Matrix commuting differential operators of rank 2

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Let us consider two differential operators

$$L_n = \sum_{i=0}^{n} u_i(x)\partial_x^i, \quad L_m = \sum_{i=0}^{m} v_i(x)\partial_x^i,$$

where coefficients $u_i(x)$ and $v_i(x)$ are scalar or matrix valued functions. The commutativity condition $L_nL_m = L_mL_n$ is equivalent to a very complicated system of nonlinear differential equations.

If two differential operators with scalar or matrix valued coefficients commute, then there exists a nonzero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$. The curve Γ defined by $R(z, w) = 0$ is called the spectral curve. If

$$L_n\psi = z\psi, \quad L_m\psi = w\psi,$$

then $(z, w) \in \Gamma$.

Some facts about scalar commuting differential operators

If coefficients are scalar functions, then for almost all $(z, w) \in \Gamma$, the dimension of the space of common eigenfunctions ψ is the same. The dimension of the space of common eigenfunctions of two commuting scalar differential operators is called the rank of this pair. The rank is a common divisor of m and n. The genus of the spectral curve of a pair of commuting operators is called the genus of this pair.
If the rank of two commuting scalar differential operators equals 1, then there are explicit formulas for coefficients of commutative operators in terms of Riemann theta-functions.

If the rank equals 1, then common eigenfunction $\psi(x, P)$ is Baker-Akhiezer function, where $P \in \Gamma$. Baker-Akhiezer function is a function with the following properties.

1) Function $\psi(x, P)$ has one essential singularity at a fixed point $q \in \Gamma$

$$\psi(x, P) = e^{kx}(1 + \frac{\xi_1(x)}{k} + \frac{\xi_2(x)}{k^2} + ...),$$

where k^{-1} is a local parameter in a neighborhood of q.

2) Function ψ has simple poles at some points $\gamma_1, ..., \gamma_g$, where g is the genus of Γ.
The set $\{\Gamma, q, \gamma_1, ..., \gamma_g\}$ is called the spectral data. If we take the spectral data where $D = \gamma_1 + ... + \gamma_g$ is a non-special divisor, then there is a unique function $\psi(x, P)$ satisfying the conditions 1) and 2).

The case when rank is greater than one is much more difficult. The first examples of commuting ordinary scalar differential operators of the nontrivial ranks 2 and 3 and the nontrivial genus g=1 were constructed by Dixmier for the nonsingular elliptic spectral curve $w^2 = z^3 - \alpha$, where α is arbitrary nonzero constant:

$$L = (\partial_x^2 + x^3 + \alpha)^2 + 2x,$$

$$M = (\partial_x^2 + x^3 + \alpha)^3 + 3x\partial_x^2 + 3\partial_x + 3x(x^2 + \alpha),$$

where L and M is the commuting pair of the Dixmier operators of rank 2, genus 1.

A general classification of commuting scalar differential operators was obtained by Krichever. The general form of commuting scalar operators of rank 2 for an arbitrary elliptic spectral curve was found by Krichever and Novikov. The general form of scalar commuting operators of rank 3 with arbitrary elliptic spectral curve was found by Mokhov. Mironov developed theory of scalar operators of rank 2 and found examples of commuting scalar operators of rank 2 and arbitrary genus. Using Mironov's method many examples of scalar commuting operators of rank 2 and arbitrary genus were found. Moreover, examples of commuting scalar differential operators of arbitrary genus and arbitrary rank with polynomial coefficients were constructed by Mokhov.

Commuting differential operators with polynomial coefficients are very important because of the following reasons. Let us consider differential operators with polynomial coefficients of n variables. The algebra of differential operators with polynomial coefficients of n variables is isomorphic to the Weyl algebra A_n. We say that map $f : A_n \to A_n$ is endomorphism if $[f(\partial_{x_i}), f(x_j)] = [\partial_{x_i}, x_j] = \delta_{ij}$. Dixmier conjecture asserts that any endomorhism of the algebra A_n is isomorphism. Dixmier conjecture is open for $n \geqslant 1$. Kontsevich, Belov-Kanel and Tsushimonto proved that Dixmier conjecture for A_n implies Jacobian conjecture for $\mathbb{C}^n$ and Jacobian conjecture for $\mathbb{C}^{2n}$ implies Dixmier conjecture for A_n.

