfrak{so}(3,\mathbb{R})$, $\mathfrak{so}(1,2)$ of $\mathfrak{sl}(2,\mathbb{C})$ can be defined by the basis u, v, w and commutating relations

$$[u,v]{=}w,\ [v,w]{=}\pm u,\ [w,u]{=}v.$$

Now we consider two subalgebras $\mathfrak{n}_1^{\pm}$ in loop algebras $\mathfrak{so}(3,\mathbb{R})\otimes\mathbb{R}[t]$ and $\mathfrak{so}(1,2)\otimes\mathbb{R}[t]$ respectively. They are defined by basic elements

$$\begin{matrix} u\otimes t^1 \\ v\otimes t^1 \end{matrix},\ w\otimes t^2,\ \begin{matrix} u\otimes t^3 \\ v\otimes t^3 \end{matrix},\ w\otimes t^4,\ \begin{matrix} u\otimes t^5 \\ v\otimes t^5 \end{matrix},\ w\otimes t^6,\ldots$$


$\ldots \quad \ldots$

Theorem

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *be a real naturally graded Lie algebra such that:*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \forall i \in \mathbb{N}.$$

Then $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *is isomorphic to the only one Lie algebra from the following list:*

$$\mathfrak{m}_0, \mathfrak{n}_1^{\pm},\ \mathfrak{n}_2, \left\{\mathfrak{m}_0^S \mid S \subset \{3,5,7,9,\dots\}\right\},$$

where $\left\{\mathfrak{m}_0^S \mid S \subset \{3,5,7,9,\dots\}\right\}$ *are central extensions of* $\mathfrak{m}_0$ *that correspond to the sequence* S *of two-cocycles.*

Remark

The Lie algebras $\mathfrak{n}_1^{\pm}$ are isomorphic over $\mathbb{C}$.

$H^2(\mathfrak{m}_0, \mathbb{K})$.

The second cohomology space $H^2(\mathfrak{m}_0, \mathbb{K})$ is graded and generated by cocycles of odd gradings

$$e^2 \wedge e^3, e^2 \wedge e^5 - e^3 \wedge e^4, e^2 \wedge e^7 - e^3 \wedge e^6 + e^4 \wedge e^5, \dots$$

Thank you!