Theory of commuting differential operators helps to find solutions of nonlinear partial differential equations from mathematical physics. Also there are deep connections between theory of commuting scalar differential operators and Schottky problem.

What can we say about common eigenfunctions of commuting differential operators of rank greater than 1?

Let us consider operator

$$L = (\partial_x^2 + A_6 x^6 + A_2 x^2 + \frac{A_{-2}}{x^2} + \frac{A_{-6}}{x^6})^2 + 16g(g+1)A_6 x^4$$

where $g \in \mathbb{N}$, $A_6 \neq 0$, and A_2 is arbitrary constants. It is proved that operator L commutes with a differential operator M of order $4g+2$. The spectral curve has the form $w^2 = z^{2m+1} + a_{2m} z^{2m} + ... + a_1 z + a_0$. It is possible to find common eigenfunctions of operators L and M in particular cases. Common eigenfunctions are found in terms of Bessel and Heun equations. When the spectral curve is nonsingular, then there are no other known explicitly found eigenfunctions of commuting operators of rank 2.

There is an interesting question. Are there Riemann surface which can not be spectral curve of commuting differential operators with polynomial or rational coefficients?

Be careful!

Any elliptic curve of the form $w^2 = z^3 + g_2 z + g_3$ and any hyperelliptic curve of the form $w^2 = z^5 + \beta_4 z^4 + \beta_3 z^3 + \beta_2 z^2 + \beta_1 z + \beta_0$, where g_i and β_i are arbitrary constants, is spectral curve of commuting operators with rational coefficients.

Matrix commuting differential operators

A general classification of commuting matrix differential operators was obtained by Grinevich. Grinevich considered two differential operators

$$L = \sum_{i=0}^{m} U_i \partial_x^i, \quad M = \sum_{i=0}^{n} V_i \partial_x^i,$$

where U_i and V_i are smooth and complex-valued $s \times s$ matrices. Let us suppose the following conditions

1) $det(U_m) \neq 0$.
2) Eigenvalues $\lambda_1(x), ..., \lambda_s(x)$ of U_m are distinct.

Without loss of generality we can suppose that

$$(U_m)_{ij} = \delta_{ij}\lambda_i, \quad (V_n)_{ij} = \delta_{ij}\mu_i, \quad tr(U_{m-1}) = 0$$

Let Γ be the spectral curve of commuting matrix operators L, M. Spectral curve of matrix commuting operators can be reducible. Let Γ_i be an irreducible component of the spectral curve. The dimension of the space of common eigenfunctions

$$L\psi = z\psi, \quad M\psi = w\psi, \quad (z, w) \in \Gamma_i$$

is called the rank of commuting pair on Γ_i. Grinevich discovered that the spectral curve Γ has s points at infinity. So, $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$, where $k \leqslant s$. Let l_i be the rank of operators on Γ_i. Operators L, M are called commuting operators of vector rank $(l_1, ..., l_k)$, where $k \leqslant s$. Numbers l_i are common divisors of m and n.

If the rank of commuting matrix differential operators equals 1, then there exists explicit formulas for coefficients in terms of Riemann theta-functions

It is very difficult to construct explicit examples of matrix commuting differential operators of rank greater than 1. In this talk we propose an effective method for constructing matrix commuting differential operators of rank 2 and vector rank $(2, 2)$.

Let us consider the operator

$$L = E(x)\partial_x^2 + R(x)\partial + Q(x),$$

where

$$E = \begin{pmatrix} \lambda_1(x) & \lambda_3(x) \\ 0 & \lambda_2(x) \end{pmatrix}, R = \begin{pmatrix} r_1(x) & r_3(x) \\ r_2(x) & -r_1(x) \end{pmatrix}, Q = \begin{pmatrix} q_1(x) & q_3(x) \\ q_2(x) & q_4(x) \end{pmatrix}$$

We want to find an operator M of order $2g$ such that $[L, M] = 0$.

System of differential equations $LM - ML = 0$ is very complicated. But we can connect the system $LM - ML = 0$ with recurrence relations

$$\begin{cases} a_i^{m+1} = g_i(a_1^m, a_2^m, a_3^m, a_4^m, b_1^m, b_2^m, b_3^m, b_4^m, r_1, r_2, r_3, q_1, q_2, q_3, q_4) \\ b_i^{m+1} = h_i(a_1^m, a_2^m, a_3^m, a_4^m, b_1^m, b_2^m, b_3^m, b_4^m, r_1, r_2, r_3, q_1, q_2, q_3, q_4), \end{cases}$$

where $i = 1, 2, 3, 4$, r_j and q_j are coefficients of L.

It is possible to prove that if there exists g such that

$$\begin{cases} a_i^{g+1} = 0 \\ b_i^{g+1} = 0 \end{cases} \quad i = 1, ..., 4$$

then the operator L commutes with an operator M of order $2g$.

Let us suppose that $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$, $r_2(x) = r_3(x) = 0$, $q_3(x) = q_2(x)$ and $q_4(x) = -q_1(x)$.
Then recurrence relations become much more easier. Using them we can find the following examples and many others.

The operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} \alpha_2 x^2 + \alpha_0 & 0 \\ 0 & -\alpha_2 x^2 - \alpha_0 \end{pmatrix} \partial_x + \begin{pmatrix} \beta x^2 + \alpha_2 x & \gamma x \\ \gamma x & -\beta x^2 - \alpha_2 x \end{pmatrix},$$

where

$$\gamma^2 = -n^2 \alpha_2^2, \quad n \in \mathbb{N}$$

and $\alpha_2, \alpha_0, \beta$ are arbitrary constants, commutes with differential operator of order $4n$, where $g = 2n$. The order of operator M equals $4n$.

Calculations show that if $n \leqslant 3$, then the spectral curve of operators L, M is nonsingular for almost all $\alpha_0, \alpha_2, \beta$ and is hyperelliptic. Hence L and M are operators of rank 2. In some cases spectral curve is reducible and we get commuting operators of rank $(2, 2)$.

If $n = 1$, then the operator L commutes with operator M of order 2. The spectral curve of operators L, M has the form

$$w^2 = z^4 - (\alpha_0\alpha_2 - 2\beta)z^2 - \alpha_2\alpha_0\beta + \beta^2.$$

This spectral curve is nonsingular if $\alpha_2\alpha_0\beta(\alpha_2\alpha_0 - \beta) \neq 0$. So in nonsingular case we get that operators L, M are operators of rank 2. If $\alpha_0 = 0$, then the spectral curve has the form

$$w^2 = (z^2 + \beta)^2 \Leftrightarrow (w - z^2 - \beta)(w + z^2 + \beta) = 0 \qquad (1)$$

We see that if $\alpha_0 = 0$, then the spectral curve is reducible. Note that $M \neq L^2 + \beta$ and $M \neq -L^2 - \beta$ but $(M - L^2 - \beta)(M + L^2 + \beta) = 0$ and we have operators of vector rank (2,2).

Let $\wp(x)$ be the Weierstrass elliptic function satisfying the equation $(\wp'(x))^2 = 4\wp^3(x) + g_2\wp(x)$. The operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} 0 & \alpha\wp(x) \\ \alpha\wp(x) & 0 \end{pmatrix},$$
$$\alpha^2 = 64n^4 - 4n^2, \quad n \in \mathbb{N}$$

commutes with a differential operator (14), of order $4n$, where $g = 2n$. The order of operator M equals $4n$.

It is possible to construct many other examples of matrix commuting operators of rank 2. It is very easy to do using recurrence relations.

Thank you for attention